SOME NEW CLASSES OF ANALYTIC FUNCTIONS DEFINED BY MEAN OF A CONVOLUTION OPERATOR

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Abstract. In this paper we use the inverse of most famous Carlson Shaffer operator and introduce some new classes of analytic functions. Some inclusion results, a radius problem are discussed. We also show that these classes are closed under convolution with a convex function.

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1. Introduction

Let A_p denotes the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k}\ z^{p+k}, \tag{1.1}$$

which are analytic in the open unit disc $E = \{z : |z| < 1\}$ and $p \in N = \{1, 2, 3, ...\}$. Further for $0 \leqslant \alpha < p, S_p^*(\alpha)$ and $K_p(\alpha)$ denotes the classes of all $p-$valently starlike and convex functions of order α respectively. Also the class of $p-$valently close-to-convex of order α type γ is denoted by $B_p(\alpha, \gamma)$ and is given by

$$B_p(\alpha,\gamma) = \left\{ f \in A_p : \exists\ g \in S_p^*(\gamma)\ s.t. \operatorname{Re}\left(\frac{zf'(z)}{g(z)}\right) > \alpha,\ z \in E, ,\ 0 \leqslant \alpha, \gamma < 1 \right\}.$$

The classes S_p^* and K_p was introduced by Goodman [3] and $B_p(\alpha, \gamma)$ was studied by Aouf [1]. Clearly

$$f \in K_p(\alpha) \Longleftrightarrow \frac{zf'}{f} \in S_p^*(\alpha). \tag{1.2}$$

The convolution (or Hadamard product) is denoted and defined by

$$f(z) * g(z) = z^p + \sum_{k=1}^{\infty} a_{p+k}\ b_{p+k}\ z^{p+k},$$

where

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k}\ z^{p+k} \text{ and } g(z) = z^p + \sum_{k=1}^{\infty} b_{p+k}\ z^{p+k}.$$

The generalized Bernadi operator is denoted and defined as,

$$J_{c,p}(f(z)) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t)\ dt,\ c > -p. \tag{1.3}$$

Inspiring from carlson shaffer, Saitoh [7] introduced a linear operator, $L_p(a,c)$, $(a \in R,\ c \in C - \{0,-1,-2,...\})$ as:

$$L_p(a,c)f(z) = \phi_p(a,c;z) * f(z) = z^p + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} a_{k+p}\ z^{p+k}, \tag{1.4}$$

where

$$\phi_p(a,c;z) = z^p + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k}\ z^{p+k},$$

and $(a)_k$ is Pochhammer symbol.

Al-Kharasani and Al-Hajiry [2] defined the linear operator $L_p^*(a,c)$ as

$$L_p^*(a,c)f(z) = \phi_p^*(a,c;z) * f(z), \tag{1.5}$$

where

$$\phi_p(a,c;z) * \phi_p^*(a,c;z) = \frac{z^p}{(1-z)^{p+1}}. \tag{1.6}$$

From (1.5) and (1.6) the following identity can be easily verified

$$L_p^*(a,c+1)f(z) = z\left(L_p^*(a,c)f(z)\right)' + (c-1)L_p^*(a,c)f(z). \tag{1.7}$$

Using the operator $L_p^*(a,c)$, we introduce the following new classes of analytic functions.

Definition 1.1

$$S_p^*(a,c,\alpha) = \left\{ f \in A_p : \operatorname{Re}\left[\frac{z(L_p^*(a,c)f(z))'}{L_p^*(a,c)f(z)}\right] > \alpha, 0 \le \alpha < 1 \right\}.$$

Definition 1.2

$$K_p^*(a,c,\alpha) = \left\{ f \in A_p : \operatorname{Re}\left[\frac{\left(z(L_p^*(a,c)f(z))'\right)'}{\left(L_p^*(a,c)f(z)\right)'}\right] > \alpha, 0 \le \alpha < 1 \right\}.$$

Definition 1.3

$$B_p^*(a,c,\alpha,\gamma)=\left\{f\in A_p:\exists\, g\in S_p^*(a,c,\gamma)\,\mathrm{Re}\left[\frac{z(L_p^*(a,c)f(z))'}{L_p^*(a,c)g(z)}\right]>\alpha, 0\le\alpha,\gamma<1\right\}.$$

2.Preliminary Results

We shall need the following lemmas in the proof of our main results:

Lemma 2.1[4] *Let* $u=u_1+iu_2, v=v_1+iv_2$ *and* $\psi(u,v)$ *be a complex valued function satisfying the conditions:*
(i) $\psi(u,v)$ *is continuous in a domain* $D\subseteq C\times C$.
(ii) $(1,0)\in D$ *and* $\psi(1,0)>0$.
(iii) $Re\psi(iu_2,v_1)\le 0$, *whenever* $(iu_2,v_1)\in D$ *and* $v_1\le\frac{1}{2}(1+u_2^2)$.

If $h(z)=1+\sum\limits_{k=1}^{\infty}a_k\,z^k$ is analytic in E such that $(h(z),zh'(z))\in D$ and $\mathrm{Re}\,\psi(h(z),zh'(z))>0$ in E, then $\mathrm{Re}\,h(z)>0$ in E.

Lemma 2.2 [5] *Let* ψ *be a convex and* g *be a starlike in* E. *Then for* F *analaytic in* E *with* $F(0)=1$, *then* $\frac{\psi*Fg}{\psi*g}$ *is contained in convex hull of* $F(E)$.

Lemma 2.3 [6] *Let* p *be an analytic function in* E *with* $p(0)=1$ *and* $\mathrm{Re}(p(z))>0$, $z\in E$, *then for* $s>0$ *and* $\mu\neq-1$ (complex),

$$\mathrm{Re}\left[p(z)+\frac{szp'(z)}{p(z)+\mu}\right]>0\qquad \textit{for } |z|<r_0,$$

where r_0 *is given by*

$$r_0=\frac{|\mu+1|}{\sqrt{A+(A^2-|\mu^2-1|^2)^{\frac{1}{2}}}}. \tag{2.1}$$

and $A=2(s+1)^2+|\mu|^2-1$.

3.Main results

Theorem 3.1 *For* $0\le\alpha<p, c\ge p$,

$$S_p^*(a,c+1,\alpha)\subseteq S_p^*(a,c,\beta),$$

where

$$\beta = \frac{2[p-2\alpha(p-c)]}{\sqrt{(2c-2p-2\alpha+1)^2+8(p-2\alpha(p-c))}+(2c-2p-2\alpha+1)}. \tag{3.1}$$

Proof. Let

$$\frac{z(L_p^*(a,c)f(z))'}{L_p^*(a,c)f(z)} = H(z) = (p-\beta)h(z)+\beta. \tag{3.2}$$

We want to show that $H(z) \in P(\beta)$ or $h(z) \in P$.

From $(1.7), (3.2)$and after some simplification, we have

$$\frac{z(L_p^*(a,c)f(z))'}{L_p^*(a,c)f(z)} - \alpha = (\beta-\alpha)+(p-\beta)h(z)+\frac{(p-\beta)zh'(z)}{(p-\beta)h(z)+(\beta+c-p)}.$$

Now by taking $u = h(z)$ and $v = zh'(z)$, we formulate a functional $\psi(u,v)$ as

$$\psi(u,v) = (\beta-\alpha)+(p-\beta)u(z)+\frac{(p-\beta)v(z)}{(p-\beta)u(z)+(\beta+c-p)}.$$

Then obviously $\psi(u,v)$ satisfies conditions (i) and (ii) of Lemma 2.1 in the domain $D \subseteq C \times \left(C - \left\{\frac{\beta+c-p}{\beta-p}\right\}\right)$.

For the third condition we proceed as follows:

$$\operatorname{Re}\psi(iu_2,v_1) = (\beta-\alpha)+\frac{(p-\beta)(\beta+c-p)v_1}{(p-\beta)^2\ u_2^2+(\beta+c-p)^2}$$

When we put $v_1 \le \frac{1}{2}(1+u_2^2)$, then $\operatorname{Re}\psi(iu_2,v_1) \le \frac{A+Bu_2^2}{2C}$, where

$$\begin{aligned} A &= 2(\beta-\alpha)(\beta+c-p)^2-(\beta+c-p)(p-\beta) \\ B &= 2(\beta-\alpha)(p-\beta)^2-(\beta+c-p)(p-\beta) \\ C &= (\beta+c-p)^2+(p-\beta)^2u_2^2 > 0. \end{aligned}$$

We note that $\operatorname{Re}\psi(iu_2,v_1) \le 0$ if and only if $A \le 0$ and $B \le 0$. From $A \le 0$,we obtain β, as given by (3.1), and from $B \le 0$ gives $0 \le \beta < p$. Hence $H(z) \in P(\beta)$ and consequently $f(z) \in S_p^*(a,c,\beta)$.

Special Case. For $\alpha = 0, a = 1, c = 1, p = 1$, we obtain a well known result that every convex function is starlike of order $\frac{1}{2}$.

Theorem 3.2. *For* $0 \le \alpha < p, c \ge p$,

$$K_p^*(a,c+1,\alpha) \subseteq K_p^*(a,c,\alpha).$$

Proof. The proof follows immediately by using (1.2) and Theorem 3.1.

Theorem 3.3. *For* $0 \leq \alpha < p, c \geq p$,

$$B_p^*(a, c+1, \alpha, \gamma) \subseteq B_p^*(a, c, \alpha, \gamma).$$

Proof. Let

$$\frac{z(L_p^*(a,c)f(z))'}{L_p^*(a,c)g(z)} = H(z) = (p-\alpha)h(z) + \alpha. \tag{3.3}$$

Using (1.7), (3.3) and after some simplification, we have

$$\frac{z(L_p^*(a,c+1)f(z))'}{L_p^*(a,c+1)g(z)} - \alpha = (p-\alpha)h(z) + \frac{(p-\alpha)zh'(z)}{(c-p)+H_0(z)},$$

where

$$H_0(z) = \frac{z(L_p^*(a,c)g(z))'}{L_p^*(a,c)g(z)}.$$

Now by taking $u = h(z)$ and $v = zh'(z)$, we formulate the function $\psi(u,v)$ as

$$\psi(u,v) = (p-\alpha) + \frac{(p-\beta)v(z)}{(c-p)+H_0(z)}.$$

Then clearly $\psi(u,v)$ satisfies all the conditions of Lemma 2.1. Hence $H(z) \in P(\alpha)$ and consequently $f \in B_p^*(a,c,\alpha,\gamma)$.

Theorem 3.4. *Let* $f \in S_p^*(a,c,\alpha)$, *then* $J_{c,p}f \in S_p^*(a,c,\alpha)$.

Proof. Let

$$\frac{z(L_p^*(a,c)J_{c,p}f(z))'}{L_p^*(a,c)J_{c,p}f(z)} = H(z) = (p-\alpha)h(z) + \alpha. \tag{3.4}$$

Using (2.6), (3.4) and after some simplification, we have

$$\frac{z(L_p^*(a,c)f(z))'}{L_p^*(a,c)f(z)} = (p-\alpha)h(z) + \frac{(p-\alpha)zh'(z)}{(p-\alpha)h(z)+(\alpha+c)} \in P(\alpha).$$

Now by taking $u = h(z)$ & $v = zh'(z)$, we formulate the function $\psi(u,v)$ as

$$\psi(u,v) = (p-\alpha) + \frac{(p-\alpha)v(z)}{(p-\alpha)h(z)+(\alpha+c)}.$$

Then clearly $\psi(u,v)$ satisfies all the conditions of Lemma 2.1. Hence $H(z) \in P(\alpha)$ and consequently $J_{c,p}f \in S_p^*(a,c,\alpha)$.

Theorem 3.5. *If* $\phi \in C$ *and* $f \in S_p^*(a,c,\alpha)$ *then* $\phi * f \in S_p^*(a,c,\alpha)$.

Proof. Let $G = \phi * f$. Then

$$L_p^*(a,c)G = \phi * (L_p^*(a,c) * f). \tag{3.5}$$

By logarithmic differentiation of (3.5) and after some simplification, we have

$$\frac{z(L_p^*(a,c)G)'}{L_p^*(a,c)G} = \frac{\phi * FL_p^*(a,c)f}{\phi * L_p^*(a,c)f},$$

where

$$F = \frac{z(L_p^*(a,c) * f)'}{L_p^*(a,c) * f}.$$

As $f \in S_p^*(a,c,\alpha)$, therefore by using Lemma 2.2, it follows at once that $\phi * f \in S_p^*(a,c,\alpha)$.

Theorem 3.6. *Let* $f(z) \in S_p^*(a,c,\alpha)$, *then* $f(z) \in S_p^*(a,c+1,\alpha)$ *for* $|z| < r$, *where* r *is given by* (2.1), *with* $s = \frac{1}{p-\alpha}, \mu = \frac{\alpha+p-1}{p-\alpha}$.

Proof. Let $f(z) \in S_p^*(a,c,\alpha)$ then

$$\frac{z(L_p^*(a,c)f(z))'}{L_p^*(a,c)f(z)} = H(z) = \alpha + (p-\alpha)h, \quad h \in P. \tag{3.6}$$

Working in the same way as in Theorem 3.1, we have

$$\frac{1}{p-\alpha}\left\{\frac{z(L_p^*(a,c+1)f(z))'}{L_p^*(a,c+1)f(z)} - \alpha\right\} = h(z) + \frac{\left(\frac{1}{p-\alpha}\right) zh'(z)}{h(z) + \frac{\alpha+p-1}{p-\alpha}}$$

Then by using Lemma 2.3 with $s = \frac{1}{p-\alpha}$,and $\mu = \frac{\alpha+p-1}{p-\alpha} \neq -1$, we have $f(z) \in S_p^*(a,c+1,\alpha)$ for $|z| < r$, where r is given by (2.1) and this radius is best possible.

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