CONJUGATELY DENSE SUBGROUPS IN GENERALIZED FC-GROUPS

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Abstract. A subgroup H of a group G is called conjugately dense in G if H has nonempty intersection with each class of conjugate elements in G. The knowledge of conjugately dense subgroups is related with an unsolved problem in group theory, as testified in the Kourovka Notebook. Here we point out the role of conjugately dense subgroups in generalized FC-groups, generalized soluble groups and generalized nilpotent groups.

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1. Introduction and Statement of Results

If H is a subgroup of a group G, H is said to be *conjugately dense* in G if for each element g of G the intersection of H with the conjugates of g in G is nonempty. These subgroups arouse interest in various situations such as in [27, Problem 8.8b], where it is discussed the question of the existence of a noncyclic finitely presented group with a conjugately dense cyclic subgroup.

The notion of conjugately dense subgroup of a given group has been introduced in [19] by V. M. Levchuk and the importance of this notion is related to an open problem which P. M. Neumann has posed in [27, Problem 6.38a].

Here is conjectured that each conjugately dense irreducible subgroup of $GL(n, K)$, the general linear group of degree n over an arbitrary field K, where n is a fixed positive integer, coincides always with the whole group $GL(n, K)$, except when K is quadratically closed, $n = 2$ and the characteristic of K is 2.

Most of the literature on conjugately dense subgroups is not available in English and it mainly consists of [19, 20, 32, 33, 34]. The validity of the conjecture in the general case would imply that each conjugately dense subgroups of $GL(n, K)$

is parabolic, but [34, Theorem 2] shows that this is not true. From the argument of [34, Theorem 2], a positive answer to Neumann's conjecture depends on a hand by the choose of K, by the characteristic of K and by n. On an other hand by the properties of stability of the conjugately dense subgroups of $GL(n, K)$ with respect to certain usual operations between subgroups such as intersections, products and conjugation. A great problem to manage conjugately dense subgroups is due to the fact that they do not constitute a formation of groups (see [8, Chapter IV]). This difficulty has been often mentioned in [19, 20, 32, 33] by means of the remark that the intersection of two conjugately dense subgroups of a group G can not be always a conjugately dense subgroup of G.

From [20, Lemma 3], if H is a conjugately dense subgroup of a group G, then $G = \bigcup_{x \in G} H^x$. Therefore [6] is related with [19, 20, 32, 33], since the description of conjugately dense subgroups can be given from the following point of view:

$(*)$ Assume that G is a group and H is a subgroup of G such that $G = \bigcup_{x \in G} H^x$. When is it possible to deduce that $H = G$?

This simple remark allows us to state that the knowledge of the conjugately dense subgroups in [19, 20, 32, 33, 34] can be formulated as in $(*)$. Thanks to [6, Theorems 1–4], we may note that conjugately dense subgroups have been described for locally finite groups, locally supersoluble groups and groups which belongs to $\mathfrak{Z}$, where $\mathfrak{Z}$ denotes the class of groups defined as it follows: a group G belongs to $\mathfrak{Z}$ if it is not covered by conjugates of any proper subgroup.

The properties of the class $\mathfrak{Z}$ can be found in [6, 30, 31]. In particular, $\mathfrak{Z}$ is closed under extensions and (restricted) direct products (though not cartesian products), as well as containing all hypercentral and soluble groups. Furthermore, $\mathfrak{Z}$ contains the class of all finite groups, but $\mathfrak{Z}$ contains neither all locally nilpotent periodic groups nor infinite transitive groups of finitary permutations (see [6, Section 1]).

It seems opportune to recall that the notion of *density* between subgroups has been studied by several authors in different contexts. [21] represents the first work which introduces the concept of dense subgroup. Following [21], a group G has *dense ascendant* (*subnormal*) *subgroups* if whenever $H < K \leq G$ and H is not maximal in K, there is an ascendant (subnormal) subgroup of G lying strictly between the subgroups H and K of G. If we point out on special families of subgroups of G, then the notion of density can be extended and many structural properties on the whole group G can be obtained. For instance [7] studies the dense subgroups which are related to the class of modular subgroups. Here we will say that a group G has *dense modular subgroups* if whenever $H < K \leq G$ and H is not maximal in K, there is a modular subgroup of G lying strictly between the subgroups H and K of G. In a similar way [11] is related to the class of infinite invariant abelian subgroups, [13] is related to the class of subnormal subgroups, [14] is related to the class of almost normal subgroups and [28] is related to the class of pronormal subgroups. The literature which is cited in [7, 11, 13, 14, 21, 28] shows that dense subgroups have interested many group-theoretical properties. In our case the reason of the terminology *dense* is due to the fact that a group G has conjugately dense subgroups if whenever $H < K \leq G$ and H is not maximal in K, there is a subgroup L of G such that for each element g of G the intersection of L with the conjugates of g in G is nonempty and L lies strictly between the subgroups H and K of G.

A common feature of [7, 11, 13, 14, 21, 28] can be found in the restrictions on the index of the dense subgroup which is discussed. Also the conjugately dense subgroups of a given group are subjected to restrictions of this type as we will see.

Notions of Theory of Covering of Groups are naturally involved in our topic (see for instance [2, 3, 4, 5, 8, 25]), because the existence of conjugately dense subgroups can be formulated as in $(*)$ and because most of the results in [7, 11, 13, 14, 21, 28] furnishes information on classes of groups which can be covered by finitely many subgroups. Relations between conjugately dense subgroups and Theory of Covering of Groups will be found as corollaries of our main results.

Therefore the present paper improves the knowledge of conjugately dense subgroups for the main classes of generalized FC-groups, generalized soluble groups and generalized nilpotent groups. Our main results have been summarized as follows.

Main Theorem. *Let G be a group. Then G has no proper conjugately dense subgroups, if one of the following conditions holds :*

(i) *G is either an FC-group or a CC-group or a PC-group or an MC-group;*

(ii) *G is locally nilpotent and either FC-hypercentral or CC-hypercentral or PC-hypercentral or MC-hypercentral.*

In order to introduce our results, Section 2 describes the general properties of the conjugately dense subgroups and Section 3 recalls the definitions of the classes of generalized FC-groups which are mentioned.

Most of our notation is standard and referred to [8] and [25].

2. Properties of Conjugately Dense Subgroups

The present Section lists some properties of a conjugately dense subgroup of a given group. We start with the following interesting two lemmas.

Lemma 2.1. *Let H be a subgroup of a group G and n be a positive integer.*

(a) *If H is conjugately dense in G, then $Z(G) \leq H$.*

(b) *If N is a normal subgroup of G such that $N \leq H \leq G$, then H is conjugately dense in G if and only if H/N is conjugately dense in G/N.*

(c) *If G is abelian or finite and H is conjugately dense in G, then $H = G$.*

(d) *If H is a proper conjugately dense in G and N is a normal subgroup of G such that G/N has no proper conjugately dense subgroup. Then $H \cap N$ is proper conjugately dense in N.*

(e) *If K is subgroup of G and H is conjugately dense in G, then $\langle H, K\rangle$ is a conjugately dense subgroup of G.*

(f) *If H is conjugately dense in G, then the index of H in G is not finite.*

(g) *Assume that H is conjugately dense in G. If G is central-by-finite, then $H = G$.*

(h) *If H is conjugately dense in G and H is an extension of a perfect group by a soluble group of derived length n, then $G = HP$, where P is a perfect normal subgroup of G of derived length at most n. Moreover, if $H < G$, then $H \cap P < P$.*

Proof. (a) and (b) are immediate consequence of the definitions. The abelian part of (c) follows from (a), and the finite case comes from $|G| = |\bigcup_{x\in G} H^x|$ and that the number of conjugates of H is $|G : N_G(H)|$. For (d), clearly HN/N is a conjugately dense in G/N. Thus $G = HN$ by assumption and the proof follows from (b). Part (h) follows from [6, Lemma 6]. So we will prove the statements (e), (f) and (g).

(e) For each element g in G we have that the conjugacy class g^G of g in G has $H \cap g^G \neq \varnothing$ and in particular, $\langle H, K\rangle \cap g^G \supseteq \langle H\rangle \cap g^G \neq \varnothing$. Therefore $\langle H, K\rangle$ is a conjugately dense subgroup of G.

(f) Suppose that $H \neq G$ is a conjugately dense subgroup of G with finite index $|G : H|$. Then $N_G(H) \geq H$ and $|G : N_G(H)|$ is finite. It follows that H has only finitely many conjugates $H^{g_1}, \ldots, H^{g_n}$ in G, where $g_1, \ldots, g_n \in G$ and n is a positive integer. Then $\bigcap_{i=1}^n H^{g_i} = K$ is normal in G and $|G : K|$ is finite. In particular G/K is a finite group and (b) implies that H/K is a conjugately dense subgroup of G/K. Now (c) implies that $H/K = G/K$ so $G = H$ and we have a contradiction.

(g) Suppose that $H \neq G$ is conjugately dense in G. Then $H \geq Z(G)$ from (a). Since G is central-by-finite, so the index $|G : H|$ is finite and (f) gives a contradiction. $\square$

We recall some results of [6] which are useful for our aims.

Lemma 2.2. *Let H be a conjugately dense subgroup of a group G.*

(a) *If G is locally finite and H is finite, then $H = G$.*

(b) *Assume that G has a finite series whose factors are either locally supersoluble or belonging to $\mathfrak{Z}$. If H has finite abelian section rank, then $H = G$.*

(c) *If G is a residually-$\mathfrak{Z}$ group, then $H = G$.*

(d) *If G is locally soluble and H is soluble with max-n, then $H = G$.*

Proof. (a) and (b) follow from [6, Theorems 2, 4]. (c) and (d) follow from [6, Corollaries 7, 8]. □

The following corollaries can be obtained directly from the above lemmas.

Corollary 2.3. *If G has a finite series whose factors are abelian or finite, then G has no proper conjugately dense subgroups. In particular, if G is soluble or hypercentral, then G has no proper conjugately dense subgroups.*

Proof. This follows by (b) of Lemma 2.2.□

Corollary 2.4. *Let H be a conjugately dense subgroup of a group G.*

(a) *If G is central-by-Chernikov, then $H = G$.*

(b) *If G is central-by-(polycyclic-by-finite), then $H = G$.*

(c) *If G is central-by-(soluble minimax-by-finite), then $H = G$.*

Proof. By (a) and (b) of Lemma 2.1, we may assume without loss of generality that $Z(G) = 1$. So, e may assume G is a Chernikov group, polycyclic-by-finite group and soluble minimax-by-finite group according to the proofs of (a), (b) and (c) respectively. Thus the results follow from Corollary 2.3. □

Corollary 2.5. *Assume that K is an arbitrary field, $GL(n, K)$ is a linear group of dimension n over K, n is a positive integer.*

(a) *If G is a locally nilpotent subgroup of $GL(n, K)$, then G has no proper conjugately dense subgroups;*

(b) *If G is a locally supersoluble subgroup of $GL(n, K)$, then G has no proper conjugately dense subgroups.*

Proof. (a) [25, Theorem 6.32, (iv)] implies that G is hypercentral, then the result follows by Corollary 2.3.

(b) This is true from (b) of Lemma 2.2. Equivalently we could use [29, Theorem B] and Corollary 2.3. □

The following example shows that the usual operations of intersection, homomorphic images and conjugation are not respected by the subgroups which are conjugately dense in a given group.

Examples 2.6.
(i). Let $G = PSL(2, \mathbb{Z})$ be the projective special linear group of dimension 2 over the ring of the integers. The subgroup U generated by the upper triangular matrices of G and the subgroup L generated by the lower triangular matrices of G are conjugately dense subgroups in G. The intersection $D = U \cap L$ is the diagonal subgroup of G and D is not conjugately dense in G.

D can be easily seen to be an homomorphic image of G and G is trivially a conjugately dense subgroup, but D is not conjugately dense in G.

Moreover [34, Theorem 2, Corollary 1] shows that there are subgroups conjugately dense in G which are not conjugate and they are uncountably many. We note also that the property to have no proper conjugately dense subgroups is not closed with respect to extensions. G is a product with amalgamation of a cyclic subgroup H of order 2 by a cyclic subgroup K of order 3 (see [34, Corollary 1, Proof]), H and K have no proper conjugately dense subgroups by (c) of Lemma 2.1. But, G has uncountably many proper conjugately dense subgroups. Clearly we may not state in this situation that $G = H$.

(ii). If G is the Tarski monster (see [12]), then it contains infinitely many proper conjugately dense subgroups H of G. In particular, each H is cyclic of order p, where p is a prime, and $G = \bigcup_{x \in G} H^x$. Clearly we may not state in this situation that $G = H$.

(iii). In [12] it has been furnished a countable group H containing an element of 'big enough' (possibly infinite) order but none of order 2 such that it is possible to construct a simply 2-generated group $G = \bigcup_{x \in G} H^x$. Such group has clearly each H which is conjugately dense in G and $H \neq G$. □

Remark 2.7. There is a classical Baer's Theorem [25, Theorem 4.16], which implies that a central-by-finite group is characterized to have a finite covering of abelian subgroups. Hence those groups which have a finite covering of abelian subgroups have no proper conjugately dense subgroups. Now it is reasonable to ask if a group which admits a finite covering of normal subgroups has no proper conjugately dense subgroups.

Example (i) shows a group G which is covered by two normal subgroups U and L, and G admits proper conjugately dense subgroups. Then it is not easy to answer this easy question.

However if G is a group which is covered by finitely many normal subgroups H, then the main results of [2, 3, 4, 5] give conditions to state whether G is either central-by-finite or nilpotent-by-finite or soluble-by-finite. In these situations Corollaries 2.3 and 2.4 imply that G has no proper conjugately dense subgroups.

Finally, [26] shows that a group G, which is covered by finitely many proper subgroups K which are are not normal in G, can not have an ascending series of normal subgroups. This gives further obstacles in order to apply Lemma 2.2 or Corollaries 2.3 and 2.4. Thus another situation where it is not easy to answer whether a group which is covered by finitely many subgroups has no proper conjugately dense subgroups. $\square$

3. Generalized FC-groups

Let $\mathfrak{X}$ be a class of groups. An element x of a group G is said to be an $\mathfrak{X}C$-*element* if $G/C_G(\langle x\rangle^G)$ satisfies $\mathfrak{X}$. A group whose elements are all $\mathfrak{X}C$-elements is said to be an $\mathfrak{X}C$-group. Sometimes $G/C_G(\langle x\rangle^G)$ is denoted by $Aut_G(\langle x\rangle^G)$ to recall that $G/C_G(\langle x\rangle^G)$ is a group of automorphisms of $\langle x\rangle^G$ (see [25, Chapter 3]).

If $\mathfrak{X}$ is the class of finite groups, we find the notion of FC-element and FC-group (see [25, Chapter 4]). If $\mathfrak{X}$ is the class of Chernikov groups, we find the notion of CC-element and CC-group (see [23, 24]). If $\mathfrak{X}$ is the class of polycyclic-by-finite groups, we find the notion of PC-element and PC-group (see [9]). If $\mathfrak{X}$ is the class of (soluble minimax)-by-finite groups, we find the notion of MC-element and MC-group (see [15, 16, 17]).

It could be opportune to recalling that a group H is said to be (soluble minimax)-by-finite if it contains a normal subgroup K such that K has a finite characteristic series $1 = K_0 \triangleleft K_1 \triangleleft \ldots \triangleleft K_n = K$ whose factors are abelian minimax and H/K is finite. An abelian minimax group A is an abelian group which has a finitely generated subgroup B such that A/B is a direct product of finitely many quasicyclic groups. They are described by [25, Lemma 10.31] and consequently soluble minimax groups are well known (see [25, vol.II, Sections 10.3 and 10.4]).

Let $\mathfrak{Y}$ be one of the following classes of groups: finite groups, Chernikov groups, polycyclic-by-finite groups, (soluble minimax)-by-finite groups.

Since $\mathfrak{Y}$ is a formation of groups (see [8]), given two $\mathfrak{Y}C$-elements x and y of G, then both $G/C_G(\langle x\rangle^G)$ and $G/C_G(\langle y\rangle^G)$ satisfy $\mathfrak{Y}$, so that the quotient group

$$G/(C_G(\langle x\rangle^G) \cap C_G(\langle y\rangle^G))$$

satisfies $\mathfrak{Y}$. But the intersection of $C_G(\langle x\rangle^G)$ with $C_G(\langle y\rangle^G)$ lies in $C_G(\langle xy^{-1}\rangle^G)$, so $G/C_G(\langle xy^{-1}\rangle^G)$ satisfies $\mathfrak{Y}$ and xy^{-1} is an $\mathfrak{Y}C$-element of G.

Hence the $\mathfrak{Y}C$-elements of G form a subgroup $Y(G)$ and $Y(G)$ is characteristic in G.

This simple remark allows us to define the series

$$1 = Y_0 \triangleleft Y_1 \triangleleft \ldots \triangleleft Y_\alpha \triangleleft Y_{\alpha+1} \triangleleft \ldots,$$

where $Y_1 = Y(G)$, the factor $Y_{\alpha+1}/Y_\alpha$ is the subgroup of G/Y_α generated by the YC-elements of G/Y_α and

$$Y_\lambda = \bigcup_{\alpha<\lambda} Y_\alpha,$$

with α ordinal and λ limit ordinal. This series is a characteristic ascending series of G and it is called *upper YC-central series* of G. The last term of the upper YC-central series of G is called *YC-hypercenter* of G and it is denoted by $Y_\lambda(G)$. If $G = Y_\beta$, for some ordinal β, we say that G is an *YC-hypercentral* group of type at most β and this is equivalent to say that $G = Y_\lambda(G)$.

The *YC-length* of an YC-hypercentral group is defined to be the least ordinal β such that $G = Y_\beta$, in particular when $G = Y_c$ for some positive integer c, we say that G is *YC-nilpotent* of length c. An YC-group is characterized to have YC-length at most 1.

In analogy with FC-groups, the first term $Y(G)$ of the upper YC-central series of G is said to be the *YC-center* of G and the α-th term of the upper YC-central series of G is said to be the *YC-center of length α* of G. Roughly speaking, the upper YC-central series of G measures the distance of G to be an YC-group. By definitions, it happens that $Z(G) \leq F(G) \leq Y(G)$, where $F(G)$ is the FC-center of G, that is, $F(G)$ is the subgroup of G which is generated by all elements which have only a finite number of conjugates.

For the class of finite groups we have FC-hypercentral groups, using the symbols F_α, F_β, F_λ, F_c instead of Y_α, Y_β, Y_λ, Y_c in the previous definitions. For the class of Chernikov groups we have CC-hypercentral groups, using the symbols $C_\alpha, C_\beta, C_\lambda, C_c$ instead of $Y_\alpha, Y_\beta, Y_\lambda, Y_c$ in the previous definitions. For the class of polycyclic-by-finite groups we have PC-hypercentral groups, using the symbols $P_\alpha, P_\beta, P_\lambda, P_c$ instead of $P_\alpha, P_\beta, P_\lambda, P_c$ in the previous definitions. For the class of (soluble minimax)-by-finite groups we have MC-hypercentral groups, using the symbols $M_\alpha, M_\beta, M_\lambda, M_c$ instead of $Y_\alpha, Y_\beta, Y_\lambda, Y_c$ in the previous definitions.

[1] and [18] investigate PC-hypercentral and CC-hypercentral groups satisfying finiteness conditions. For instance [1, Theorem A, Theorem B] adapt some classical McLain's Theorems on generalized nilpotent series [25, Theorems 4.37 and 4.38] with related finiteness conditions [25, Theorems 4.39, 4.39.1 and 4.39.2]. On the other hand the conditions of hypercentrality and YC-hypercentrality can be different in

a same group as the consideration of the infinite dihedral group shows (see also [1, 18]).

We end this Section giving a definition of another class of generalized FC-groups whose relations with the linearity have been studied in [22].

A group G is said to be a locally FC-group if each finitely generated subgroup of G is an FC-group. A locally FC-hypercentral group is a group whose finitely generated subgroups are FC-hypercentral.

4. Proof of Main Theorem and Consequences

Proof of (i) Main Theorem. Let G be a group. If G is a FC-group, then from [25, Theorem 4.32, (i)], $G/Z(G)$ is locally finite. So, from (a) and (b) of Lemma 2.1 we may suppose without loss of generality that $Z(G) = 1$. Therefore we may assume that G is a finite group. Now the result follows from Corollary 2.3. If G is a CC-group, then as similar as above we may assume that $Z(G) = 1$. Now [10, Theorem 3.2], allows us to conclude that G is isomorphic to a subgroup of a direct product of Chernikov groups with trivial centers. The class $\mathfrak{Z}$ contains all Chernikov groups, since it contains the class of soluble groups. Furthermore, $\mathfrak{Z}$ is closed under direct products. Then G belongs to $\mathfrak{Z}$ and obviously (c) of Lemma 2.2 implies that $H = G$. When G is a PC-group, then [9, Theorem 3.2] allows us to conclude that G is isomorphic to a subgroup of a direct product of polycyclic-by-finite groups with trivial centers. Similarly, if G is MC-group, then G is residually (soluble minimax)-by-finite as testified in [15]. Since the class $\mathfrak{Z}$ contains all polycyclic-by-finite groups and (soluble minimax)-by-finite groups and it is closed under direct products. Then G belongs to $\mathfrak{Z}$ and obviously (c) of Lemma 2.2 implies that $H = G$.□

We note that a generalized FC-group can be neither locally nilpotent nor locally soluble a priori (see [9, 15, 16, 23, 24, 25]). This fact allows us to note that generalized soluble groups, generalized nilpotent groups and generalized FC-groups can have a complicated structure. However the structure of generalized FC-groups which are either locally soluble or locally nilpotent is well-known. For instance locally soluble MC-groups are hyperabelian and locally nilpotent MC-groups are hypercentral thanks to [15, Theorems 3, 4]. Locally soluble FC-hypercentral, respectively CC-hypercentral, respectively PC-hypercentral, groups are hyperabelian by means of [9, Theorem 3.2] and [10, Theorem 2.1]. Moreover locally nilpotent FC-hypercentral, respectively CC-hypercentral, respectively PC-hypercentral, groups are hypercentral applying again [9, Theorem 3.2] and [10, Theorem 2.1].

Proof of (ii) Main Theorem. Let G be a locally nilpotent. If G is PC-hypercentral group, then from [18, Corollary 2.3], we have that the α-th term P_α

of the upper PC-central series of G lies in the $\omega\alpha$-th term $Z_{\omega\alpha}(G)$ of the upper central series of G for each ordinal α. In particular, $G = P_\lambda(G)$ coincides with the hypercenter $Z_{\omega\lambda}(G)$ of G. Therefore, G is hypercentral. Now the result follows by Corollary 2.3.

[18, Corollary 2.3], continues to be valid for CC-hypercentral groups, FC-hypercentral groups and MC-hypercentral groups. We have respectively that $G = C_\alpha \leq Z_{\omega\alpha}(G)$, $G = F_\alpha \leq Z_{\omega\alpha}(G)$ and $G = M_\alpha \leq Z_{\omega\alpha}(G)$ for each ordinal α. As before we have respectively that $G = C_\lambda(G)$, $G = F_\lambda(G)$ and $G = M_\lambda(G)$ coincide with the hypercenter $Z_{\omega\lambda}(G)$ of G. Now the result again follows by Corollary 2.3 an the proof of theorem is completed. □

The consideration of the group $G = PSL(2, \mathbb{Z})$ in (i) of Examples 2.6 shows that there exists a FC-hypercentral group which has proper conjugately dense subgroups. Here $Z(G) \neq 1$, $Z(G) \leq F(G)$ and we may proceed as it is described in Section 3 in order to construct the upper FC-central series of G. Therefore the hypothesis of locally nilpotence in part (b) of the main theorem can not be omitted.

Corollary 4.3. *Let K be an arbitrary field and $GL(n, K)$ be a linear group of dimension n over K. If G is a subgroup of $GL(n, K)$ and G is locally FC-hypercentral, then G has no proper conjugately dense subgroups.*

Proof From [22, Theorem 3], G is soluble-by-finite. Therefore we apply Lemma 2.2 and the result follows. □

The terminology of the classes of generalized nilpotent groups which are listed in the following result can be found in [25, Chapter 6].

Corollary 4.4. *Let K be an arbitrary field and $GL(n, K)$ be a linear group of dimension n over K. If G is a subgroup of $GL(n, K)$ and one of the following conditions holds*

(a) *G is a Baer-nilpotent group,*

(b) *G is an Engel-group,*

(c) *G is a weakly nilpotent group,*

(d) *G is a V-group,*

(e) *G is an U-group,*

(f) *G is an $\tilde{N}$-groups,*

(g) *G is a $\bar{\bar{Z}}$-group,*

then G has no proper conjugately dense subgroups.

Proof. A Baer-nilpotent linear group is hypercentral from [25, Vol.2, p.35]. The classes of generalized groups which are listed in the successive statements (b)-(g) are subclasses of the class of Baer-nilpotent groups (see [25, Vol.2, Chapter 6]). Then these groups are again hypercentral and the result follows by Lemma 2.2. □

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References

[1] J. C. Beidleman, A. Galoppo and A. M. Manfredino, *On PC-hypercentral and CC-hypercentral groups*, Comm. Alg., **26**, (1998), 3045–3055.

[2] M. A. Brodie, R. F. Chamberlain and L. C. Kappe, *Finite coverings by normal subgroups*, Proc. Amer. Math. Soc., **104**, (1988), 669–674.

[3] M. A. Brodie and L. C. Kappe, *Finite coverings by subgroups with a given property*, Glasg. Math. J., **35**, (1993), 179–188.

[4] M. A. Brodie, *Finite n-coverings of groups*, Arch. Math., **63**, (1994), 385–392.

[5] M. A. Brodie and R. F. Morse, *Finite subnormal coverings of certain solvable groups*, Comm. Alg., **30**, (2002), 2569–2581.

[6] G. Cutolo, H. Smith and J. Wiegold, *Groups covered by conjugates of proper subgroups*, J. Algebra, **293**, (2005), 261–268.

[7] F. De Mari, *Groups with dense modular subgroups*, In: Proceedings of the intensive bimester dedicated to the memory of Reinhold Baer (1902-1979), Napoli, Italy, May-June 2002. (Advances in Group Theory 2002, Rome, Aracne, 2003), pp.85-91.

[8] K. Doerk and T. Hawkes, *Finite Soluble Groups*, de Gruyter, Berlin, 1992.

[9] S. Franciosi, F. de Giovanni and M. J. Tomkinson, *Groups with polycyclic-by-finite conjugacy classes*, Boll. Un. Mat. Ital., **4B**, (1990), 35–55.

[10] S. Franciosi, F. de Giovanni and M. J. Tomkinson, *Groups with Chernikov conjugacy classes*, J. Austral. Math. Soc., **50**, (1991), 1–14.

[11] V. Eh. Goretskij, *Groups with a dense system of infinite invariant Abelian subgroups*, in: Group-theoretical studies (Work-Collect., Kiev, 1978), pp.127–138.

[12] S. V. Ivanov and A. Yu. Ol'shanskii, *Some applications of graded diagrams in combinatorial group theory.* In: Groups St.Andrews 1989 (London Math. Soc. Lecture Note Ser., vol. 160, Cambridge Univ. Press, Cambridge, 1991), pp.258-308.

[13] L. A. Kurdachenko, N. F. Kuzennyj and V. V. Pylaev, *Infinite groups with a generalized dense system of subnormal subgroups*, *Ukr. Math. J.*, **33**, (1982), 313–316.

[14] L. A. Kurdachenko, N. F. Kuzennyj and N. N. Semko, *Groups with a dense system of infinite almost normal subgroups*, Ukr. Math. J., **43**, (1991), 904–908.

[15] L. Kurdachenko, *On groups with minimax conjugacy classes*, in: Infinite groups and adjoining algebraic structures (Kiev, Naukova Dumka, 1993), pp.160–177.

[16] L. Kurdachenko, *On Normal Closures of Elements in Generalized FC-groups*, in: Infinite groups 94 (Ravello) (Eds. de Giovanni/Newell, de Gruyter, Berlin, 1995), pp.141–151.

[17] L. Kurdachenko and J. Otal. *Frattini properties of groups with minimax conjugacy classes*. In: Topics in Infinite Groups (Quad. di Mat., vol.8, Caserta, 2000), pp.223–235.

[18] T. Landolfi, *On generalized central series of groups*, Ricerche di Mat., **XLIV**, (1995), 337–347.

[19] V. M. Levchuck and S. A. Zyubin, *Conjugately dense subgroups of the group $GL_2(K)$ over a locally finite field K*, in: The International Conference Symmetry and Differential Equation (IVM SO RAN, Krasnoyarsk, 1999), pp.110–112.

[20] V. M. Levchuck and S. A. Zyubin, *Conjugately dense subgroups of locally finite field Chevalley groups of Lie rank 1*, Sibirsk. Mat. Z., **44**, (2003), 581–586.

[21] A. Mann, *Groups with dense normal subgroups*, Israel J. Math., **6**, (1968), 13–25.

[22] M. M. Murach, *Some generalized FC-groups of matrices*, Ukr. Math. J., **28**, (1974), 92–97.

[23] Ya. D. Polovicky, *Groups with extremal classes of conjugate elements*, Sibirsk. Mat. Z., **5**, (1964), 891–895.

[24] Ya. D. Polovicky, *The periodic groups with extremal classes of conjugate abelian subgroups*, Izvestija VUZ, ser Math., **4**, (1977), 95–101.

[25] D. J. S. Robinson, *Finiteness conditions and generalized soluble groups*, Springer, Berlin, 1972.

[26] J. Sonn, *Groups That are the Union of Finitely Many Proper Subgroups*, Amer. Math. Monthly, **83**, (1976), 263–265.

[27] Unsolved Problems in Group Theory, The Kourovka Notebook, No. 14 (Inst. Mat. Novosibirsk, Novosibirsk, 1999).

[28] G. Vincenzi, *Groups with dense pronormal subgroups*. Ricerche di Mat., **40**, (1991), 75–79.

[29] B. A. F. Wehrfritz, *Supersoluble and locally supersoluble linear groups*, J. Algebra, **17**, (1971), 41–58.

[30] J. Wiegold, *Groups of finitary permutations*, Arch. Math. (Basel), **25**, (1974), 466–469.

[31] J. Wiegold, *Transitive groups with fixed-points free permutations*, Arch. Math. (Basel), **27**, (1976), 473–475.

[32] S. A. Zyubin, *Conjugately dense subgroups of the group $PSL_2(K)$ over a locally finite field K*, in: Proceedings of the 34th Students Scientific Conference (Krasnoyarsk Univ., Krasnoyarsk, 2001), pp.64–69.

[33] S. A. Zyubin, *Conjugately dense subgroups of the Suzuki group over a locally finite field*, in: Study on Mathematical Analysis and Algebra (Tomsk Univ., Tomsk, 2001, vol.3), pp.102–105.

[34] S. A. Zyubin, *On conjugately dense subgroups of free products of groups with amalgamation*, Algebra Logika, **45** (2006), 296–305.

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