A SUBCLASS OF ANALYTIC FUNCTIONS DEFINED BY RUSCHEWEYH DERIVATIVE

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Abstract. By means of the Ruscheweyh derivative we define a new class $\mathcal{BR}_{p,n}(m,\mu,\alpha)$ involving functions $f \in A(p,n)$. Parallel results, for some related classes including the class of starlike and convex functions respectively, are also obtained.

2000 *Mathematics Subject Classification*: 30C45

1. Introduction and definitions

Denote by U the unit disc of the complex plane, $U = \{z \in \mathbb{C} : \ |z| < 1\}$ and $\mathcal{H}(U)$ the space of holomorphic functions in U.

Let

$$A(p,n) = \{f \in \mathcal{H}(U) : \ f(z) = z^p + \sum_{j=p+n}^{\infty} a_j z^j, \quad z \in U\}, \tag{1}$$

with $A(1,n) = A_n$ and

$$\mathcal{H}[a,n] = \{f \in \mathcal{H}(U) : \ f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots, \ z \in U\},$$

where $p, n \in \mathbb{N}$, $a \in \mathbb{C}$.

Let $\mathcal{S}$ denote the subclass of functions that are univalent in U.

By $\mathcal{S}_n^*(p,\alpha)$ we denote a subclass of $A(p,n)$ consisting of p-valently starlike functions of order α, $0 \le \alpha < p$ which satisfies

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad z \in U. \tag{2}$$

Further, a function f belonging to $\mathcal{S}$ is said to be p-valently convex of order α in U, if and only if

$$\mathrm{Re}\left(\frac{zf''(z)}{f'(z)}+1\right)>\alpha,\quad z\in U \tag{3}$$

for some α, $(0\leq\alpha<p)$. We denote by $\mathcal{K}_n(p,\alpha)$ the class of functions in $\mathcal{S}$ which are p-valently convex of order α in U and denote by $\mathcal{R}_n(p,\alpha)$ the class of functions in $A(p,n)$ which satisfy

$$\mathrm{Re}f'(z)>\alpha,\quad z\in U. \tag{4}$$

It is well known that $\mathcal{K}_n(p,\alpha)\subset\mathcal{S}_n^*(p,\alpha)\subset\mathcal{S}$.

If f and g are analytic functions in U, we say that f is subordinate to g, written $f\prec g$, if there is a function w analytic in U, with $w(0)=0$, $|w(z)|<1$, for all $z\in U$ such that $f(z)=g(w(z))$ for all $z\in U$. If g is univalent, then $f\prec g$ if and only if $f(0)=g(0)$ and $f(U)\subseteq g(U)$.

In [5] Ruscheweyh has defined the operator $D^m: A(p,n)\to A(p,n)$, $n\in\mathbb{N}$, $m\in\mathbb{N}\cup\{0\}$,

$$\begin{aligned}D^0f(z)&=f(z)\\ D^1f(z)&=zf'(z)\\ (m+1)D^{m+1}f(z)&=z\left[D^mf(z)\right]'+mD^mf(z),\quad z\in U.\end{aligned}$$

We note that if $f\in A(p,n)$, then

$$D^mf(z)=z^p+\sum_{j=n+p}^{\infty}C_{m+j-1}^{m}a_jz^j,\quad z\in U.$$

To prove our main theorem we shall need the following lemma.

Lemma 1.[4]. *Let u be analytic in U with $u(0)=1$ and suppose that*

$$\mathrm{Re}\left(1+\frac{zu'(z)}{u(z)}\right)>\frac{3\alpha-1}{2\alpha},\quad z\in U. \tag{5}$$

Then $\mathrm{Re}u(z)>\alpha$ *for* $z\in U$ *and* $1/2\leq\alpha<1$.

2. Main results

Definition 1. *We say that a function $f\in A(p,n)$ is in the class $\mathcal{BR}_{p,n}(m,\mu,\alpha)$, $p,n\in\mathbb{N}$, $m\in\mathbb{N}\cup\{0\}$, $\mu\geq0$, $\alpha\in[0,1)$ if*

$$\left|\frac{D^{m+1}f(z)}{z^p}\left(\frac{z^p}{D^mf(z)}\right)^{\mu}-p\right|<p-\alpha,\qquad z\in U. \tag{6}$$

Remark. The family $\mathcal{BR}_{p,n}(m,\mu,\alpha)$ is a new comprehensive class of analytic functions which includes various new classes of analytic univalent functions as well as some very well-known ones. For example, $\mathcal{BR}_{1,n}(0,1,\alpha)\equiv\mathcal{S}_n^*(1,\alpha)$, $\mathcal{BR}_{1,n}(1,1,\alpha)\equiv\mathcal{K}_n(1,\alpha)$ and $\mathcal{BR}_{1,n}(0,0,\alpha)\equiv\mathcal{R}_n(1,\alpha)$. Another interesting subclass is the special case $\mathcal{BR}_{1,1}(0,2,\alpha)\equiv\mathcal{B}(\alpha)$ which has been introduced by Frasin and Darus [3] and also the class $\mathcal{BR}_{1,1}(0,\mu,\alpha)\equiv\mathcal{B}(\mu,\alpha)$ which has been introduced by Frasin and Jahangiri [4].

In this note we provide a sufficient condition for functions to be in the class $\mathcal{BR}_{p,n}(m,\mu,\alpha)$. Consequently, as a special case, we show that convex functions of order 1/2 are also members of the above defined family.

Theorem 2. *For the function* $f\in A(p,n)$, $p,n\in\mathbb{N}$, $m\in\mathbb{N}\cup\{0\}$, $\mu\geq 0$, $1/2\leq\alpha<1$ *if*

$$(m+2)\frac{D^{m+2}f(z)}{D^{m+1}f(z)}-\mu(m+1)\frac{D^{m+1}f(z)}{D^{m}f(z)}+\mu(m+p)-(m+p)\prec 1+\beta z,\quad z\in U, \tag{7}$$

where $\beta=\frac{3\alpha-1}{2\alpha}$, *then* $f\in\mathcal{BR}_{p,n}(m,\mu,\alpha)$.

*Proof.*If we consider

$$u(z)=\frac{D^{m+1}f(z)}{z^p}\left(\frac{z^p}{D^m f(z)}\right)^{\mu}, \tag{8}$$

then $u(z)$ is analytic in U with $u(0)=1$. A simple differentiation yields

$$\frac{zu'(z)}{u(z)}=(m+2)\frac{D^{m+2}f(z)}{D^{m+1}f(z)}-\mu(m+1)\frac{D^{m+1}f(z)}{D^{m}f(z)}+\mu(m+p)-(m+p+1). \tag{9}$$

Using (7) we get $\mathrm{Re}\left(1+\frac{zu'(z)}{u(z)}\right)>\frac{3\alpha-1}{2\alpha}$.

Thus, from Lemma 1 we deduce that $\mathrm{Re}\left\{\frac{D^{m+1}f(z)}{z^p}\left(\frac{z^p}{D^m f(z)}\right)^{\mu}\right\}>\alpha$.

Therefore, $f\in\mathcal{BR}_{p,n}(m,\mu,\alpha)$, by Definition 1.

As a consequence of the above theorem we have the following interesting corollaries.

Corollary 3.*If* $f\in A_n$ *and*

$$\mathrm{Re}\left\{\frac{6zf'(z)+6z^2f''(z)+z^3f'''(z)}{2zf'(z)+z^2f''(z)}-\frac{zf''(z)}{f'(z)}\right\}>\frac{3}{2},\quad z\in U, \tag{10}$$

then

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\frac{3}{2},\quad z\in U. \tag{11}$$

That is, f is convex of order $\frac{3}{2}$.

Corollary 4. *If $f \in A_n$ and*

$$\mathrm{Re}\left\{\frac{4zf'(z)+5z^2f''(z)+z^3f'''(z)}{2zf'(z)+z^2f''(z)}\right\} > \frac{1}{2}, \quad z \in U, \tag{12}$$

then

$$\mathrm{Re}\left\{f'(z)+\frac{1}{2}zf''(z)\right\} > \frac{1}{2}, \quad z \in U. \tag{13}$$

Corollary 5. *If $f \in A_n$ and*

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \frac{1}{2}, \quad z \in U, \tag{14}$$

then

$$\mathrm{Re} f'(z) > \frac{1}{2}, \quad z \in U. \tag{15}$$

In another words, if the function f is convex of order $\frac{1}{2}$, then $f \in \mathcal{BR}_{1,n}(0,0,\frac{1}{2}) \equiv \mathcal{R}_n\left(1,\frac{1}{2}\right)$.

Corollary 6. *If $f \in A_n$ and*

$$\mathrm{Re}\left\{\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right\} > -\frac{3}{2}, \quad z \in U, \tag{16}$$

then f is starlike of order $\frac{1}{2}$, hence $f \in \mathcal{BR}_{1,n}(0,1,\frac{1}{2})$.

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