# UNIVALENCE CONDITION FOR A NEW GENERALIZATION OF THE FAMILY OF INTEGRAL OPERATORS

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Abstract. In [3], Breaz et al. gave an univalence condition of the integral operator $G_{n,\alpha}$ introduced in [2]. The purpose of this paper is to generalize the definition of $G_{n,\alpha}$ by means of the Al-Oboudi differential operator and investigate univalence condition of this generalized integral operator. Our results generalize the results of [3].

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## 1.Introduction

Let $\mathcal{A}$ denote the class of all functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, and

$$\mathcal{S} = \{f \in \mathcal{A} : f \text{ is univalent in } \mathbb{U}\}.$$

For $f \in \mathcal{A}$, Al-Oboudi [1] introduced the following operator:

$$D^0 f(z) = f(z), \tag{2}$$

$$D^1 f(z) = (1-\delta)f(z) + \delta z f'(z) = D_\delta f(z), \quad \delta \geq 0 \tag{3}$$

$$D^n f(z) = D_\delta(D^{n-1} f(z)), \quad (n \in \mathbb{N} := \{1, 2, 3, \ldots\}). \tag{4}$$

If $f$ is given by (1), then from (3) and (4) we see that

$$D^n f(z) = z + \sum_{k=2}^{\infty} [1 + (k-1)\delta]^n a_k z^k, \quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}), \tag{5}$$

with $D^n f(0) = 0$.

**Remark 1.** When $\delta = 1$, we get Sălăgean's differential operator [8].

The following results will be required in our investigation.

**Schwarz Lemma** [4]. *Let the analytic function $f$ be regular in the open unit disk $\mathbb{U}$ and let $f(0) = 0$. If*

$$|f(z)| \leq 1 \quad (z \in \mathbb{U}),$$

*then*

$$|f(z)| \leq |z| \quad (z \in \mathbb{U}),$$

*where the equality holds true only if*

$$f(z) = Kz \quad (z \in \mathbb{U}) \quad \text{and} \quad |K| = 1.$$

**Theorem A** [6]**.** *Let*

$$\alpha \in \mathbb{C} \quad (\operatorname{Re} \alpha > 0)$$

*and*

$$c \in \mathbb{C} \quad (|c| \leq 1;\ c \neq -1).$$

*Suppose also that the function $f(z)$ given by* (1) *is analytic in $\mathbb{U}$. If*

$$\left| c\,|z|^{2\alpha} + \left(1 - |z|^{2\alpha}\right) \frac{zf''(z)}{\alpha f'(z)} \right| \leq 1 \quad (z \in \mathbb{U}),$$

*then the function $F_\alpha(z)$ defined by*

$$F_\alpha(z) := \left\{ \alpha \int_0^z t^{\alpha-1} f'(t)dt \right\}^{\frac{1}{\alpha}} = z + \cdots \tag{6}$$

*is analytic and univalent in $\mathbb{U}$.*

**Theoem B** [5]**.** *Let $f \in \mathcal{A}$ satisfy the following inequality:*

$$\left| \frac{z^2 f'(z)}{[f(z)]^2} - 1 \right| \leq 1 \quad (z \in \mathbb{U}). \tag{7}$$

*Then $f$ is univalent in $\mathbb{U}$.*

**Theorem C** [7]. *Let the function $g \in \mathcal{A}$ satisfies the inequality* (7). *Also let*

$$\alpha \in \mathbb{R} \quad \left(\alpha \in \left[1, \frac{3}{2}\right]\right) \quad \text{and} \quad c \in \mathbb{C}.$$

*If*

$$|c| \leq \frac{3-2\alpha}{\alpha} \quad (c \neq -1)$$

*and*

$$|g(z)| \leq 1 \quad (z \in \mathbb{U}),$$

*then the function $G_\alpha(z)$ defined by*

$$G_\alpha(z) := \left\{\alpha \int_0^z (g(t))^{\alpha-1}\, dt\right\}^{\frac{1}{\alpha}} \tag{8}$$

*is in the univalent function class $\mathcal{S}$.*

In [2], Breaz and Breaz considered the integral operator

$$G_{n,\alpha}(z) := \left\{[n(\alpha-1)+1] \int_0^z (g_1(t))^{\alpha-1} \cdots (g_n(t))^{\alpha-1}\, dt\right\}^{\frac{1}{n(\alpha-1)+1}} \quad (g_1, \ldots, g_n \in \mathcal{A}) \tag{9}$$

and proved that the function $G_{n,\alpha}(z)$ is univalent in $\mathbb{U}$.

**Remark 2.** We note that for $n = 1$, we obtain the integral operator in (8).

In [3], Breaz et al. proved the following theorem.

**Theorem D** [3]. *Let $M \geq 1$ and suppose that each of the functions $g_j \in \mathcal{A}$ $(j \in \{1, \ldots, n\})$ satisfies the inequality* (7). *Also let*

$$\alpha \in \mathbb{R} \quad \left(\alpha \in \left[1, \frac{(2M+1)n}{(2M+1)n-1}\right]\right) \quad \text{and} \quad c \in \mathbb{C}.$$

*If*

$$|c| \leq 1 + \left(\frac{1-\alpha}{\alpha}\right)(2M+1)n$$

*and*

$$|g_j(z)| \leq M \quad (z \in \mathbb{U};\ j \in \{1, \ldots, n\}),$$

*then the function* $G_{n,\alpha}(z)$ *defined by* (9) *is in the univalent function class* $\mathcal{S}$.

Now we introduce a new general integral operator by means of the Al-Oboudi differential operator.

**Definition 1.** Let $n \in \mathbb{N}$, $m \in \mathbb{N}_0$ and $\alpha \in \mathbb{C}$. We define the integral operator $G_{n,m,\alpha} : \mathcal{A}^n \to \mathcal{A}$ by

$$G_{n,m,\alpha}(z) := \left\{ [n(\alpha - 1) + 1] \int_0^z \prod_{j=1}^{n} \left(D^m g_j(t)\right)^{\alpha-1} dt \right\}^{\frac{1}{n(\alpha-1)+1}} \qquad (z \in \mathbb{U}), \tag{10}$$

where $g_1, \ldots, g_n \in \mathcal{A}$ and $D^m$ is the Al-Oboudi differential operator.

**Remark 3.** In the special case $n = 1$, we obtain the integral operator

$$G_{m,\alpha}(z) := \left\{ \alpha \int_0^z \left(D^m g(t)\right)^{\alpha-1} dt \right\}^{\frac{1}{\alpha}} \qquad (z \in \mathbb{U}). \tag{11}$$

**Remark 4.** If we set $m = 0$ in (10) and (11), then we obtain the integral operators defined in (9) and (8), respectively.

## 2.Main results

**Theorem 1.** *Let* $M \geq 1$ *and suppose that each of the functions* $g_j \in \mathcal{A}\ (j \in \{1, \ldots, n\})$ *satisfies the inequality*

$$\left| \frac{z^2 \left(D^m g_j(z)\right)'}{\left(D^m g_j(z)\right)^2} - 1 \right| \leq 1 \quad (z \in \mathbb{U};\ m \in \mathbb{N}_0). \tag{12}$$

*Also let*

$$\alpha \in \mathbb{R} \quad \left( \alpha \in \left[ 1, \frac{(2M+1)n}{(2M+1)n - 1} \right] \right) \quad \text{and} \quad c \in \mathbb{C}.$$

*If*

$$|c| \leq 1 + \left( \frac{1 - \alpha}{n(\alpha - 1) + 1} \right) (2M + 1)n$$

*and*

$$|D^m g_j(z)| \leq M \quad (z \in \mathbb{U};\ j \in \{1, \ldots, n\}),$$

*then the integral operator* $G_{n,m,\alpha}(z)$ *defined by* (10) *is in the univalent function class* $\mathcal{S}$.

*Proof.* Since $g_j \in \mathcal{A}$ $(j \in \{1, \ldots, n\})$, by (5), we have

$$\frac{D^m g_j(z)}{z} = 1 + \sum_{k=2}^{\infty} [1 + (k-1)\delta]^m \, a_{k,j} z^{k-1} \quad (m \in \mathbb{N}_0)$$

and

$$\frac{D^m g_j(z)}{z} \neq 0$$

for all $z \in \mathbb{U}$.

Also we note that

$$G_{n,m,\alpha}(z) = \left\{ [n(\alpha-1)+1] \int_0^z t^{n(\alpha-1)} \prod_{j=1}^{n} \left( \frac{D^m g_j(t)}{t} \right)^{\alpha-1} dt \right\}^{\frac{1}{n(\alpha-1)+1}}.$$

Define a function

$$f(z) = \int_0^z \prod_{j=1}^{n} \left( \frac{D^m g_j(t)}{t} \right)^{\alpha-1} dt.$$

Then we obtain

$$f'(z) = \prod_{j=1}^{n} \left( \frac{D^m g_j(z)}{z} \right)^{\alpha-1}. \tag{13}$$

It is clear that $f(0) = f'(0) - 1 = 0$.

The equality (13) implies that

$$\ln f'(z) = (\alpha - 1) \sum_{j=1}^{n} \ln \frac{D^m g_j(z)}{z}$$

or equivalently

$$\ln f'(z) = (\alpha - 1) \sum_{j=1}^{n} \left( \ln D^m g_j(z) - \ln z \right).$$

By differentiating above equality, we get

$$\frac{f''(z)}{f'(z)} = (\alpha - 1)\sum_{j=1}^{n}\left(\frac{(D^m g_j(z))'}{D^m g_j(z)} - \frac{1}{z}\right).$$

Hence we obtain

$$\frac{zf''(z)}{f'(z)} = (\alpha - 1)\sum_{j=1}^{n}\left(\frac{z\,(D^m g_j(z))'}{D^m g_j(z)} - 1\right),$$

which readily shows that

$$\begin{aligned}
&\left| c\,|z|^{2[n(\alpha-1)+1]} + \left(1 - |z|^{2[n(\alpha-1)+1]}\right)\frac{zf''(z)}{[n(\alpha-1)+1]\,f'(z)} \right| \\
= &\left| c\,|z|^{2[n(\alpha-1)+1]} + \left(1 - |z|^{2[n(\alpha-1)+1]}\right)\left(\frac{\alpha-1}{n(\alpha-1)+1}\right)\sum_{j=1}^{n}\left(\frac{z\,(D^m g_j(z))'}{D^m g_j(z)} - 1\right) \right| \\
\leq &\; |c| + \left(\frac{\alpha-1}{n(\alpha-1)+1}\right)\sum_{j=1}^{n}\left(\left|\frac{z^2\,(D^m g_j(z))'}{(D^m g_j(z))^2}\right|\left|\frac{D^m g_j(z)}{z}\right| + 1\right).
\end{aligned}$$

From the hypothesis, we have $|g_j(z)| \leq M$ $(j \in \{1, \ldots, n\}\ ;\ z \in \mathbb{U})$, then by the Schwarz lemma, we obtain that

$$|g_j(z)| \leq M\,|z| \quad (j \in \{1, \ldots, n\}\ ;\ z \in \mathbb{U}).$$

Then we find

$$\begin{aligned}
&\left| c\,|z|^{2[n(\alpha-1)+1]} + \left(1 - |z|^{2[n(\alpha-1)+1]}\right)\frac{zf''(z)}{[n(\alpha-1)+1]\,f'(z)} \right| \\
\leq &\; |c| + \left(\frac{\alpha-1}{n(\alpha-1)+1}\right)\sum_{j=1}^{n}\left(\left|\frac{z^2\,(D^m g_j(z))'}{(D^m g_j(z))^2}\right| M + 1\right) \\
\leq &\; |c| + \left(\frac{\alpha-1}{n(\alpha-1)+1}\right)\sum_{i=1}^{n}\left(\left|\frac{z^2\,(D^m g_j(z))'}{(D^m g_j(z))^2} - 1\right| M + M + 1\right) \\
\leq &\; |c| + \left(\frac{\alpha-1}{n(\alpha-1)+1}\right)(2M+1)n \leq 1
\end{aligned}$$

since $|c| \leq 1 + \left(\frac{1-\alpha}{n(\alpha-1)+1}\right)(2M+1)n$. Applying Theorem A, we obtain that $G_{n,m,\alpha}$ is in the univalent function class $\mathcal{S}$.

**Remark 5.** If we set $m = 0$ in Theorem 1, then we have Theorem D.

**Corollary 2.** *Let each of the functions* $g_j \in \mathcal{A}$ $(j \in \{1, \ldots, n\})$ *satisfies the inequality* (12). *Suppose also that*

$$\alpha \in \mathbb{R} \quad \left(\alpha \in \left[1, \frac{3n}{3n-1}\right]\right) \quad \textit{and} \quad c \in \mathbb{C}.$$

*If*

$$|c| \leq 1 + 3\left(\frac{1-\alpha}{n(\alpha-1)+1}\right)n$$

*and*

$$|D^m g_j(z)| \leq 1 \quad (z \in \mathbb{U};\ j \in \{1, \ldots, n\}),$$

*then the integral operator* $G_{n,m,\alpha}(z)$ *defined by* (10) *is in the univalent function class* $\mathcal{S}$.

*Proof.* In Theorem 1, we consider $M = 1$.

**Remark 6.** If we set $m = 0$ in Corollary 2, then we have Corollary 1 in [3].

**Corollary 3.** *Let* $M \geq 1$ *and suppose that the functions* $g \in \mathcal{A}$ *satisfies the inequality* (12). *Also let*

$$\alpha \in \mathbb{R} \quad \left(\alpha \in \left[1, \frac{2M+1}{2M}\right]\right) \quad \textit{and} \quad c \in \mathbb{C}.$$

*If*

$$|c| \leq 1 + \left(\frac{1-\alpha}{\alpha}\right)(2M+1)$$

*and*

$$|D^m g(z)| \leq M \quad (z \in \mathbb{U}),$$

*then the integral operator* $G_{m,\alpha}(z)$ *defined by* (11) *is in the univalent function class* $\mathcal{S}$.

*Proof.* In Theorem 1, we consider $n = 1$.

**Remark 7.** If we set $m = 0$ in Corollary 3, then we have Corollary 2 in [3].

**Remark 8.** If we set $M = n = 1$ and $m = 0$ in Theorem 1, then we obtain Theorem C.

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