BOUNDEDNESS OF MULTILINEAR COMMUTATOR OF SINGULAR INTEGRAL IN MORREY SPACES ON HOMOGENEOUS SPACES

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Abstract: In this paper, we prove the boundedness of the multilinear commutator related to the singular integral operator in Morrey and Morrey-Herz spaces on homogeneous spaces.

2000 *Mathematics Subject Classification*: 42B20, 42B25.

1. Preliminaries

Sawano and Tanka(see [13]) introduced the Morrey spaces on the non-homogeneous spaces and proved the boundedness of Hardy-Littlewood maximal operators, Calderón-Zygmund operators and fractional integral operators in Morrey spaces. On the base of the above results, Yang and Meng(see [14]) considered the boundedness of the commutators generated by Calderón-Zygmund operators or fractional integral operators with RBMO(μ) in Morrey spaces. Motivated by these results, in this paper, we will introduce the multilinear commutator related to the singular operator on homogeneous spaces, and prove the boundedness properties of the operator in Morrey and Morrey-Herz spaces on homogeneous spaces.

Give a set X, a function $d : X \times X \to R^+$ is called a quasi-distance on X if the following conditions are satisfied:
(i) for every x and y in X, $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$,
(ii) for every x and y in X, $d(x, y) = d(y, x)$,
(iii) there exists a constant $l \geq 1$ such that

$$d(x, y) \leq l(d(x, z) + d(z, y)) \tag{1}$$

for every x, y and z in X.

Let μ be a positive measure on the σ-algebra of subsets of X which contains the r-balls $B_r(x) = \{y : d(x, y) < r\}$. We assume that μ satisfies a doubling condition, that is, there exists a constant A such that

$$0 < \mu(B_{2r}(x)) \le A\mu(B_r(x)) < \infty \tag{2}$$

holds for all $x \in X$ and $r > 0$.

A structure (X, d, μ), with d and μ as above, is called a space of homogeneous type. The constants k and A in (1) and (2) will be called the constants of the space.

A homogeneous space (X, d, μ) is said to be normal if there exist positive constants c_1, c_2 and $\alpha > 0$ such that $c_1 r^\alpha \le \mu(B_r(x)) \le c_2 r^\alpha$ for every $x \in X$ and every r satisfying that $\mu(x) < r < \mu(X)$.

By [11], we know that for any homogeneous space (X, d, μ), there exists a normal homogeneous space (X, δ, μ) such that δ and d are topologically equivalent.

Then let us introduce some notations(see [1][4][11]). In this paper, B will denote a ball of X, and for a ball B and a function b, let $b_B = \mu(B)^{-1} \int_B b(x) d\mu(x)$ and the sharp function of b is defined by

$$b^{\#}(x) = \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |b(y) - b_B| d\mu(y).$$

It is well-known that (see [4])

$$b^{\#}(x) \approx \sup_{B \ni x} \inf_{c \in C} \frac{1}{\mu(B)} \int_B |b(y) - C| d\mu(y).$$

We say that b belongs to $BMO(X)$ if $b^{\#}$ belongs to $L^\infty(X)$ and define $||b||_{BMO} = ||b^{\#}||_{L^\infty}$. It has been known that(see [4])

$$||b - b_{2^k B}||_{BMO} \le Ck||b||_{BMO}.$$

Definition 1. *Suppose b_i $(i = 1, \cdots, m)$ are the fixed locally integrable functions on X. Let T be the singular integral as*

$$T(f)(x) = \int_X K(x, y) f(y) d\mu(y),$$

where K is a locally integrable function on $X \times X \setminus \{(x, y) : x = y\}$ and satisfies the following properties:

(1) $|K(x, y)| \le \dfrac{C}{\mu(B(x, d(x, y)))}$,

(2) $|K(x, y) - K(x, y')| + |K(y, x) - K(y', x)| \le C \dfrac{(d(y, y'))^\delta}{\mu(B(y, d(x, y)))(d(x, y))^\delta}$,

when $d(x, y) \ge 2d(y, y')$, with some $\delta \in (0, 1]$.

The multilinear commutator of the singular integral is defined by

$$T_{\vec{b}}(f)(x) = \int_X \prod_{i=1}^m (b_i(x) - b_i(y)) K(x, y) f(y) d\mu(y)$$

Given a positive integer m and $1 \leq i \leq m$, we denote by C_i^m the family of all finite subsets $\sigma = \{\sigma(1), \cdots, \sigma(i)\}$ of $\{1, \cdots, m\}$ of i different elements. For $\sigma \in C_i^m$, set $\sigma^c = \{1, \cdots, m\} \setminus \sigma$. For $\vec{b} = (b_1, \cdots, b_m)$ and $\sigma = \{\sigma(1), \cdots, \sigma(i)\} \in C_i^m$, set $\vec{b}_\sigma = (b_{\sigma(1)}, \cdots, b_{\sigma(i)})$, $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(i)}$ and $||\vec{b}_\sigma||_{BMO} = ||b_{\sigma(1)}||_{BMO} \cdots ||b_{\sigma(i)}||_{BMO}$.

Fixed $x_0 \in X$, $B_k = \{x \in X : d(x_0, x) < 2^k\}$, $A_k = B_k \setminus B_{k-1}$, for any $k \in Z$. The notation $\chi_k(x) = \chi_{A_k}(x)$ is the characteristic function of the set A_k. In addition, for a function $f \in L^1_{loc}(X)$, denote $f_k = f\chi_k$.

In what follows, $C > 0$ always denotes a constant that is independent of main parameters involved but whose value may differ from line to line. For any index $p \in [1, \infty]$, we denote by p' its conjugate index, namely, $1/p + 1/p' = 1$.

2. Boundedness on Generalized Morrey space

In this section, the generalized Morrey spaces will be introduced and the boundedness in generalized Morrey spaces for the multilinear commutator of the singular integral will be discussed.

Definition 2. *For a positive growth function ϕ on X, which satisfies $\phi(2r) \leq D\phi(r)$ for all $r > 0$ with $D > 0$ being a constant independent of r. The generalized Morrey space $L^{p,\phi}$ with $1 \leq p < \infty$ is defined as follows:*

$$L^{p,\phi}(X) = \{f \in L^p_{loc}(X) : ||f||_{L^{p,\phi}} < \infty\}$$

where

$$||f||_{L^{p,\phi}} = \sup_{x \in X, r > 0} \left(\frac{1}{\phi(r)} \int_{B_r(x)} |f(y)|^p d\mu(y) \right)^{1/p}.$$

Remark. If $\phi(r) = r^\delta$, $\delta \geq 0$, then $L^{p,\phi} = L^{p,\delta}$, which is the classical Morrey space(see[12]).

Lemma 1. *Let $1 < r < \infty$, $b_i \in BMO(X)$ for $i = 1, \cdots, k$ and $k \in N$. Then, we have*

$$\frac{1}{\mu(B)} \int_B \prod_{i=1}^k |b_i(y) - (b_i)_B| d\mu(y) \leq C \prod_{i=1}^k ||b_i||_{BMO}$$

and

$$\left(\frac{1}{\mu(B)} \int_B \prod_{i=1}^k |b_i(y) - (b_i)_B|^r d\mu(y) \right)^{1/r} \leq C \prod_{i=1}^k ||b_i||_{BMO}.$$

Lemma 2. (see[3]) *Let (X, d, μ) be a normal homogeneous space and $p \in (1, \infty)$, $1 \leq D < 2^\alpha$. If a sublinear operator T is bounded on $L^p(X)$ and for any $f \in L^1(X)$ with compact support and $x \notin c_0 suppf$,*

$$|Tf(x)| \leq C \int_X \frac{|f(y)|}{\mu(B(x, y))} d\mu(y),$$

where $c_0 \geq 1$ *and* $c > 0$ *are positive constants and* $B(x,y)$ *is the ball* $\{z \in X : d(z,x) < d(x,y)\}$*, then* T *is also bounded on* $L^{p,\phi}(X)$*.*

Theorem 1. *Let* $b_j \in BMO(X)$ *for* $j = 1, \cdots, m$*,* $p \in (1,\infty)$*,* (X,d,μ) *be a normal homogeneous space and* T *be the singular integral as Definition 1. Suppose* ϕ *is the function as in Definition 2 with* $1 \leq D < 2^\alpha$*. Then* $T_{\vec{b}}$ *is bounded on* $L^{p,\phi}(X)$*.*

Proof. Without loss of generality, we may assume $c_0 = 1$. For any $x_0 \in X$, $r > 0$, and any complex-valued measurable function f on X, we write

$$f(y) = (f\chi_{B_{2^N r}(x_0)})(y) + \sum_{k\geq N}(f\chi_{B_{2^{k+1}r}(x_0)\setminus B_{2^k r}(x_0)}) \equiv f_0(y) + \sum_{k\geq N} f_k(y),$$

where N is a positive integer to be chosen. When $m = 1$, set $b_r = \mu(B_r(x_0))^{-1} \int_{B_r(x_0)} b_1(y)d\mu(y)$, it is to see

$$\begin{aligned}\left(\int_{B_r(x_0)} |T_{b_1} f(x)|^p d\mu(x)\right)^{1/p} &\leq \left(\int_{B_r(x_0)} |T_{b_1} f_0(x)|^p d\mu(x)\right)^{1/p} \\ &\quad + \sum_{k\geq N}\left(\int_{B_r(x_0)} |T_{b_1} f_k(x)|^p d\mu(x)\right)^{1/p} = I + II\end{aligned}$$

For I, we have

$$\begin{aligned} I &= \left(\int_{B_r(x_0)} |T_{b_1} f_0(x)|^p d\mu(x)\right)^{1/p} \\ &= \left(\int_{B_r(x_0)} |\int_X (b_1(x) - b_1(y))K(x,y)f_0(y)d\mu(y)|^P d\mu(x)\right)^{1/p} \\ &\leq C\left(\int_{B_r(x_0)} \mu(B_{2^N r}(x_0))^{-p}|\int_{B_{2^N r}(x_0)} (b_1(x) - b_1(y))f(y)d\mu(y)|^P d\mu(x)\right)^{1/p} \\ &\leq C\mu(B_{2^N r}(x_0))^{-1}\left(\int_{B_r(x_0)}\left(\int_{B_{2^N r}(x_0)} |b_1(x) - b_1(y)|^p|f(y)|^p d\mu(y)\right)d\mu(x)\right)^{1/p} \\ &\leq C\mu(B_{2^N r}(x_0))^{-1}\Bigg[\left(\int_{B_r(x_0)} |b_1(x) - b_r|^p d\mu(x)\right)^{1/p}\left(\int_{B_{2^N r}(x_0)} |f(y)|d\mu(y)\right) \\ &\quad + \mu(B_r(x_0))^{1/p}\int_{B_{2^N r}(x_0)} |b_1(y) - b_r||f(y)|d\mu(y)\Bigg] \\ &= \nu_1 + \nu_2. \end{aligned}$$

For ν_1, we get

$$\begin{aligned}
\nu_1 &\leq C\left(\frac{\mu(B_r(x_0))}{\mu(B_{2^Nr}(x_0))}\right)^{1/p}\left(\frac{1}{\mu(B_r(x_0))}\int_{B_r(x_0)}|b_1(x)-b_r|^p d\mu(x)\right)^{1/p}\\
&\quad\times\left(\int_{B_{2^Nr}(x_0)}|f(y)|^p d\mu(y)\right)^{1/p}\\
&\leq C||b_1||_{BMO}||f||_{L^{p,\phi}}\phi^{1/p}(2^Nr)\\
&\leq CD^N||b_1||_{BMO}||f||_{L^{p,\phi}}\phi^{1/p}(r)\\
&\leq C||b_1||_{BMO}||f||_{L^{p,\phi}}.
\end{aligned}$$

For ν_2, by Hölder's inequality, we get

$$\begin{aligned}
\nu_2 &\leq C\frac{\mu(B_r(x_0))^{1/p}}{\mu(B_{2^Nr}(x_0))}\left(\int_{B_{2^Nr}(x_0)}|b_1(y)-b_r|^q d\mu(y)\right)^{1/q}\left(\int_{B_{2^Nr}(x_0)}|f(y)|^p d\mu(y)\right)^{1/p}\\
&\leq C\frac{\mu(B_r(x_0))^{1/p}\mu(B_{2^Nr}(x_0))^{1/q}}{\mu(B_{2^Nr}(x_0))}\left(\frac{1}{\mu(B_{2^Nr}(x_0))}\int_{B_{2^Nr}(x_0)}|b_1(y)-b_r|^q d\mu(y)\right)^{1/q}\\
&\quad\times||f||_{L^{p,\phi}}\phi^{1/p}(2^Nr)\\
&\leq CN||b_1||_{BMO}D^N||f||_{L^{p,\phi}}\phi^{1/p}(r)\\
&\leq C||b_1||_{BMO}||f||_{L^{p,\phi}}.
\end{aligned}$$

For II, $k \geq N$, we have

$$\begin{aligned}
&\left(\int_{B_r(x_0)}|T_{b_1}f_k(x)|^p d\mu(x)\right)^{1/p}\\
&\leq C\left(\int_{B_r(x_0)}\left(\int_{B_{2^{k+1}r}(x_0)\setminus B_{2^kr}(x_0)}\frac{|b_1(x)-b_1(y)||f(y)|}{\mu(B(x,y))}d\mu(y)\right)^p d\mu(x)\right)^{1/p}.
\end{aligned}$$

Similar to [3], we choose N be a fixed large positive integer such that

$$\frac{1}{l^2}-\frac{l+1}{2^Nl}>0$$

and there is an integer $k_0 \in [1, N]$ satisfying

$$2^{-k_0}<\frac{1}{l^2}-\frac{l+1}{2^Nl}.$$

For these k_0 and N chosen, we claim that $B(x_0, 2^{k-k_0}r) \subseteq B(x,y)$ for any $x \in B_r(x_0)$ and $y \in B_{2^{k+1}}(x_0) \setminus B_{2^k r}(x_0)$ if $k = N, N+1, N+2, \cdots$. Therefore, we obtain that

$$\begin{aligned}
&\left(\int_{B_r(x_0)} |T_{b_1} f_k(x)|^p d\mu(x)\right)^{1/p} \\
\leq\ & C\left(\int_{B_r(x_0)}\left(\frac{1}{\mu(B_{2^{k-k_0}r}(x_0))}\int_{B_{2^{k+1}r}(x_0)} |b_1(x)-b_1(y)||f(y)|d\mu(y)\right)^p d\mu(x)\right)^{1/p} \\
\leq\ & \frac{C}{\mu(B_{2^{k-k_0}r}(x_0))}\Bigg[\left(\int_{B_r(x_0)} |b_1(x)-b_r|^p d\mu(x)\right)^{1/p}\left(\int_{B_{2^{k+1}r}(x_0)} |f(y)|d\mu(y)\right) \\
& +\mu(B_r(x_0))^{1/p}\int_{B_{2^{k+1}r}(x_0)} |b_1(y)-b_r||f(y)|d\mu(y)\Bigg] \\
=\ & S_1 + S_2.
\end{aligned}$$

For S_1, we have

$$\begin{aligned}
S_1 \leq\ & C\frac{\mu(B_r(x_0))^{1/p}\mu(B_{2^{k+1}r}(x_0))^{1/q}}{\mu(B_{2^{k-k_0}r}(x_0))}\left(\frac{1}{\mu(B_r(x_0))}\int_{B_r(x_0)} |b_1(x)-b_r|^p d\mu(x)\right)^{1/p} \\
& \times\left(\int_{B_{2^{k+1}r}(x_0)} |f(y)|^p d\mu(y)\right)^{1/p} \\
\leq\ & C r^{\alpha/p}(2^{k+1}r)^{-\alpha/p}||b_1||_{BMO}\phi^{1/p}(2^{k+1}r)||f||_{L^{p,\phi}(X)} \\
\leq\ & C\left(\frac{D}{2^\alpha}\right)^{(k+1)/p}||b_1||_{BMO}\phi^{1/p}(r)||f||_{L^{p,\phi}}
\end{aligned}$$

For S_2, by Hölder's inequality, we have

$$\begin{aligned}
S_2 \leq\ & C\frac{\mu(B_r(x_0))^{1/p}\mu(B_{2^{k+1}r}(x_0))^{1/q}}{\mu(B_{2^{k-k_0}r}(x_0))}\left(\frac{1}{\mu(B_{2^{k+1}r}(x_0))}\int_{B_{2^{k+1}r}(x_0)} |b_1(y)-b_r|^q d\mu(y)\right)^{1/q} \\
& \times\left(\int_{B_{2^{k+1}r}(x_0)} |f(y)|^p d\mu(y)\right)^{1/p} \\
\leq\ & C\left(\frac{D}{2^\alpha}\right)^{(k+1)/p} k||b_1||_{BMO}\phi^{1/p}(r)||f||_{L^{p,\phi}}.
\end{aligned}$$

Thus

$$\sum_{k\geq N}\left(\int_{B_r(x_0)} |T_{b_1} f_k(x)|^p d\mu(x)\right)^{1/p}$$

$$
\begin{aligned}
&\leq C\sum_{k\geq N}\left(\frac{D}{2^{\alpha}}\right)^{(k+1)/p}(k+1)||b_1||_{BMO}\phi^{1/p}(r)||f||_{L^{p,\phi}(X)}\\
&\leq C||b_1||_{BMO}||f||_{L^{p,\phi}(X)}.
\end{aligned}
$$

This completes the proof of the case $m=1$.

When $m>1$, set $\vec{b}_r=((b_1)_{B_r(x_0)},\cdots,(b_m)_{B_r(x_0)})$, where $(b_i)_{B_r(x_0)}=\mu(B_r(x_0))^{-1}$ $\int_{B_r(x_0)} b_j(y)d\mu(y)$, $1\leq i\leq m$, it is easy to see

$$
\begin{aligned}
\left(\int_{B_r(x_0)}|T_{\vec{b}}f(x)|^p d\mu(x)\right)^{1/p} &\leq \left(\int_{B_r(x_0)}|T_{\vec{b}}f_0(x)|^p d\mu(x)\right)^{1/p}\\
&+\sum_{k\geq N}\left(\int_{B_r(x_0)}|T_{\vec{b}}f_k(x)|^p d\mu(x)\right)^{1/p}=H_1+H_2.
\end{aligned}
$$

Since

$$
\begin{aligned}
&T_{\vec{b}}(f)(x)=\int_X\prod_{i=1}^m(b_i(x)-b_i(y))K(x,y)f(y)d\mu(y)\\
=&\int_X\prod_{i=1}^m\left[(b_j(x)-(b_j)_{B_r(x_0)})-(b_j(y)-(b_j)_{B_r(x_0)})\right]K(x-y)f(y)d\mu(y)\\
=&\sum_{i=0}^m\sum_{\sigma\in C_i^m}(-1)^{m-i}(\vec{b}(x)-\vec{b}_r)_\sigma\int_X(\vec{b}(y)-\vec{b}_r)_{\sigma^c}K(x,y)f(y)d\mu(y),
\end{aligned}
$$

so

$$
\begin{aligned}
&H_1(x)\\
=&\left(\int_{B_r(x_0)}|T_{\vec{b}}f_0(x)|^p d\mu(x)\right)^{1/p}\\
=&\left(\int_{B_r(x_0)}|\int_X\prod_{i=1}^m(b_i(x)-b_i(y))K(x,y)f_0(y)d\mu(y)|^p d\mu(x)\right)^{1/p}\\
\leq&\left(\int_{B_r(x_0)}\left(\sum_{i=0}^m\sum_{\sigma\in C_i^m}|(\vec{b}(x)-\vec{b}_r)_\sigma|\int_{B_{2^N r}(x_0)}|(\vec{b}(y)-\vec{b}_r)_{\sigma^c}||K(x,y)||f(y)|d\mu(y)\right)^p d\mu(x)\right)^{1/p}\\
\leq&C\mu(B_{2^N r}(x_0))^{-1}\sum_{i=0}^m\sum_{\sigma\in C_i^m}\left(\int_{B_r(x_0)}|(\vec{b}(x)-\vec{b}_r)_\sigma|^p d\mu(x)\right)^{1/p}\int_{B_{2^N r}(x_0)}|(\vec{b}(y)-\vec{b}_r)_{\sigma^c}||f(y)|d\mu(y)\\
\leq&C\frac{\mu(B_r(x_0))^{1/p}\mu(B_{2^N r}(x_0))^{1/q}}{\mu(B_{2^N r}(x_0))}\sum_{i=0}^m\sum_{\sigma\in C_i^m}\left(\frac{1}{\mu(B_r(x_0))}\int_{B_r(x_0)}|(\vec{b}(x)-\vec{b}_r)_\sigma|^p d\mu(x)\right)^{1/p}
\end{aligned}
$$

$$
\begin{aligned}
&\times\left(\frac{1}{\mu(B_{2^N r}(x_0))}\int_{B_{2^N r}(x_0)}|(\vec{b}(y)-\vec{b}_r)_{\sigma^c}|^q d\mu(y)\right)^{1/q}\left(\int_{B_{2^N r}(x_0)}|f(y)|^p d\mu(y)\right)^{1/p}\\
\le\ & C||\vec{b}_\sigma||_{BMO}N^m||\vec{b}_{\sigma^c}||_{BMO}\phi^{1/p}(2^N r)||f||_{L^{p,\phi}(X)}\\
\le\ & CN^m D^N||\vec{b}||_{BMO}||f||_{L^{p,\phi}(X)}\\
\le\ & C||\vec{b}||_{BMO}||f||_{L^{p,\phi}}
\end{aligned}
$$

and

$$
\begin{aligned}
H_2(x) &= \sum_{k\ge N}\left(\int_{B_r(x_0)}|T_{\vec{b}}f_k(x)|^p d\mu(x)\right)^{1/p}\\
&= C\sum_{k\ge N}\left(\int_{B_r(x_0)}|\int_X\prod_{i=1}^m(b_i(x)-b_i(y))K(x,y)f_k(y)d\mu(y)|^p d\mu(x)\right)^{1/p}\\
&\le C\sum_{k\ge N}\left(\int_{B_r(x_0)}\left(\sum_{i=0}^m\sum_{\sigma\in C_i^m}|(\vec{b}(x)-\vec{b}_r)_\sigma|\int_{B_{2^{k+1}r}(x_0)\setminus B_{2^k r}(x_0)}|(\vec{b}(y)-\vec{b}_r)_{\sigma^c}|\right.\right.\\
&\qquad\left.\left.\times|K(x,y)||f(y)|d\mu(y)\right)^p d\mu(x)\right)^{1/p}\\
&\le C\sum_{k\ge N}\mu(B_{2^{k-k_0}r}(x_0))^{-1}\sum_{i=0}^m\sum_{\sigma\in C_i^m}\left(\int_{B_r(x_0)}|(\vec{b}(x)-\vec{b}_r)_\sigma|^p d\mu(x)\right)^{1/p}\\
&\qquad\times\int_{B_{2^{k+1}r}(x_0)}|(\vec{b}(y)-\vec{b}_r)_{\sigma^c}||f(y)|d\mu(y)\\
&\le C\sum_{k\ge N}\frac{\mu(B_r(x_0))^{1/p}\mu(B_{2^{k+1}r}(x_0))^{1/q}}{\mu(B_{2^{k-k_0}r}(x_0))}\sum_{i=0}^m\sum_{\sigma\in C_i^m}\left(\frac{1}{\mu(B_r(x_0))}\int_{B_r(x_0)}|(\vec{b}(x)-\vec{b}_r)_\sigma|^p d\mu(x)\right)^{1/p}\\
&\qquad\times\left(\frac{1}{\mu(B_{2^{k+1}r}(x_0))}\int_{B_{2^{k+1}r}(x_0)}|(\vec{b}(y)-\vec{b}_r)_{\sigma^c}|^q d\mu(y)\right)^{1/q}\left(\int_{B_{2^{k+1}r}(x_0)}|f(y)|^p d\mu(y)\right)^{1/p}\\
&\le C\sum_{k\ge N}\left(\frac{D}{2^\alpha}\right)^{(k+1)/p}k^m||\vec{b}||_{BMO}\phi^{1/p}||f||_{L^{p,\phi}(X)}\\
&\le C||\vec{b}||_{BMO}||f||_{L^{p,\phi}}.
\end{aligned}
$$

This finishes the proof of Theorem 1.

3. Boundedness on homogeneous Morrey-Herz space

In this section, the homogeneous Morrey-Herz spaces will be introduced and the boundedness in homogeneous Morrey-Herz spaces for the multilinear commutator of the singular integral will be discussed.

Definition 3. *Let $\alpha \in R$, $0 \le \lambda < \infty$, $0 < p < \infty$ and $1 \le q < \infty$. The homogeneous Morrey-Herz space $M\dot{K}^{\alpha,\lambda}_{p,q}(X)$ is defined by*

$$M\dot{K}^{\alpha,\lambda}_{p,q}(X) = \{f \in L^q(X \setminus \{x_0\}) : ||f||_{M\dot{K}^{\alpha,\lambda}_{p,q}} < \infty\},$$

where

$$||f||_{M\dot{K}^{\alpha,\lambda}_{p,q}} = \sup_{k_0 \in \mathbf{Z}} \mu(B_{k_0})^{-\lambda} \left(\sum_{k=-\infty}^{k_0} \mu(B_k)^{\alpha p} ||f\chi_k||^p_{L^q(X)} \right)^{1/p}$$

with the usual modifications made when $p = \infty$.

Compare the homogeneous Morrey-Herz space $M\dot{K}^{\alpha,\lambda}_{p,q}(X)$ with the homogeneous Herz space $\dot{K}^{\alpha,p}_q(X)$(see [9]) and the Morrey space $M^\lambda_q(X)$(see [10]), where $\dot{K}^{\alpha,p}_q(X)$ is defined by

$$\dot{K}^{\alpha,p}_q(X) = \left\{ f \in L^q_{loc}(X \setminus \{x_0\}) : \left(\sum_{k=-\infty}^{\infty} \mu(B_k)^{\alpha p} ||f\chi_k||^p_{L^q} \right)^{1/p} < \infty \right\};$$

and $M^\lambda_q(X)$ is defined by

$$M^\lambda_q(X) = \left\{ f \in L^q_{loc}(X) : \sup_{t>0} \frac{1}{t^\lambda} \int_{d(x,y)<t} |f(y)|^q d\mu(y) < \infty \right\}.$$

Obviously, $M\dot{K}^{\alpha,0}_{p,q}(X) = \dot{K}^{\alpha,p}_q(X)$ and $M^\lambda_q(X) \subset M\dot{K}^{\alpha,0}_{p,q}(X)$.

We can see that when $\lambda = 0$, $M\dot{K}^{\alpha,0}_{p,q}(X)$ is just the homogeneous Herz space. So in this paper, we only give the results when $\lambda > 0$.

Theorem 2. *Let T be the singular integral as Definition 1, $b_i \in BMO(X)$ for $i = 1, \cdots, m$, $0 < \lambda < \infty$, $0 < p < \infty$, $1 < q < \infty$ and $-1/q + \lambda < \alpha < 1/q' + \lambda$. If $T_{\vec{b}}$ is bounded on $L^q(X)$, then $T_{\vec{b}}$ is also bounded on $M\dot{K}^{\alpha,\lambda}_{p,q}(X)$.*

Proof. Let $f \in M\dot{K}^{\alpha,\lambda}_{p,q}(X)$ and write it as

$$f(x) = \sum_{j=-\infty}^{\infty} f(x)\chi_j(x) \equiv \sum_{j=-\infty}^{\infty} f_j(x).$$

When $m = 1$

$$||T_{b_1}(f)||_{M\dot{K}^{\alpha,\lambda}_{p,q}(X)} = \sup_{k_0 \in \mathbf{Z}} \mu(B_{k_0})^{-\lambda} \left(\sum_{k=-\infty}^{k_0} \mu(B_k)^{\alpha p} ||T_{b_1}(f)\chi_k||^p_{L^q(X)} \right)^{1/p}$$

$$\begin{aligned}
&\le C\sup_{k_0\in\mathbf{Z}}\mu(B_{k_0})^{-\lambda}\left\{\sum_{k=-\infty}^{k_0}\mu(B_k)^{\alpha p}\left(\sum_{j=-\infty}^{k-3}||T_{b_1}(f)\chi_k||_{L^q(X)}\right)^p\right\}^{1/p}+\\
&\le C\sup_{k_0\in\mathbf{Z}}\mu(B_{k_0})^{-\lambda}\left\{\sum_{k=-\infty}^{k_0}\mu(B_k)^{\alpha p}\left(\sum_{j=k-2}^{k+2}||T_{b_1}(f)\chi_k||_{L^q(X)}\right)^p\right\}^{1/p}+\\
&\le C\sup_{k_0\in\mathbf{Z}}\mu(B_{k_0})^{-\lambda}\left\{\sum_{k=-\infty}^{k_0}\mu(B_k)^{\alpha p}\left(\sum_{j=k+3}^{\infty}||T_{b_1}(f)\chi_k||_{L^q(X)}\right)^p\right\}^{1/p}\\
&= E_1+E_2+E_3.
\end{aligned}$$

We first estimate E_1. Note that $j-k\le -3$ and $x\in A_k$, denote $b_j=\mu(B)^{-1}\int_B b(y)d\mu(y)$, where B means the smallest doubling ball which is like $2^kB(k\in\mathbf{N}\cup\{0\})$.

$$\begin{aligned}
||T_{b_1}(f_j)\chi_k||_{L^q} &= \left\{\int_{A_k}\left(\int_{A_j}|b_1(x)-b_1(y)f(y)K(x,y)|d\mu(y)\right)^q d\mu(x)\right\}^{1/q}\\
&\le C\left\{\int_{A_k}\left(\int_{A_j}|b_1(x)-b_1(y)||f(y)||K(x,y)|d\mu(y)\right)^q d\mu(x)\right\}^{1/q}\\
&\le C\left\{\int_{A_k}\mu(B(x,d(x,y)))^{-q}\left(\int_{A_j}|b_1(x)-b_1(y)||f(y)|d\mu(y)\right)^q d\mu(x)\right\}^{1/q}\\
&\le C\mu(B_k)^{-1}\left\{\int_{A_k}\left(\int_{A_j}|b_1(x)-b_1(y)||f(y)|d\mu(y)\right)^q d\mu(x)\right\}^{1/q}\\
&\le C\Bigg\{\mu(B_k)^{-1}||f_j||_{L^1}\left(\int_{A_k}|b_1(x)-b_j|^q d\mu(x)\right)^{1/q}\\
&\quad+\mu(B_k)^{-1}\mu(B_k)^{1/q}||f_j||_{L^q}\left(\int_{A_j}|b_1(y)-b_j|^{q'}d\mu(y)\right)^{1/q'}\Bigg\}\\
&\le C\Big\{\mu(B_k)^{-1}||f_j||_{L^q}\mu(B_j)^{1/q'}\mu(B_k)^{1/q}||b_1||_{BMO}\\
&\quad+\mu(B_k)^{-1}\mu(B_k)^{1/q}\mu(B_j)^{1/q'}||f_j||_{L^q}||b_1||_{BMO}\Big\}\\
&\le C\left(\frac{\mu(B_j)}{\mu(B_k)}\right)^{1/q'}||f_j||_{L^q}||b_1||_{BMO},
\end{aligned}$$

where we used the condition $|K(x,y)|\le\dfrac{C}{\mu(B(x,d(x,y)))}$ and Hölder's inequality.

Note that $\alpha < 1/q' + \lambda$, we get

$$\begin{aligned} E_1 &\le C\sup_{k_0\in\mathbf{Z}}\mu(B_{k_0})^{-\lambda}\Big\{\sum_{k=-\infty}^{k_0}\mu(B_k)^{\alpha p}\Big[\sum_{j=-\infty}^{k-3}\Big(\frac{\mu(B_j)}{\mu(B_k)}\Big)^{1/q'}||f_j||_{L^q}\Big]^p\Big\}^{1/p} \\ &\le C\sup_{k_0\in\mathbf{Z}}\mu(B_{k_0})^{-\lambda}\Big\{\sum_{k=-\infty}^{k_0}\mu(B_k)^{\lambda p}\Big[\sum_{j=-\infty}^{k-3}\Big(\frac{\mu(B_j)}{\mu(B_k)}\Big)^{1/q'-\alpha+\lambda}\mu(B_j)^{-\lambda}\Big(\sum_{l=-\infty}^{j}\mu(B_j)^{\alpha p}||f_j||^p_{L^q)}\Big)^{1/p}\Big]^p\Big\}^{1/p} \\ &\le C\sup_{k_0\in\mathbf{Z}}\mu(B_{k_0})^{-\lambda}\Big\{\sum_{k=-\infty}^{k_0}\mu(B_k)^{\lambda p}\Big[\sum_{j=-\infty}^{k-3}2^{(j-k)(\lambda-\alpha+1/q')}||f||_{M\dot K^{\alpha,\lambda}_{p,q}}\Big]^p\Big\}^{1/p} \\ &\le C||f||_{M\dot K^{\alpha,\lambda}_{p,q}}. \end{aligned}$$

Next we estimate E_2, by the $L^q(X)$ boundedness of T_{b_1}, we have

$$\begin{aligned} E_2 &\le C\sup_{k_0\in\mathbf{Z}}\mu(B_{k_0})^{-\lambda}\Big\{\sum_{k=-\infty}^{k_0}\mu(B_k)^{\alpha p}\Big(\sum_{j=k-2}^{k+2}||f_j||_{L^q}\Big)^p\Big\}^{1/p} \\ &\le C\sup_{k_0\in\mathbf{Z}}\mu(B_{k_0})^{-\lambda}\Big\{\sum_{k=-\infty}^{k_0}\mu(B_k)^{\alpha p}||f_j||^p_{L^q}\Big\}^{1/p} \\ &\le C||f||_{M\dot K^{\alpha,\lambda}_{p,q}}. \end{aligned}$$

Finally, we estimate E_3. Similar to the proof of E_1, we easily obtain

$$\begin{aligned} ||T_{b_1}(f_j)\chi_k||_{L^q} &= \left\{\int_{A_k}\left(\int_{A_j}|b_1(x)-b_1(y)f(y)K(x,y)|d\mu(y)\right)^q d\mu(x)\right\}^{1/q} \\ &\le C\left\{\int_{A_k}\left(\int_{A_j}|b_1(x)-b_1(y)||f(y)||K(x,y)|d\mu(y)\right)^q d\mu(x)\right\}^{1/q} \\ &\le C\left\{\int_{A_k}\left(\int_{A_j}|b_1(x)-b_1(y)||f(y)|\mu(B(x,d(x,y)))^{-1}d\mu(y)\right)^q d\mu(x)\right\}^{1/q} \\ &\le C\mu(B_j)^{-1}\left\{\int_{A_k}\left(\int_{A_j}|b_1(x)-b_1(y)||f(y)|d\mu(y)\right)^q d\mu(x)\right\}^{1/q} \\ &\le C\Big\{\mu(B_j)^{-1}||f_j||_{L^1}\left(\int_{A_k}|b_1(x)-b_j|^q d\mu(x)\right)^{1/q} \\ &\quad +\mu(B_j)^{-1}\mu(B_k)^{1/q}||f_j||_{L^q(X)}\left(\int_{A_j}|b_1(y)-b_j|^{q'}d\mu(y)\right)^{1/q'}\Big\} \\ &\le C\Big\{\mu(B_j)^{-1}||f_j||_{L^q}\mu(B_j)^{1/q'}\mu(B_k)^{1/q}||b_1||_{BMO} \end{aligned}$$

$$+\mu(B_j)^{-1}\mu(B_k)^{1/q}\mu(B_j)^{1/q'}||f_j||_{L^q}||b_1||_{BMO}\Big\}$$
$$\leq C\left(\frac{\mu(B_k)}{\mu(B_j)}\right)^{1/q}||f_j||_{L^q}||b_1||_{BMO}.$$

Note that $\alpha > -1/q+\lambda$ and $k-j \leq -3$, we get

$$\begin{aligned}
E_3 &\leq C\sup_{k_0\in\mathbf{Z}}\mu(B_{k_0})^{-\lambda}\Big\{\sum_{k=-\infty}^{k_0}\mu(B_k)^{\alpha p}\Big[\sum_{j=k+3}^{\infty}\Big(\frac{\mu(B_k)}{\mu(B_j)}\Big)^{1/q}||f_j||_{L^q}\Big]^p\Big\}^{1/p}\\
&\leq C\sup_{k_0\in\mathbf{Z}}\mu(B_{k_0})^{-\lambda}\Big\{\sum_{k=-\infty}^{k_0}\mu(B_k)^{\lambda p}\Big[\sum_{j=k+3}^{\infty}\Big(\frac{\mu(B_k)}{\mu(B_j)}\Big)^{1/q+\alpha-\lambda}\mu(B_j)^{-\lambda}\Big(\sum_{l=-\infty}^{j}\mu(B_j)^{\alpha p}||f_j||_{L^q}^p\Big)^{1/p}\Big]^p\Big\}^{1/p}\\
&\leq C\sup_{k_0\in\mathbf{Z}}\mu(B_{k_0})^{-\lambda}\Big\{\sum_{k=-\infty}^{k_0}\mu(B_k)^{\lambda p}\Big[\sum_{j=k+3}^{\infty}2^{(k-j)(1/q+\alpha-\lambda)}||f||_{M\dot{K}_{p,q}^{\alpha,\lambda}}\Big]^p\Big\}^{1/p}\\
&\leq C||f||_{M\dot{K}_{p,q}^{\alpha,\lambda}(X)}.
\end{aligned}$$

This completes the proof of the case $m=1$.

When $m>1$,

$$\begin{aligned}
||T_{\vec{b}}(f)||_{M\dot{K}_{p,q}^{\alpha,\lambda}} &= \sup_{k_0\in\mathbf{Z}}\mu(B_{k_0})^{-\lambda}\left(\sum_{k=-\infty}^{k_0}\mu(B_k)^{\alpha p}||T_{\vec{b}}(f)\chi_k||_{L^q}^p\right)^{1/p}\\
&\leq C\sup_{k_0\in\mathbf{Z}}\mu(B_{k_0})^{-\lambda}\left\{\sum_{k=-\infty}^{k_0}\mu(B_k)^{\alpha p}\left(\sum_{j=-\infty}^{k-3}||T_{\vec{b}}(f)\chi_k||_{L^q}\right)^p\right\}^{1/p}+\\
&\leq C\sup_{k_0\in\mathbf{Z}}\mu(B_{k_0})^{-\lambda}\left\{\sum_{k=-\infty}^{k_0}\mu(B_k)^{\alpha p}\left(\sum_{j=k-2}^{k+2}||T_{\vec{b}}(f)\chi_k||_{L^q}\right)^p\right\}^{1/p}+\\
&\leq C\sup_{k_0\in\mathbf{Z}}\mu(B_{k_0})^{-\lambda}\left\{\sum_{k=-\infty}^{k_0}\mu(B_k)^{\alpha p}\left(\sum_{j=k+3}^{\infty}||T_{\vec{b}}(f)\chi_k||_{L^q}\right)^p\right\}^{1/p}\\
&= G_1+G_2+G_3.
\end{aligned}$$

For G_1, similar to the proof of E_1, set $\vec{b}=(b_1,\cdots,b_m)$, where $b_i=\mu(B_i)^{-1}\int_{B_i}b_i(y)d\mu(y)$, $1\leq i\leq m$, we have

$$||T_{\vec{b}}(f_j)\chi_k||_{L^q} = \left\{\int_{A_k}\left(\int_{A_j}\prod_{i=1}^{m}|b_i(x)-b_i(y)f(y)K(x,y)|d\mu(y)\right)^q d\mu(x)\right\}^{1/q}$$

$$\begin{aligned}
&\leq C\mu(B_k)^{-1}\left\{\int_{A_k}\sum_{i=0}^{m}\sum_{\sigma\in C_i^m}|(\vec{b}(x)-\vec{b})_\sigma|^q d\mu(x)\right\}^{1/q}\int_{A_j}|(\vec{b}(y)-\vec{b})_{\sigma^c}||f(y)|d\mu(y)\\
&\leq C\mu(B_k)^{-1}\mu(B_k)^{1/q}k^m||\vec{b}_\sigma||_{BMO}\mu(B_j)^{1/q'}k^m||\vec{b}_{\sigma^c}||_{BMO}||f_j||_{L^q}\\
&\leq C\left(\frac{\mu(B_j)}{\mu(B_k)}\right)^{1/q'}k^{2m}||f_j||_{L^q}||\vec{b}||_{BMO},
\end{aligned}$$

thus, notice $\alpha < 1/q' + \lambda$,

$$\begin{aligned}
G_1 &\leq C\sup_{k_0\in\mathbf{Z}}\mu(B_{k_0})^{-\lambda}\left\{\sum_{k=-\infty}^{k_0}\mu(B_k)^{\alpha p}\Big[\sum_{j=-\infty}^{k-3}\Big(\frac{\mu(B_j)}{\mu(B_k)}\Big)^{1/q'}k^{2m}||f_j||_{L^q}\Big]^p\right\}^{1/p}\\
&\leq C\sup_{k_0\in\mathbf{Z}}\mu(B_{k_0})^{-\lambda}\left\{\sum_{k=-\infty}^{k_0}\mu(B_k)^{\lambda p}\Big[\sum_{j=-\infty}^{k-3}k^{2m}\Big(\frac{\mu(B_j)}{\mu(B_k)}\Big)^{1/q'-\alpha+\lambda}\right.\\
&\quad\left.\times\mu(B_j)^{-\lambda}\Big(\sum_{l=-\infty}^{j}\mu(B_j)^{\alpha p}||f_j||_{L^q}^p\Big)^{1/p}\Big]^p\right\}^{1/p}\\
&\leq C\sup_{k_0\in\mathbf{Z}}\mu(B_{k_0})^{-\lambda}\left\{\sum_{k=-\infty}^{k_0}\mu(B_k)^{\lambda p}\Big[\sum_{j=-\infty}^{k-3}2^{(j-k)(\lambda-\alpha+1/q')}k^{2m}||f||_{M\dot{K}_{p,q}^{\alpha,\lambda}}\Big]^p\right\}^{1/p}\\
&\leq C||f||_{M\dot{K}_{p,q}^{\alpha,\lambda}}.
\end{aligned}$$

For G_2, similar to the proof of E_2, by the $L^q(X)$ boundedness of $T_{\vec{b}}$, we get

$$\begin{aligned}
G_2 &\leq C\sup_{k_0\in\mathbf{Z}}\mu(B_{k_0})^{-\lambda}\left\{\sum_{k=-\infty}^{k_0}\mu(B_k)^{\alpha p}\Big(\sum_{j=k-2}^{k+2}||f_j||_{L^q(X)}\Big)^p\right\}^{1/p}\\
&\leq C\sup_{k_0\in\mathbf{Z}}\mu(B_{k_0})^{-\lambda}\left\{\sum_{k=-\infty}^{k_0}\mu(B_k)^{\alpha p}||f_j||_{L^q}^p\right\}^{1/p}\\
&\leq C||f||_{M\dot{K}_{p,q}^{\alpha,\lambda}}.
\end{aligned}$$

For G_3, similar to the proof of G_1, we have

$$\begin{aligned}
||T_{\vec{b}}(f_j)\chi_k||_{L^q} &= \left\{\int_{A_k}\Big(\int_{A_j}\prod_{i=1}^{m}|b_i(x)-b_i(y)f(y)K(x,y)|d\mu(y)\Big)^q d\mu(x)\right\}^{1/q}\\
&\leq C\mu(B_j)^{-1}\left\{\int_{A_k}\sum_{i=0}^{m}\sum_{\sigma\in C_i^m}|(\vec{b}(x)-\vec{b})_\sigma|^q d\mu(x)\right\}^{1/q}\int_{A_j}|(\vec{b}(y)-\vec{b})_{\sigma^c}||f(y)|d\mu(y)\\
&\leq C\mu(B_j)^{-1}\mu(B_k)^{1/q}k^m||\vec{b}_\sigma||_{BMO}\mu(B_j)^{1/q'}k^m||\vec{b}_{\sigma^c}||_{BMO}||f_j||_{L^q}
\end{aligned}$$

$$\leq C\left(\frac{\mu(B_k)}{\mu(B_j)}\right)^{1/q} k^{2m}||f_j||_{L^q}||\vec{b}||_{BMO}.$$

thus, notice $\alpha > -1/q+\lambda$,

$$\begin{aligned}
G_3 &\leq C\sup_{k_0\in\mathbf{Z}}\mu(B_{k_0})^{-\lambda}\Big\{\sum_{k=-\infty}^{k_0}\mu(B_k)^{\alpha p}\Big[\sum_{j=k+3}^{\infty}\Big(\frac{\mu(B_k)}{\mu(B_j)}\Big)^{1/q}k^{2m}||f_j||_{L^q}\Big]^p\Big\}^{1/p}\\
&\leq C\sup_{k_0\in\mathbf{Z}}\mu(B_{k_0})^{-\lambda}\Big\{\sum_{k=-\infty}^{k_0}\mu(B_k)^{\lambda p}\Big[\sum_{j=k+3}^{\infty}k^{2m}\Big(\frac{\mu(B_k)}{\mu(B_j)}\Big)^{1/q+\alpha-\lambda}\\
&\quad\times\mu(B_j)^{-\lambda}\Big(\sum_{l=-\infty}^{j}\mu(B_j)^{\alpha p}||f_j||^p_{L^q}\Big)^{1/p}\Big]^p\Big\}^{1/p}\\
&\leq C\sup_{k_0\in\mathbf{Z}}\mu(B_{k_0})^{-\lambda}\Big\{\sum_{k=-\infty}^{k_0}\mu(B_k)^{\lambda p}\Big[\sum_{j=k+3}^{\infty}2^{(k-j)(1/q+\alpha-\lambda)}k^{2m}||f||_{M\dot{K}^{\alpha,\lambda}_{p,q}}\Big]^p\Big\}^{1/p}\\
&\leq C||f||_{M\dot{K}^{\alpha,\lambda}_{p,q}}.
\end{aligned}$$

The estimates for G_1, G_2 and G_3 indicate that $||T_{\vec{b}}||_{M\dot{K}^{\alpha,\lambda}_{p,q}} \leq C||f||_{M\dot{K}^{\alpha,\lambda}_{p,q}}$. And we complete the proof of the Theorem 2.

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