BOUNDEDNESS FOR MULTILINEAR OPERATORS OF MULTIPLIER OPERATORS ON TRIEBEL-LIZORKIN AND LEBESGUE SPACES

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Abstract. The boundedness for the multilinear operators associated to some multiplier operators and the Lipschitz functions on Triebel-Lizorkin and Lebesgue spaces are obtained.

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1. Introduction

As the development of singular integral operators, their commutators and multilinear operators have been well studied (see [1-7]). From [2][13], we know that the commutators and multilinear operators generated by the singular integral operators and the Lipschitz functions are bounded on the Triebel-Lizorkin and Lebesgue spaces. The purpose of this paper is to introduce some multilinear operators associated to some multiplier operators and the Lipschitz functions, and prove the boundedness properties for the multilinear operators on the Triebel-Lizorkin and Lebesgue spaces.

2. Preliminaries and Theorems

First, let us introduce some notations. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For a locally integrable function f, let $f(Q) = \int_Q f(x)dx$, $f_Q = |Q|^{-1} \int_Q f(x)dx$ and

$$f^{\#}(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy.$$

It is well-known that(see [14][15])

$$f^{\#}(x) \approx \sup_{Q \ni x} \inf_{c \in C} \frac{1}{|Q|} \int_Q |f(y) - c| dy.$$

For $1 \leq p < \infty$ and $0 \leq \delta < n$, let

$$M_{\delta,p}(f)(x) = \sup_{Q \ni x} \left(\frac{1}{|Q|^{1-p\delta/n}} \int_Q |f(y)|^p dy \right)^{1/p},$$

which is the Hardy-Littlewood maximal function when $p = 1$ and $\delta = 0$.

For $\beta > 0$ and $p > 1$, let $\dot{F}_p^{\beta,\infty}(R^n)$ be the homogeneous Triebel-Lizorkin space (see [13]). The Lipschitz space $\dot{\Lambda}_\beta(R^n)$ is the space of functions f such that

$$||f||_{\dot{\Lambda}_\beta} = \sup_{\substack{x, h \in R^n \\ h \neq 0}} \left| \Delta_h^{[\beta]+1} f(x) \right| / |h|^\beta < \infty,$$

where Δ_h^k denotes the k-th difference operator (see [13]).

In this paper, we will study a class of multilinear operators associated to some multiplier operators as follows.

A bounded measurable function k defined on $R^n \setminus \{0\}$ is called a multiplier. The multiplier operator T associated with k is defined by

$$T(f)(x) = k(x)\hat{f}(x), \text{ for } f \in S(R^n),$$

where $\hat{f}$ denotes the Fourier transform of f and $S(R^n)$ is the Schwartz test function class. Now, we recall the definition of the class $M(s,l)$. Denote by $|x| \sim t$ the fact that the value of x lies in the annulus $\{x \in R^n : at < |x| < bt\}$, where $0 < a \leq 1 < b < \infty$ are values specified in each instance.

Definition 1. ([11]) *Let $l \geq 0$ be a real number and $1 \leq s \leq 2$. we say that the multiplier k satisfies the condition $M(s,l)$, if*

$$\left(\int_{|\xi| \sim R} |D^\alpha k(\xi)|^s d\xi \right)^{\frac{1}{s}} < CR^{n/s - |\alpha|}$$

for all $R > 0$ and multi-indices α with $|\alpha| \leq l$, when l is a positive integer, and, in addition, if

$$\left(\int_{|\xi| \sim R} |D^\alpha k(\xi) - D^\alpha k(\xi - z)|^s d\xi \right)^{\frac{1}{s}} \leq C(\frac{|z|}{R})^\gamma R^{\frac{n}{s} - |\alpha|}$$

for all $|z| < R/2$ *and all multi-indices* α *with* $|\alpha| = [l]$*, the integer part of* l*,i.e.,* $[l]$ *is the greatest integer less than or equal to* l*, and* $l = [l] + \gamma$ *when* l *is not an integer.*

Denote $D(R^n) = \{\phi \in S(R^n) : \text{supp}(\phi) \text{ is compact}\}$ and $\hat{D}_0(R^n) = \{\phi \in S(R^n) : \hat{\phi} \in D(R^n) \text{ and } \hat{\phi} \text{ vanishes in a neighbourhood of the origin}\}$. The following boundedness property of T on $L^p(R^n)$ is proved by Strömberg and Torkinsky (see [11-14]).

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Lemma 1.([11]) *Let* $k \in M(s,l), 1 \le s \le 2$*, and* $l > \frac{n}{s}$*. Then the associated mapping* T*, defined a priori for* $f \in \hat{D}_0(R^n)$*,* $T(f)(x) = (f * K)(x)$*, extends to a bounded mapping from* $L^p(R^n)$ *into itself for* $1 < p < \infty$ *and* $K(x) = \check{k}(x)$*.*

Definition 2.([11]) *For a real number* $\tilde{l} \ge 0$ *and* $1 \le \tilde{s} < \infty$*, we say that* K *verifies the condition* $\tilde{M}(\tilde{s},\tilde{l})$*, and write* $K \in \tilde{M}(\tilde{s},\tilde{l})$*, if*

$$\left(\int_{|x|\sim R} |D^{\tilde{\alpha}}K(x)|^{\tilde{s}}dx\right)^{\frac{1}{\tilde{s}}} \le CR^{n/\tilde{s}-n-|\tilde{\alpha}|}, \quad R > 0$$

for all multi-indices $|\tilde{\alpha}| \le \tilde{l}$ *and, in addition, if*

$$\left(\int_{|x|\sim R} |D^{\tilde{\alpha}}K(x) - D^{\tilde{\alpha}}K(x-z)|^{\tilde{s}}dx\right)^{\frac{1}{\tilde{s}}} \le C(\frac{|z|}{R})^v R^{\frac{n}{\tilde{s}}-n-u}, \qquad if \quad 0 < v < 1,$$

$$\left(\int_{|x|\sim R} |D^{\tilde{\alpha}}K(x) - D^{\tilde{\alpha}}K(x-z)|^{\tilde{s}}dx\right)^{\frac{1}{\tilde{s}}} \le C(\frac{|z|}{R})(log\frac{R}{|z|})R^{\frac{n}{\tilde{s}}-n-u}, \qquad if \quad v = 1,$$

for all $|z| < \frac{R}{2}, R > 0$*, and all multi-indices* $\tilde{\alpha}$ *with*$|\tilde{\alpha}| = u$*, where* u *denotes the largest integer strictly less than* $\tilde{l}$ *with* $\tilde{l} = u + v$*.*

Lemma 2. ([11]) *Suppose* $k \in M(s,l)$*,* $1 \le s \le 2$*. Given* $1 \le \tilde{s} < \infty$*, let* $r \ge 1$ *be such that* $\frac{1}{r} = \max\{\frac{1}{s}, 1 - \frac{1}{\tilde{s}}\}$*. Then* $K \in \tilde{M}(\tilde{s},\tilde{l})$*, where* $\tilde{l} = l - \frac{n}{r}$ *.*

Lemma 3. *Let* $1 \le s \le 2$*, suppose that* l *is a positive real number with* $l > n/r$*,* $1/r = \max\{1/s, 1-1/\tilde{s}\}$*, and* $k \in M(s,l)$*. Then there is a positive constant* a*, such that*

$$\left(\int_{Q_k} |K(x-z) - K(x_Q - z)|^{\tilde{s}}dz\right)^{1/\tilde{s}} \le C2^{-ka}(2^k h)^{-n/\tilde{s}'}.$$

Proof. We split our proof into two cases:

Case 1. $1 \le s \le 2$ and $0 < l - n/s \le 1$. We choose a real number $1 < \tilde{s} < \infty$ such that $s \le \tilde{s}$, and set $\tilde{l} = l - \frac{n}{s} > 0$. Since $k \in M(s,l)$, then by Lemma 3, there is $K \in \tilde{M}(\tilde{s},\tilde{l})$.

When $\tilde{l} = l - \frac{n}{s} < 1$, noting that l is a positive real number and $l > \frac{n}{s}$. Applying the condition $K \in \tilde{M}(\tilde{s}, \tilde{l})$ for $v = l - \frac{n}{s}$ and $u = 0$, one has

$$\left(\int_{Q_k} |K(x-z) - K(x_Q - z)|^{\tilde{s}} dz\right)^{\frac{1}{\tilde{s}}} \leq C2^{-k(l-\frac{n}{s})}(2^k h)^{-\frac{n}{\tilde{s}'}},$$

let $a = l - \frac{n}{s}$,

$$\left(\int_{Q_k} |K(x-z) - K(x_Q - z)|^{\tilde{s}} dz\right)^{\frac{1}{\tilde{s}}} \leq C2^{-ka}(2^k h)^{-\frac{n}{\tilde{s}'}}.$$

When $\tilde{l} = l - \frac{n}{s} = 1$, we choose $0 < \xi < 1$, such that $t^{1-\xi}\log(1/t) \leq C$ for $0 < t < 1/2$. Noting that $K \in \tilde{M}(\tilde{s}, \tilde{l})$, by Definition 2, for $u = 0, v = 1$,

$$\begin{aligned} &\left(\int_{Q_k} |K(x-z) - K(x_Q - z)|^{\tilde{s}} dz\right)^{\frac{1}{\tilde{s}}} \\ \leq\ & C\left(\frac{|y - x_Q|}{2^k h}\right)^{\xi}\left(\frac{|y - x_Q|}{2^k h}\right)^{1-\xi}\left(\log \frac{2^k h}{|y - x_Q|}\right)(2^k h)^{n/\tilde{s} - n} \\ \leq\ & C2^{-k\xi}(2^k h)^{-n/\tilde{s}'}, \end{aligned}$$

let $a = \xi$, then

$$\left(\int_{Q_k} |K(x-z) - K(x_Q - z)|^{\tilde{s}} dz\right)^{\frac{1}{\tilde{s}}} \leq C2^{-ka}(2^k h)^{-n/\tilde{s}'}.$$

Case 2. $1 \leq s \leq 2$ and $l - n/s > 1$. Set $d = [l - n/s]$, if $l - n/s > 1$ is not an integer, and $d = l - n/s - 1$ if $l - n/s > 1$ is an integer. Choose $l_1 = l - d$; then $0 < l_1 - n/s \leq 1$ and $0 < l_1 < l$. So, from $k \in M(s, l)$ we know $k \in M(s, l_1)$. Set $\tilde{l} = l_1 - n/s$; by Lemma 3, $K \in \tilde{M}(\tilde{s}, \tilde{l})$. Repeating the proof of **Case 1**, except for replacing l by l_1, we can obtain the same result under the assumption $l - n/s > 1$. We omit the details here.

Certainly when $0 < \tilde{s}' < s$, which is the same as the above.

Now we can define the multilinear operator associated to the multiplier operator T. Let m_j be the positive integers($j = 1, \cdots, l$), $m_1 + \cdots + m_l = m$ and A_j be the functions on $R^n (j = 1, \cdots, l)$. Set

$$R_{m_j+1}(A_j; x, y) = A_j(x) - \sum_{|\alpha| \leq m_j} \frac{1}{\alpha!} D^{\alpha} A_j(y)(x-y)^{\alpha}.$$

By Lemma 1, $T(f)(x) = (K * f)(x)$ for $K(x) = \check{k}(x)$. The multilinear operator associated to T is defined by

$$T^A(f)(x) = \int_{R^n} \frac{\prod_{j=1}^{l} R_{m_j+1}(A_j; x, y)}{|x-y|^m} K(x-y) f(y) dy.$$

Note that when $m = 0$, T^A os just the multilinear commutators of T and A (see [20-21]). While when $m > 0$, it is non-trivial generalizations of the commutators. It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [4-8][10]). The purpose of this paper is to study the boundedness properties for the multilinear operator T^A. We shall prove the following theorems in Section 3.

Now we can state our theorems as following.

Theorem 1. *Let $0 < \beta < \min(1/l, a/l)$ and $D^\alpha A_j \in \dot\Lambda_\beta(R^n)$ for all α with $|\alpha| = m_j$ and $j = 1, \cdots, l$. Then T^A is bounded from $L^p(R^n)$ to $\dot F_p^{l\beta,\infty}(R^n)$ for any $1 < p < \infty$.*

Theorem 2. *Let $0 < \beta \le 1$ and $D^\alpha A_j \in \dot\Lambda_\beta(R^n)$ for all α with $|\alpha| = m_j$ and $j = 1, \cdots, l$. Then T^A is bounded from $L^p(R^n)$ to $L^q(R^n)$ for any $1 < p < n/l\beta$ and $1/p - 1/q = l\beta/n$.*

3. Proof of Theorem

To prove the theorem, we need the following lemmas.

Lemma 4(see [13]). *For $0 < \beta < 1$, $1 < p < \infty$, we have*

$$||f||_{\dot F_p^{\beta,\infty}} \approx \left|\left|\sup_Q \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - f_Q| dx\right|\right|_{L^p} \approx \left|\left|\sup_{\cdot\in Q} \inf_c \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - c| dx\right|\right|_{L^p}.$$

Lemma 5(see [13]). *For $0 < \beta < 1, 1 \le p \le \infty$, we have*

$$||b||_{\dot\Lambda_\beta} \approx \sup_Q \frac{1}{|Q|^{1+\beta/n}} \int_Q |b(x) - b_Q| dx \approx \sup_Q \frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|} \int_Q |b(x) - b_Q|^p dx\right)^{1/p}.$$

Lemma 6(see [1]). *Suppose that $0 \le \delta < n$, $1 \le r < p < n/\delta$ and $1/q = 1/p - \delta/n$. Then*

$$||M_{\delta,r}(f)||_{L^q} \le C||f||_{L^p}.$$

Lemma 7(see [5]). *Let A be a function on R^n and $D^\alpha A \in L^q(R^n)$ for $|\alpha| = m$ and some $q > n$. Then*

$$|R_m(A; x, y)| \le C|x - y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde Q(x, y)|} \int_{\tilde Q(x,y)} |D^\alpha A(z)|^q dz\right)^{1/q},$$

where $\tilde Q(x, y)$ is the cube centered at x and having side length $5\sqrt{n}|x - y|$.

Lemma 8 (see [16]). *For $b \in \dot\Lambda_\beta$, $0 < \beta < 1$, $0 \le \delta < n$ and $1 < r < \infty$, we have*

$$||(b - b_Q) f \chi_Q||_{L^r} \le C||b||_{\dot\Lambda_\beta} |Q|^{1/r+\beta/n-\delta/n} M_{\delta,r}(f).$$

Lemma 9.([3]) *Let T be the multiplier operator. Then, for every $f \in L^p(R^n), 1 < p < \infty$,*

$$||T(f)||_{L^p} \leq C||f||_{L^p}.$$

Proof of Theorem 1. We first prove the sharp estimate for T^A as following

$$\frac{1}{|Q|^{1+l\beta/n}} \int_Q |T^A(f)(x) - C_0| dx \leq C \prod_{j=1}^{l} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{BMO} \right) M_s(f)(\tilde{x}).$$

for $1 < s < p$ and some constant C_0. Without loss of generality, we may assume $l = 2$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{A}_j(x) = A_j(x) - \sum_{|\alpha|=m_j} \frac{1}{\alpha!}(D^\alpha A_j)_{\tilde{Q}} x^\alpha$, then $R_{m_j+1}(A_j; x, y) = R_{m_j+1}(\tilde{A}_j; x, y)$ and $D^\alpha \tilde{A}_j = D^\alpha A_j - (D^\alpha A_j)_{\tilde{Q}}$ for $|\alpha| = m_j$. We write, for $f_1 = f\chi_{\tilde{Q}}$ and $f_2 = f\chi_{R^n \setminus \tilde{Q}}$,

$$\begin{aligned}
T^A(f)(x) \ = \ & \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{A}_j; x, y)}{|x-y|^m} K(x-y) f_2(y) dy \\
& + \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j; x, y)}{|x-y|^m} K(x-y) f_1(y) dy \\
& - \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \frac{R_{m_2}(\tilde{A}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^m} D^{\alpha_1} \tilde{A}_1(y) K(x-y) f_1(y) dy \\
& - \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \frac{R_{m_1}(\tilde{A}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^m} D^{\alpha_2} \tilde{A}_2(y) K(x-y) f_1(y) dy \\
& + \sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2} \frac{1}{\alpha_1! \alpha_2!} \int_{R^n} \frac{(x-y)^{\alpha_1+\alpha_2} D^{\alpha_1} \tilde{A}_1(y) D^{\alpha_2} \tilde{A}_2(y)}{|x-y|^m} \\
& \cdot K(x-y) f_1(y) dy,
\end{aligned}$$

then

$$\begin{aligned}
& \frac{1}{|Q|^{1+2\beta/n}} \int_Q \left| T^A(f)(x) - T^{\tilde{A}}(f_2)(x_0) \right| dx \\
& \frac{1}{|Q|^{1+2\beta/n}} \int_Q \left| \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j; x, y)}{|x-y|^m} K(x-y) f_1(y) dy \right| dx \\
& + \frac{C}{|Q|^{1+2\beta/n}} \int_Q \left| \sum_{|\alpha_1|=m_1} \int_{R^n} \frac{R_{m_2}(\tilde{A}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^m} D^{\alpha_1} \tilde{A}_1(y) K(x-y) f_1(y) dy \right| dx \\
& + \frac{C}{|Q|^{1+2\beta/n}} \int_Q \left| \sum_{|\alpha_2|=m_2} \int_{R^n} \frac{R_{m_1}(\tilde{A}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^m} D^{\alpha_2} \tilde{A}_2(y) K(x-y) f_1(y) dy \right| dx
\end{aligned}$$

$$+\frac{C}{|Q|^{1+2\beta/n}}\int_Q\left|\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\int_{R^n}\frac{(x-y)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{A}_1(y)D^{\alpha_2}\tilde{A}_2(y)}{|x-y|^m}K(x-y)f_1(y)dy\right|dx$$
$$+\frac{1}{|Q|^{1+2\beta/n}}\int_Q\left|T^{\tilde{A}}(f_2)(x)-T^{\tilde{A}}(f_2)(x_0)\right|dx$$
$$:=I_1+I_2+I_3+I_4+I_5.$$

Now, let us estimate I_1, I_2, I_3, I_4 and I_5, respectively. First, by Lemma 7 and Lemma 5, we get, for $x\in Q$ and $y\in\tilde{Q}$,

$$\begin{aligned}|R_m(\tilde{A}_j;x,y)| &\le C|x-y|^m\sum_{|\alpha|=m}\sup_{x\in\tilde{Q}}|D^\alpha A_j(x)-(D^\alpha A_j)_{\tilde{Q}}|\\ &\le C|x-y|^m|Q|^{\beta/n}\sum_{|\alpha|=m}||D^\alpha A_j||_{\dot{\Lambda}_\beta}.\end{aligned}$$

Now, by the Hölder's inequality and L^s-boundedness of T, we obtain

$$\begin{aligned}I_1 &\le C\prod_{j=1}^2\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{\dot{\Lambda}_\beta}\right)\frac{1}{|Q|}\int_Q|T(f_1)(x)|dx\\ &\le C\prod_{j=1}^2\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{\dot{\Lambda}_\beta}\right)\left(\frac{1}{|Q|}\int_{R^n}|T(f_1)(x)|^sdx\right)^{1/s}\\ &\le C\prod_{j=1}^2\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{\dot{\Lambda}_\beta}\right)\left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}}|f(x)|^sdx\right)^{1/s}\\ &\le C\prod_{j=1}^2\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{\dot{\Lambda}_\beta}\right)M_s(f)(\tilde{x}).\end{aligned}$$

For I_2, by Lemma 8, we get

$$\begin{aligned}I_2 &\le C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2||_{\dot{\Lambda}_\beta}|Q|^{-\beta/n}\sum_{|\alpha_1|=m_1}\frac{1}{|Q|}\int_Q|T(D^{\alpha_1}\tilde{A}f_1)(x)|dx\\ &\le C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2||_{\dot{\Lambda}_\beta}\sum_{|\alpha_1|=m_1}|Q|^{-\beta/n-1/s}||T((D^{\alpha_1}A-(D^{\alpha_1}A)_{\tilde{Q}})f_1)||_{L^s}\\ &\le C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2||_{\dot{\Lambda}_\beta}|Q|^{-\beta/n-1/s}\sum_{|\alpha_1|=m}||(D^{\alpha_1}A-(D^{\alpha_1}A)_{\tilde{Q}})f_1||_{L^s}\\ &\le C\prod_{j=1}^2\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{\dot{\Lambda}_\beta}\right)M_s(f)(\tilde{x}).\end{aligned}$$

For I_3, similar to the proof of I_2, we get

$$I_3 \leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{\dot{\Lambda}_\beta} \right) M_s(f)(\tilde{x}).$$

Similarly, for I_4, set $s = pq_3$ with $p, q_1, q_2, q_3 > 1$ and $1/q_1 + 1/q_2 + 1/q_3 = 1$, we obtain

$$\begin{aligned}
I_4 &\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \frac{1}{|Q|^{1+2\beta/n}} \int_Q |T(D^{\alpha_1}\tilde{A}_1 D^{\alpha_2}\tilde{A}_2 f_1)(x)| dx \\
&\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} |Q|^{-2\beta/n} \left(\frac{1}{|Q|} \int_{R^n} |T(D^{\alpha_1}\tilde{A}_1 D^{\alpha_2}\tilde{A}_2 f_1)(x)|^p dx \right)^{1/p} \\
&\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} |Q|^{-2\beta/n-1/p} \left(\int_{R^n} |D^{\alpha_1}\tilde{A}_1(x) D^{\alpha_2}\tilde{A}_2(x) f_1(x)|^p dx \right)^{1/p} \\
&\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} |\tilde{Q}|^{-2\beta/n} \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} |f(x)|^{pq_3} dx \right)^{1/pq_3} \\
&\quad \times \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} |D^{\alpha_1}\tilde{A}_1(x)|^{pq_1} dx \right)^{1/pq_1} \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} |D^{\alpha_2}\tilde{A}_2(x)|^{pq_2} dx \right)^{1/pq_2} \\
&\leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{\dot{\Lambda}_\beta} \right) M_s(f)(\tilde{x}).
\end{aligned}$$

For I_5, we write

$$\begin{aligned}
& T^{\tilde{A}}(f_2)(x) - T^{\tilde{A}}(f_2)(x_0) \\
&= \int_{R^n} \left(\frac{K(x-y)}{|x-y|^m} - \frac{K(x_0-y)}{|x_0-y|^m} \right) \prod_{j=1}^{2} R_{m_j}(\tilde{A}_j; x, y) f_2(y) dy \\
&+ \int_{R^n} \left(R_{m_1}(\tilde{A}_1; x, y) - R_{m_1}(\tilde{A}_1; x_0, y) \right) \frac{R_{m_2}(\tilde{A}_2; x, y)}{|x_0-y|^m} K(x_0-y) f_2(y) dy \\
&+ \int_{R^n} \left(R_{m_2}(\tilde{A}_2; x, y) - R_{m_2}(\tilde{A}_2; x_0, y) \right) \frac{R_{m_1}(\tilde{A}_1; x_0, y)}{|x_0-y|^m} K(x_0-y) f_2(y) dy \\
&- \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} D^{\alpha_1}\tilde{A}_1(y) f_2(y) \\
&\times \left[\frac{R_{m_2}(\tilde{A}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^m} K(x-y) - \frac{R_{m_2}(\tilde{A}_2; x_0, y)(x_0-y)^{\alpha_1}}{|x_0-y|^m} K(x_0-y) \right] dy
\end{aligned}$$

$$
\begin{aligned}
& -\sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\int_{R^n}D^{\alpha_2}\tilde{A}_2(y)f_2(y)\\
& \times\left[\frac{R_{m_1}(\tilde{A}_1;x,y)(x-y)^{\alpha_2}}{|x-y|^m}K(x-y)-\frac{R_{m_1}(\tilde{A}_1;x_0,y)(x_0-y)^{\alpha_2}}{|x_0-y|^m}K(x_0-y)\right]dy\\
& +\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\frac{1}{\alpha_1!\alpha_2!}\int_{R^n}\left[\frac{(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m}K(x-y)-\frac{(x_0-y)^{\alpha_1+\alpha_2}}{|x_0-y|^m}K(x_0-y)\right]\\
& \times D^{\alpha_1}\tilde{A}_1(y)D^{\alpha_2}\tilde{A}_2(y)f_2(y)dy\\
= & \ I_5^{(1)}+I_5^{(2)}+I_5^{(3)}+I_5^{(4)}+I_5^{(5)}+I_5^{(6)}.
\end{aligned}
$$

By Lemma 8 and the following inequality, for $b\in\dot{\Lambda}_\beta(R^n)$,

$$|b(x)-b_Q|\le\frac{1}{|Q|}\int_Q||b||_{\dot{\Lambda}_\beta}|x-y|^\beta dy\le||b||_{\dot{\Lambda}_\beta}(|x-x_0|+d)^\beta,$$

we get

$$|R_{m_j}(\tilde{A}_j;x,y)|\le\sum_{|\alpha|=m_j}||D^\alpha A_j||_{\dot{\Lambda}_\beta}(|x-y|+d)^{m_j+\beta}.$$

Note that $|x-y|\sim|x_0-y|$ for $x\in Q$ and $y\in R^n\setminus\tilde{Q}$, we obtain, by Lemma 2 and 3,

$$
\begin{aligned}
|I_5^{(1)}| \le\ & \sum_{k=0}^\infty\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}|K(x-y)-K(x_0-y)|\frac{1}{|x-y|^m}\prod_{j=1}^2|R_{m_j}(\tilde{A}_j;x,y)||f(y)|dy\\
& +\sum_{k=0}^\infty\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\left|\frac{1}{|x-y|^m}-\frac{1}{|x_0-y|^m}\right||K(x_0-y)|\prod_{j=1}^2|R_{m_j}(\tilde{A}_j;x,y)||f(y)|dy\\
\le\ & C\prod_{j=1}^2\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{\dot{\Lambda}_\beta}\right)\sum_{k=0}^\infty|2^{k+1}\tilde{Q}|^{2\beta/n}\left(\int_{2^{k+1}\tilde{Q}}|f(y)|^s dy\right)^{1/s}\\
& \times\left(\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}|K(x-y)-K(x_0-y)|^{s'}dy\right)^{1/s'}\\
& +C\prod_{j=1}^2\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{\dot{\Lambda}_\beta}\right)\sum_{k=0}^\infty|2^{k+1}\tilde{Q}|^{2\beta/n}2^{-k}\left(\int_{2^{k+1}\tilde{Q}}|f(y)|^s dy\right)^{1/s}\\
& \times\left(\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}|K(x_0-y)|^{s'}dy\right)^{1/s'}\\
\le\ & C\prod_{j=1}^2\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{\dot{\Lambda}_\beta}\right)|Q|^{2\beta/n}\sum_{k=1}^\infty(2^{k(2\beta-a)}+2^{k(2\beta-1)})
\end{aligned}
$$

$$\cdot \left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f(y)|^s dy\right)^{1/s}$$
$$\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{\dot{\Lambda}_\beta}\right)|Q|^{2\beta/n}M_s(f)(\tilde{x}).$$

For $I_5^{(2)}$, by the formula (see [5]):

$$R_{m_j}(\tilde{A}_j;x,y)-R_{m_j}(\tilde{A}_j;x_0,y)=\sum_{|\eta|<m_j}\frac{1}{\eta!}R_{m_j-|\eta|}(D^\eta\tilde{A}_j;x,x_0)(x-y)^\eta$$

and Lemma 4, we get

$$\begin{aligned}
|I_5^{(2)}| &\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^{\alpha}A_j||_{\dot{\Lambda}_\beta}\right)\sum_{k=0}^{\infty}|2^{k+1}\tilde{Q}|^{2\beta/n}\\
&\quad\cdot\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\frac{|x-x_0|}{|x_0-y|}|K(x_0-y)||f(y)|dy\\
&\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^{\alpha}A_j||_{\dot{\Lambda}_\beta}\right)\sum_{k=0}^{\infty}|2^{k+1}\tilde{Q}|^{2\beta/n}2^{-k}\left(\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}|K(x_0-y)|^{s'}dy\right)^{1/s'}\\
&\quad\times\left(\int_{2^{k+1}\tilde{Q}}|f(y)|^s dy\right)^{1/s}\\
&\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^{\alpha}b_j||_{\dot{\Lambda}_\beta}\right)\sum_{k=0}^{\infty}2^{k(2\beta-1)}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f(y)|^s dy\right)^{1/s}\\
&\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^{\alpha}A_j||_{\dot{\Lambda}_\beta}\right)|Q|^{2\beta/n}M_s(f)(\tilde{x}).
\end{aligned}$$

Similarly,

$$|I_5^{(3)}|\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^{\alpha}A_j||_{\dot{\Lambda}_\beta}\right)|Q|^{2\beta/n}M_s(f)(\tilde{x}).$$

For $I_5^{(4)}$, similar to the estimates of $I_5^{(1)}$ and $I_5^{(2)}$, we obtain,

$$\begin{aligned}
|I_5^{(4)}| &\leq C\sum_{|\alpha_1|=m_1}\int_{R^n\setminus\tilde{Q}}\left|\frac{(x-y)^{\alpha_1}K(x-y)}{|x-y|^m}-\frac{(x_0-y)^{\alpha_1}K(x_0-y)}{|x_0-y|^m}\right|\\
&\quad\times|R_{m_2}(\tilde{A}_2;x,y)||D^{\alpha_1}\tilde{A}_1(y)||f(y)|dy
\end{aligned}$$

$$
\begin{aligned}
&+C\sum_{|\alpha_1|=m_1}\int_{R^n\setminus\tilde{Q}}|R_{m_2}(\tilde{A}_2;x,y)-R_{m_2}(\tilde{A}_2;x_0,y)|\\
&\times\frac{|(x_0-y)^{\alpha_1}K(x_0-y)|}{|x_0-y|^m}|D^{\alpha_1}\tilde{A}_1(y)||f(y)|dy\\
\leq\ & C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{\dot{\Lambda}_\beta}\right)|Q|^{2\beta/n}\sum_{k=0}^{\infty}(2^{k(2\beta-a)}+2^{k(2\beta-1)})\\
&\times\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f(y)|^s dy\right)^{1/s}\\
\leq\ & C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{\dot{\Lambda}_\beta}\right)|Q|^{2\beta/n}M_s(f)(\tilde{x}).
\end{aligned}
$$

Similarly,

$$
|I_5^{(5)}|\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{\dot{\Lambda}_\beta}\right)|Q|^{2\beta/n}M_s(f)(\tilde{x}).
$$

For $I_5^{(6)}$, we get

$$
\begin{aligned}
|I_5^{(6)}|\ \leq\ & C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\int_{R^n\setminus\tilde{Q}}\left|\frac{(x-y)^{\alpha_1+\alpha_2}K(x-y)}{|x-y|^m}-\frac{(x_0-y)^{\alpha_1+\alpha_2}K(x_0-y)}{|x_0-y|^m}\right|\\
&\times|D^{\alpha_1}\tilde{A}_1(y)||D^{\alpha_2}\tilde{A}_2(y)||f(y)|dy\\
\leq\ & C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j} A_j||_{\dot{\Lambda}_\beta}\right)|Q|^{2\beta/n}\sum_{k=0}^{\infty}(2^{k(2\beta-a)}+2^{k(2\beta-1)})\\
&\cdot\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f(y)|^s dy\right)^{1/s}\\
\leq\ & C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j} A_j||_{\dot{\Lambda}_\beta}\right)|Q|^{2\beta/n}M_s(f)(x).
\end{aligned}
$$

Thus

$$
|T^{\tilde{A}}(f_2)(x)-T^{\tilde{A}}(f_2)(x_0)|\leq C\sum_{|\alpha|=m}||D^\alpha A||_{\dot{\Lambda}_\beta}|Q|^{2\beta/n}M_s(f)(\tilde{x})
$$

and

$$
I_5\leq C\sum_{|\alpha|=m}||D^\alpha A||_{\dot{\Lambda}_\beta}M_s(f)(\tilde{x}).
$$

We now put these estimates together, and taking the supremum over all Q such that $\tilde{x} \in Q$, and using Lemma 4 and 6, we obtain

$$\begin{aligned} ||T^A(f)||_{\dot{F}_p^{2\beta,\infty}} &\leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{\dot{\Lambda}_\beta} \right) ||M_s(f)||_{L^p} \\ &\leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{\dot{\Lambda}_\beta} \right) ||f||_{L^p}. \end{aligned}$$

This completes the proof of the theorem.

Proof of Theorem 2. By using the same argument as in proof of Theorem 1, we obtain, for $1 < s < p$,

$$\frac{1}{|Q|} \int_Q |T^A(f)(x) - T^{\tilde{A}}(f_2)(x_0)| dx \leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{\dot{\Lambda}_\beta} \right) M_{2\beta,s}(f),$$

thus, we get the sharp estimate of T^A as following

$$(T^A(f))^{\#} \leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{\dot{\Lambda}_\beta} \right) M_{2\beta,s}(f).$$

Now, using Lemma 6, we get

$$\begin{aligned} ||T^A(f)||_{L^q} &\leq C||(T^A(f))^{\#}||_{L^q} \leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{\dot{\Lambda}_\beta} \right) ||M_{2\beta,r}(f)||_{L^q} \\ &\leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{\dot{\Lambda}_\beta} \right) ||f||_{L^p}. \end{aligned}$$

This completes the proof of the theorem.

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