THE GENERALIZED COMMUTATIVITY DEGREE IN A FINITE GROUP

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Abstract. Extending the notion of probability of commuting pairs in [2, 6, 7, 8], it is introduced that of generalized commutativity degree. Its influence on the group structure is investigated.

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1. The definition

All the groups which we consider are finite. The notion of commutativity degree, or probability of commuting pairs, was introduced in [2, 7] and studied by many authors in different contexts. See [3, 4, 5, 6, 8]. Focusing on the role of the normalizers, instead of that of the centralizers, we define the *generalized commutativity degree* of a group G as the ratio

$$gd(G) = \frac{|\{(g, X) \in G \times \mathcal{L}(G) \mid X^g = X\}|}{|G||\mathcal{L}(G)|}, \tag{1}$$

where $\mathcal{L}(G)$ is the subgroup lattice of G. Recall that $N(G) = \bigcap_{X \in \mathcal{L}(G)} N_G(X)$ is the *norm* of G, studied in [1] and [9, Chapters 1.4, 1.5]. It is useful to consider Eq. (1) in the following form

$$|G||\mathcal{L}(G)|gd(G) = \sum_{X \in \mathcal{L}(G)} |N_G(X)|$$

$$= \sum_{X \in \mathcal{L}(N(G))} |N_G(X)| + \sum_{X \in \mathcal{L}(G) - \mathcal{L}(N(G))} |N_G(X)| \tag{2}$$

Of course, $gd(G) \in (0,1]$ and it gives a probability structure on $G \times \mathcal{L}(G)$. Note that $gd(G)=1$ if and only if the sum of all $|N_G(X)|$ for $X \in \mathcal{L}(G)$ is equal to $|G||\mathcal{L}(G)|$. By default, a hamiltonian group G has $gd(G)=1$ and their structure is well–known by [9, Theorem 2.3.12]. Then we will avoid the study of these groups from our description. Roughly speaking, we will avoid from our description the direct products of abelian groups and quaternion groups of order 8. On another hand, we will investigate the role of Eq. (1) and (1) with respect to structural properties of G and will find some bounds for $gd(G)$.

2. General properties

A concrete example will help us to visualize Eq. (1) and (1).

Example. The symmetric group S_3 has $\mathcal{L}(S_3)=\{\{1\},S_3,A_3,H,K,L\}$, where $A_3=\langle(123)\rangle=\langle a\rangle$, $H=\langle(12)\rangle=\langle h\rangle$, $K=\langle(13)\rangle=\langle k\rangle$, $L=\langle(23)\rangle=\langle l\rangle$. We may easily check that

$$|\{(g,X)\in S_3\times\mathcal{L}(S_3)\mid X^g=X\}|$$

$$=|\{(1,A_3),(a,A_3),(a^{-1},A_3),(h,A_3),(k,A_3),(l,A_3),$$

$$(1,\{1\}),(a,\{1\}),(a^{-1},\{1\}),(h,\{1\}),(k,\{1\}),(l,\{1\}),(1,S_3),(a,S_3),(a^{-1},S_3)$$

$$(h,S_3),(k,S_3),(l,S_3),(1,H),(1,K),(1,L),(h,H),(k,K),(l,L)\}|=24$$

so that $gd(S_3)=\frac{24}{36}=\frac{2}{3}$. S_3 is not hamiltonian and $Z(S_3)=N(S_3)=\{1\}$. See [9,Theorem 1.4.3]. On the other hand, we have

$$6\cdot 6\ gd(S_3)=1\cdot 6+\sum_{X\in\mathcal{L}(S_3)-\mathcal{L}(N(S_3))}|N_{S_3}(X)|=1\cdot 6+$$

$$|N_{S_3}(H)|+|N_{S_3}(K)|+|N_{S_3}(L)|+|N_{S_3}(A_3)|+|N_{S_3}(S_3)|=1\cdot 6+(2+2+2+6+6).\ \square$$

[3, Theorems 2.5, 3.3] and [8, Lemma 1.4] show that the commutativity degree is monotone. This is a mensural property which is useful in [6, Theorems 8,9,12]. For $gd(G)$ we have something of similar.

Proposition 2.1. *Let G be a group, $H \in \mathcal{L}(G)$ and N be a normal subgroup of G. Then* *(i)* $gd(H) \leq |G:H|gd(G)$ *and* $gd(G) \leq |G:H|gd(H)$*. The equality holds if and only if* $G = HN_G(X)$*, for each* $X \in \mathcal{L}(G)$*.*

(ii) $gd(G) \leq gd(G/N)gd(N)$*. The equality holds if and only if* $N_{G/N}(X/N) = N_G(X)/N$*, for each* $X \in \mathcal{L}(G)$ *containing* N*.*

Proof. (i). $N_H(X)$ lies in $N_G(X)$ for every $X \in \mathcal{L}(G)$ and so $gd(H) \leq gd(G)$. A fortiori, $gd(H) \leq |G:H|\ gd(G)$. The first inequality follows.

Note that $|N_G(X):N_H(X)| \leq |G:H|$, then $|N_G(X)| \leq |G:H||N_H(X)|$. Therefore,

$$\sum_{X\in\mathcal{L}(G)} |N_G(X)| \leq |G:H| \sum_{X\in\mathcal{L}(G)} |N_H(X)| = |G:H| \sum_{X\in\mathcal{L}(G)} |N_{G\cap H}(X)| =$$

Note that X is contained in $N_{G\cap H}(X)$ and so in H. Then we may substitute X with a variable Y running in $\mathcal{L}(H)$. But, $|N_{G\cap H}(Y)| \leq |N_G(Y)|$. Then

$$= |G:H| \sum_{Y\in\mathcal{L}(H)} |N_{G\cap H}(Y)| \leq |G:H| \sum_{Y\in\mathcal{L}(H)} |N_G(Y)|$$

and again $|N_G(Y)| \leq |G:H||N_H(Y)|$ gives

$$\leq |G:H|\big(|G:H| \sum_{Y\in\mathcal{L}(H)} |N_H(Y)|\big) = |G:H|^2 \sum_{Y\in\mathcal{L}(H)} |N_H(Y)|$$

from which the second inequality follows. Note that the equality follows if and only if $G = HN_G(X)$.

(ii). Note that $N_G(X)/N_N(X) \simeq N_G(X)N/N \leq N_{G/N}(X/N)$ with equality if and only if $N_{G/N}(X/N) = N_G(X)N$. Then

$$\sum_{X\in\mathcal{L}(G)} |N_G(X)| = \sum_{X\in\mathcal{L}(G)} |N_G(X)N/N||N_N(X)| \leq \sum_{X\in\mathcal{L}(G)} |N_{G/N}(X/N)||N_N(X)|$$

a substitution argument as in (i) above allows us to write

$$= \sum_{(X,Y)\in\mathcal{L}(G)\times\mathcal{L}(N)} |N_{G/N}(X/N)|\ \ |N_N(Y)| = \sum_{X\in\mathcal{L}(G)} \Big(\sum_{Y\in\mathcal{L}(N)} |N_{G/N}(X/N)||N_N(Y)| \Big)$$

$$= \sum_{X\in\mathcal{L}(G)} |N_{G/N}(X/N)| \sum_{Y\in\mathcal{L}(N)} |N_N(Y)| = gd(G/N)\ gd(N)$$

It is clear when the equality holds. □

Note that Proposition 2.1 is obviously satisfied in case of hamiltonian groups and in particular of abelian groups. This is true also for the next three results.

Corollary 2.2. *Let G be a group. $gd(G) \leq \prod_S gd(S)$, where S runs through the composition factors of G. Repetitions are allowed.*

Proof. This follows from repeated applications of Proposition 2.1 (ii). □

Corollary 2.3. *Let G and H be two groups. Then $gd(G \times H) = gd(G)\ gd(H)$.*

Proof. This follows from Proposition 2.1 (ii). But, we give also a direct proof of this fact for showing that $gd(G)$ is multiplicative. We have

$$gd(G \times H) = \frac{1}{|G\times H||\mathcal{L}(G\times H)|} \sum_{(X\times Y)\in\mathcal{L}(G)\times\mathcal{L}(H)} |N_{G\times H}(X \times Y)|$$

$$= \frac{1}{|G\times H||\mathcal{L}(G\times H)|} \sum_{(X\times Y)\in\mathcal{L}(G)\times\mathcal{L}(H)} |N_G(X) \times N_H(Y)|$$

$$= \frac{1}{|G||H||\mathcal{L}(G)||\mathcal{L}(H)|} \sum_{(X\times Y)\in\mathcal{L}(G)\times\mathcal{L}(H)} |N_G(X)||N_H(Y)|$$

$$= \frac{1}{|G||\mathcal{L}(G)|} \sum_{X\in\mathcal{L}(G)} \Big(\frac{1}{|H||\mathcal{L}(H)|} \sum_{Y\in\mathcal{L}(H)} |N_H(Y)|\Big)|N_G(X)|$$

$$= \frac{1}{|G||\mathcal{L}(G)|} \sum_{X\in\mathcal{L}(G)} \Big(gd(H)\Big)|N_G(X)|$$

$$= gd(H)\frac{1}{|G||\mathcal{L}(G)|} \sum_{X\in\mathcal{L}(G)} |N_G(X)| = gd(H)gd(G).\ \square$$

The next result generalizes the situation for S_3, shown in Example.

Proposition 2.4. *Assume that G is a non-hamiltonian group of order pq, where $p < q$ are primes. Then $gd(G) = \frac{4}{q+3}$*

Proof. From [9, p.26], except for $\{1\}$ and G, $\mathcal{L}(G)$ has only $q+1$ elements. Then $|\mathcal{L}(G)| = q+3$. On another hand, $Z(G) = \{1\}$, then $N(G) = \{1\}$ by [9, Theorem 1.4.3]. In particular, $|\mathcal{L}(N(G))| = 1$. The Sylow's Theorem shows there are q subgroups X of order p and only one subgroup K of order q. It is easy to check that $N_G(X) = X$ and $N_G(K) = G$. Then Eq. (2) becomes

$$pq(q+3)gd(G) = pq + \sum_{X \in \mathcal{L}(G)-\{1\}} |N_G(X)| = pq + (q|X| + |G| + |G|) = 4pq$$

and so $gd(G) = \frac{4}{q+3}$. □

From [9, pp. 26–29], or using GAP, it is easy to check that $|\mathcal{L}(D_8)| = 10$, $|\mathcal{L}(D_{16})| = 19$ and $|\mathcal{L}(Q_{16})| = 11$. This intuition, in case of D_{2^n} and Q_{2^n} for ≥ 1, can be formalized, noting that they are extra-special 2-groups. Then for $n \geq 2$

$$|\mathcal{L}(D_{2^n})| = 2 + (n-1) + \sum_{i=1}^{n-1} 2^i, \quad |\mathcal{L}(Q_{2^n})| = 2 + n + \sum_{i=1}^{n-1} 2^i. \tag{3}$$

Then, if p is the smallest prime dividing $|G|$ and $e \in \{0, 1\}$, the assumption

$$|\mathcal{L}(G)| = p + n - e + \sum_{i=1}^{n-1} p^i \tag{4}$$

is meaningful. For instance, $(p, n, e) = (2, 2, 1)$ gives $|\mathcal{L}(D_8)|$.

Lemma 2.5. *If G is a non-hamiltonian group satisfying Eq. (4) and $|N(G)| = p$, then*

$$gd(G) \leq \frac{p^{n-1} + p^{n-2} + \ldots + p^3 + p^2 + 4p + n - e - 2}{p^n + p^{n-1} + \ldots + p^3 + 2p^2 + (n-e+1)p}.$$

Proof. Note that $|N_G(X)| \leq \frac{|G|}{p}$ for every $X \in \mathcal{L}(G)$. Of course, $|\mathcal{L}(N(G))| =$

2. Then Eq. (2) becomes

$$
\begin{aligned}
&|G|\Big(p+n-e+\textstyle\sum_{i=1}^{n-1} p^i\Big) gd(G) = 2|G| + \displaystyle\sum_{X\in\mathcal{L}(G)-\mathcal{L}(N(G))} |N_G(X)| \\
&\le 2|G| + \Big(|\mathcal{L}(G)-\mathcal{L}(N(G))|\Big)\tfrac{|G|}{p} = 2|G| + \tfrac{|G|}{p}(|\mathcal{L}(G)|-2) \\
&= 2|G| + \tfrac{|G|}{p}\Big(\Big(p+n-e+\textstyle\sum_{i=1}^{n-1} p^i\Big)-2\Big) \\
&= |G|\tfrac{3p+n-e-2+\sum_{i=1}^{n-1} p^i}{p}.
\end{aligned}
$$

This gives $gd(G) \le \frac{3p+n-e-2+\sum_{i=1}^{n-1} p^i}{p\left(p+n-e+\sum_{i=1}^{n-1} p^i\right)}$, and the result follows. □

No we will show the main result of the paper. From bounds on $gd(G)$ we may deduce structural information on $N(G)$. This fact illustrates in the case of $gd(G)$ some known circumstances as [8, Theorems 3.1, 3.3, 4.3, 5.1], [3, Theorems 3.10, 5.5], [4, Theorem B], [5, Theorems A, B].

Theorem 2.6. *Let G be a non-hamiltonian p-group satisfying Eq. (4). If $1 > gd(G) \ge \frac{4}{p+3}$, then $N(G)$ is non-cyclic.*

Proof. Assume $N(G) = G$. Eq. (2) becomes $|G||\mathcal{L}(G)|gd(G) = |G||\mathcal{L}(G)|$. This case cannot hold, since $gd(G) < 1$. Then $N(G) \ne G$. Assume $N(G)$ cyclic. Since G is p-group, $\{1\} \ne Z(G) \le N(G)$. See [1] or [9, Theorem 1.4.2]. Then $|N(G)| = p$ and Lemma 2.5 implies

$$\frac{4}{p+3} \le gd(G) \le \frac{p^{n-1}+p^{n-2}+\ldots+p^3+p^2+4p+n-e-2}{p^n+p^{n-1}+\ldots+p^3+2p^2+(n-e+1)p},$$

that is,

$$
\begin{aligned}
f(p) = 4p^n + 4p^{n-1} + \ldots + 4p^3 + 8p^2 + 4(n-e+1)p \le gd(G) \\
\le p^n + 4p^{n-1} + \ldots + 2p^3 + 6p^2 + 3(n-e+1)p = g(p).
\end{aligned}
$$

But each coefficient of the polynomial $f(p)$ is greater than each coefficient of the polynomial $g(p)$, then we must have $f(p) > g(p)$. This is a contradiction. Then $|N(G)| \ne p$. We deduce that $N(G)$ cannot be cyclic. □

Remark 2.7. From [1, Theorems 1, 2] we know that in a p-group G either the group $N(G)/Z(G)$, or $[N(G), G]$, is cyclic. This shows how $N(G)$ cannot be big. Theorem 2.6 shows in a certain sense how $N(G)$ cannot be small. □

Remark 2.8. Looking at the methods in [4, 5, 7], we believe that most of the results in the present paper can be adapted in the infinite case, at least when we consider the compact groups. To the best of our knowledge, this is still an open problem. Indeed, no literature is known on this point. □

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