DIFFERENTIAL SUBORDINATIONS OBTAINED USING DZIOK-SRIVASTAVA LINEAR OPERATOR

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Abstract. By using the properties of Dziok-Srivastava linear operator we obtain differential subordinations using functions from class A.

1. Introduction and preliminaries

Let U denote the unit disc of the complex plane:

$$U = \{z \in \mathbb{C} : \ |z| < 1\}$$

and

$$\overline{U} = \{z \in \mathbb{C} : \ |z| \leq 1\}.$$

Let $\mathcal{H}(U)$ denote the space of holomorphic functions in U and let

$$\mathcal{A}_n = \{f \in \mathcal{H}(U), \ f(z) = z + a_{n+1}z^{n+1} + \ldots, \ z \in U\}$$

with $A_1 = A$.

Let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}(U), \ f(z) = z + a_n z^n + a_{n+1}z^{n+1} + \ldots, \ z \in U\},$$

$$S = \{f \in A; \ f \text{ is univalent in } U\}.$$

Let

$$K = \left\{ f \in \mathcal{A}, \ \mathrm{Re}\, \frac{zf''(z)}{f'(z)} + 1 > 0, \ z \in U \right\},$$

denote the class of convex functions in U and

$$S^* = \left\{ f \in A; \ \mathrm{Re}\, \frac{zf'(z)}{f(z)} > 0, \ z \in U \right\}$$

denote the class of starlike functions in U.

If f and g are analytic functions in U, then we say that f is subordinate to g, written $f \prec g$, if there is a function w analytic in U, with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$ such that $f(z) = g[w(z)]$ for $z \in U$.

If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

The method of differential subordinations (also known as the admissible functions method) was introduced by P.T. Mocanu and S.S. Miller in 1978 [1] and 1981 [2] it was developed in [3].

Definition 1. *Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and let h be univalent in U. If p is analytic in U and satisfies the (second-order) differential subordination*

(i) $\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z)$, $z \in U$

then p is called a solution of differential subordination. The univalent function q is called a dominant of the solution of the differential subordination, or more simply a dominant, if $p \prec q$ for all p satisfying (ii). A dominant $\widetilde{q}$ that satisfies $\widetilde{q} \prec q$ for all dominant q of (ii) is said to be the best dominant of (i). (Note that the best dominant is unique up to a rotation of U).

In [4] the authors introduce the dual problem of the differential subordination which they call differential superordination.

Definition 2. [4]*Let $f, F \in \mathcal{H}(U)$ and let F be univalent in U. The function F is said to be superordinare to f, or f is subordinate to F, written $f \prec F$, if $f(0) = F(0)$ and $f(U) \subset F(U)$.*

Definition 3. [4]*Let $\varphi : \mathbb{C}^3 \times U \to \mathbb{C}$ and let h be analytic in U. If p and $\varphi(p(z), zp'(z), z^2p''(z); z)$ are univalent in U and satisfy the (second-order) differential superordination*

(ii) $h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z)$

then p is called a solution of the differential superordination. An analytic function q is called a subordinant of the solution of the differential superordination, or more simply a subordinant if $q \prec p$ for all p satisfying (i). A univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (ii) is said to be the best subordinant. (Note that the best subordinant is unique up to a rotation of U).

Definition 4. [3, Definition 2.2b, p. 21] *We denote by Q the set of functions f that are analytic and injective on $\overline{U} \setminus E(f)$, where*

$$E(f) = \left\{ \zeta \in \partial U;\ \lim_{z \to \zeta} f(z) = \infty \right\}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$.

The subclass of Q for which $f(0)=a$ is denoted by $Q(a)$.

In order to prove the new results we shall use the following lemmas:

Lemma A. (Hallenbeck and Ruschweyh) [4, Th. 3.1.6, p. 71] *Let h be convex function, with $h(0)=\alpha$ and let $\gamma\in\mathbb{C}^*$ be a complex number with $\operatorname{Re}\gamma\geq 0$. If $p\in\mathcal{H}[a,n]$ and*

$$p(z)+\frac{1}{\gamma}zp'(z)\prec h(z),\quad z\in U$$

then

$$p(z)\prec q(z)\prec h(z),\quad z\in U$$

where

$$q(z)=\frac{\gamma}{nz^{\gamma/n}}\int_0^z h(t)t^{\frac{\gamma}{n}-1}dt,\quad z\in U.$$

The function q is convex in U and is the best dominant.

For two functions

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k \quad\text{and}\quad g(z)=z+\sum_{k=2}^{\infty}b_kz^k,$$

the Hadamard product (or convolution) of f and g is defined by

$$(f*g)(z):=z+\sum_{k=2}^{\infty}a_kb_kz^k.$$

For $\alpha_i\in\mathbb{C}$, $i=1,2,3,\dots,l$ and $\beta_j\in\mathbb{C}\setminus\{0,-1,-2,\dots\}$, $j=1,2,\dots,m$, the generalized hypergeometric function is defined by

$${}_lF_m(\alpha_1,\alpha_2,\dots,\alpha_l;\beta_1,\beta_2,\dots,\beta_m;z)=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\dots(\alpha_l)_n}{(\beta_1)_n\dots(\beta_m)_n}\cdot\frac{z^n}{n!}$$

$$(l\leq m+1,\ m\in\mathbb{N}_0=\{0,1,2,\dots\})$$

where $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n:\frac{\Gamma(a+n)}{\Gamma(a)}=\begin{cases}1, & n=0\\ a(a+1)\dots(a+n-1), & n\in\mathbb{N}:=\{1,2,\dots\}\end{cases}$$

Corresponding to the function

$$h(\alpha_1,\alpha_2,\dots,\alpha_l;\beta_1,\beta_2,\dots,\beta_m;z)=z\cdot {}_lF_m(\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_m;z)$$

Dziok-Srivastava operator [5], [6], [7] is

$$H^l_m(\alpha_1, \alpha_2, \dots, \alpha_l; \beta_1, \beta_2, \dots, \beta_m; z)$$
$$= h(\alpha_1, \alpha_2, \dots, \alpha_l; \beta_1, \beta_2, \dots, \beta_m; z) * f(z)$$
$$= z + \sum_{n=2}^{\infty} \frac{(\alpha_1)_{n-1}(\alpha_2)_{n-1}\dots(\alpha_l)_{n-1}}{(\beta_1)_{n-1}(\beta_2)_{n-1}\dots(\beta_l)_{n-1}} \cdot a_n \cdot \frac{z^n}{(n-1)!}$$

For simplicity, we write

$$H^l_m[\alpha_1]f(z) = H^l_m(\alpha_1, \alpha_2, \dots, \alpha_l; \beta_1, \beta_2, \dots, \beta_m; z).$$

It is well known [2] that

$$\alpha_1 H^l_m[\alpha_1 + 1]f(z) = z\{H^l_m[\alpha_1]f(z)\}' + (\alpha_1 - 1)H^l_m[\alpha_1]f(z). \tag{1}$$

2. Main results

Theorem 1. *Let* $l, m \in \mathbb{N}$, $l \leq m+1$, $\alpha_i \in \mathbb{C}$, $i = 1, 2, \dots, l$ *and* $\beta_j \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$, $j = 1, 2, 3, \dots, m$, $f \in A$, *and Dziok-Srivastava linear operator* $H^l_m[\alpha_1]f(z)$ *is given by (1).*

If it verifies the differential subordination

$$\{H^l_m[\alpha_1 + 1]f(z)\}' \prec h(z), \quad z \in U, \ \mathrm{Re}\,\alpha_1 > 0 \tag{2}$$

where h is a convex function, then

$$[H^l_m[\alpha_1]f(z)]' \prec q(z),$$

where

$$q(z) = \frac{\alpha_1}{z^{\alpha_1}} \int_0^z h(t) t^{\alpha_1 - 1} dt$$

q is a convex function and it is the best dominant.

Proof. Differentiating (1), we obtain:

$$\alpha_1\{H^l_m[\alpha_1 + 1]f(z)\}' = \alpha_1\{H^l_m[\alpha_1]f(z)\}' + Z\{H^l_m[\alpha_1]f(z)\}'', \quad z \in U. \tag{3}$$

we note that

$$p(z) = \{H^l_m[\alpha_1]f(z)\}' = 1 + p_1 z + p_2 z^2 + \dots, \tag{4}$$

$p_i \in \mathbb{C}$, where $p \in H[1,1]$.

Using (4) and (3) , the subordination (2) becomes:

$$p(z) + \frac{1}{\alpha_1} zp'(z) \prec h(z), \quad z \in U, \ \mathrm{Re}\,\alpha_1 > 0. \tag{5}$$

Using Lemma A, we have

$$p(z) \prec q(z) = \frac{\alpha_1}{z^{\alpha_1}} \int_0^z h(t) t^{\alpha_1 - 1} dt$$

i.e.

$$\{H_m^l[\alpha_1]f(z)\}' \prec q(z) = \frac{\alpha_1}{z^{\alpha_1}} \int_0^z h(t) t^{\alpha_1 - 1} dt, \quad z \in U,$$

q is a convex function and it is the best dominant.

Remark 1. If the function

$$h(z) = \frac{1 + (2\alpha - 1)z}{1 + z}, \quad z \in U, \ 0 \leq \alpha < 1,$$

then Theorem 1 can be expressed in the following :

Corollary 1. *Let $l, m \in \mathbb{N}$, $l \leq m + 1$, $\alpha_i \in \mathbb{C}$, $i = 1, 2, \ldots, l$ and $\beta_j \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$, $j = 1, 2, 3, \ldots, m$ let $f \in A$ and let $H_m^l[\alpha_1]f(z)$ be Dziok-Srivastava linear operator given by (1).*

If

$$\{H_m^l[\alpha_1 + 1]f(z)\}' \prec h(z) = \frac{1 + (2\alpha - 1)z}{1 + z}, \quad z \in U, \ 0 \leq \alpha < 1$$

then

$$\{H_m^l[\alpha_1]f(z)\}' \prec q(z) = \alpha_1(2\alpha - 1) + \frac{2\alpha_1(1 - \alpha)}{z^{\alpha_1}} \sigma(z, \alpha_1)$$

where

$$\sigma(z, \alpha_1) = \int_0^z \frac{t^{\alpha_1 - 1}}{1 + t} dt. \tag{6}$$

Since

$$q(z) = \alpha_1(2\alpha - 1) + 2\alpha_1(1 - \alpha) \frac{\sigma(z, \alpha_1)}{z^{\alpha_1}}$$

then, from the convexity of q, and since the fact that $q(U)$ is symmetric with respect to the real axis and from $p(z) \prec q(z)$, we deduce

$$\mathrm{Re}\, p(z) > \mathrm{Re}\, q(1) = (2\alpha - 1)\mathrm{Re}\, \alpha_1 + 2(1-\alpha)\mathrm{Re}\, \alpha_1 \cdot \sigma(1, \alpha_1) > 0$$

we obtain

$$\mathrm{Re}\, \{H_m^l[\alpha_1]f(z)\}' > 0$$

i.e.

$$H_m^l[\alpha_1]f(z) \in S.$$

Remark 2. If $m = 0$, $l = 1$, $\alpha_1 = 1$, $f \in A$ we obtain:

$$H_0^1[1]f(z) = f(z), \quad H_0^1[2]f(z) = zf'(z).$$

In this case Theorem 1 can be expressed under the form of the following corollary:

Corollary 2. *If the differential subordination*

$$zf''(z) + f'(z) \prec h(z), \quad z \in U$$

holds, then

$$f'(z) \prec \frac{1}{z}\int_0^z h(t)dt, \quad z \in U.$$

Theorem 2. *Let $l, m \in \mathbb{N}$, $l \le m + 1$, $\alpha_i \in \mathbb{C}$, $i = 1, 2, \ldots, l$, $\beta_j \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$, $j = 1, 2, \ldots, m$ let $f \in A$ and let $H_m^l[\alpha_1]f(z)$ be Dziok-Srivastava linear operator given by (1).*

If the differential subordination

$$\{H_m^l[\alpha_1]f(z)\}' \prec h(z), \quad z \in U, \ \mathrm{Re}\, \alpha_1 > 0 \tag{7}$$

holds, then

$$\frac{H_m^l[\alpha_1]f(z)}{z} \prec q(z) = \frac{1}{z}\int_0^z h(t)dt.$$

Proof. Let

$$p(z) = \frac{H_m^l[\alpha_1]f(z)}{z} = 1 + p_1 z + p_2 z^2 + \ldots, \quad z \in U, \ p \in H[1,1]. \tag{8}$$

Differentiating (8), we obtain

$$p(z) + zp'(z) = \{H_m^l[\alpha_1]f(z)\}' \tag{9}$$

Using (9), the differential subordination (7) becomes:

$$p(z) + zp'(z) \prec h(z), \quad z \in U.$$

Using Lemma A, we obtain

$$p(z) \prec q(z) = \frac{1}{z}\int_0^z h(t)dt.$$

The function q is convex and it is the best dominant.

Remark 3. If $m = 0$, $l = 1$, $\alpha_1 = 2$, $f \in A$, $h(z) = \dfrac{1+z}{1-z}$ then the Theorem 2 becomes the following corollary:

Corollary 3. *If the differential subordination:*

$$zf'(z) + f(z) \prec \frac{1+z}{1-z}, \quad z \in U$$

holds , then

$$f(z) \prec q(z) = \frac{1}{z}\int_0^z \frac{1+t}{1-t}dt = -1 + \frac{2\ln(1-z)}{z}.$$

Then function q is convex and it is the best dominant.

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