AN INTEGRAL OPERATOR ON ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract. In this paper we consider an integral operator on analytic functions and prove some preserving theorems regarding some subclasses of analytic functions with negative coefficients.

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1. Introduction

Let $\mathcal{H}(U)$ be the set of functions which are regular in the unit disc U, $A = \{f \in \mathcal{H}(U) : f(0) = f'(0) - 1 = 0\}$, $\mathcal{H}_u(U) = \{f \in \mathcal{H}(U) : f$ is univalent in $U\}$ and $S = \{f \in A : f$ is univalent in $U\}$.

We denote with T the subset of the functions $f \in S$, which have the form

$$f(z) = z - \sum_{j=2}^{\infty} a_j z^j \,,\ a_j \geq 0 \,,\ j \geq 2 \,,\ z \in U \tag{1}$$

and with $T^* = T \bigcap S^*$, $T^*(\alpha) = T \bigcap S^*(\alpha)$, $T^c = T \bigcap S^c$ and $T^c(\alpha) = T \bigcap S^c(\alpha)$, where $0 \leq \alpha < 1$.

Theorem 1.1 [5] *For a function f having the form (1) the following assertions are equivalents:*

(i) $\sum_{j=2}^{\infty} j a_j \leq 1$;

(ii) $f \in T$;

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(iii) $f \in T^*$.

Regarding the classes $T^*(\alpha)$ and $T^c(\alpha)$ with $0 \leq \alpha < 1$, we recall here the following result:

Theorem 1.2 [5] *A function f having the form (1) is in the class $T^*(\alpha)$ if and only if:*

$$\sum_{j=2}^{\infty} \frac{j-\alpha}{1-\alpha} a_j \leq 1 \,, \tag{2}$$

and is in the class $T^c(\alpha)$ if and only if:

$$\sum_{j=2}^{\infty} \frac{j(j-\alpha)}{1-\alpha} a_j \leq 1 \,. \tag{3}$$

Definition 1.1 [2] *Let $S^*(\alpha,\beta)$ denote the class of functions having the form (1) which are starlike and satisfy*

$$\left| \frac{\frac{zf'(z)}{f(z)} - 1}{\frac{zf'(z)}{f(z)} + (1-2\alpha)} \right| < \beta \tag{4}$$

for $0 \leq \alpha < 1$ and $0 < \beta \leq 1$. And let $C^*(\alpha,\beta)$ denote the class of functions such that $zf'(z)$ is in the class $S^*(\alpha,\beta)$.

Theorem 1.3 [2] *A function f having the form (1) is in the class $S^*(\alpha,\beta)$ if and only if:*

$$\sum_{j=2}^{\infty} \left\{ (j-1) + \beta(j+1-2\alpha) \right\} a_j \leq 2\beta(1-\alpha) \,, \tag{5}$$

and is in the class $C^*(\alpha,\beta)$ if and only if:

$$\sum_{j=2}^{\infty} j \left\{ (j-1) + \beta(j+1-2\alpha) \right\} a_j \leq 2\beta(1-\alpha) \,. \tag{6}$$

Let D^n be the Sălăgean differential operator (see [3]) defined as:

$$D^n : A \to A \,, \quad n \in \mathbb{N} \text{ and } D^0 f(z) = f(z)$$

$$D^1 f(z) = Df(z) = zf'(z) \,, \quad D^n f(z) = D(D^{n-1} f(z)).$$

In [4] the author define the class $T_n(\alpha,\beta)$, from which by choosing different values for the parameters we obtain variously subclasses of analytic functions with negative coefficients (for example $T_n(\alpha,1)=T_n(\alpha)$ which is the class of n-starlike of order α functions with negative coefficients and $T_0(\alpha,\beta)=S^*(\alpha,\beta)\bigcap T$, where $S^*(\alpha,\beta)$ is the class defined by (4)).

Definition 1.2 [4] *Let $\alpha\in[0,1)$, $\beta\in(0,1]$ and $n\in\mathbb{N}$. We define the class $S_n(\alpha,\beta)$ of the n-starlike of order α and type β through*

$$S_n(\alpha,\beta)=\{f\in A\,;\,|J(f,n,\alpha;z)|<\beta\}$$

where $J(f,n,\alpha;z)=\dfrac{D^{n+1}f(z)-D^nf(z)}{D^{n+1}f(z)+(1-2\alpha)D^nf(z)}$, $z\in U$. *Consequently* $T_n(\alpha,\beta)=S_n(\alpha,\beta)\bigcap T$.

Theorem 1.4 [4] *Let f be a function having the form (1). Then $f\in T_n(\alpha,\beta)$ if and only if*

$$\sum_{j=2}^{\infty}j^n\left[j-1+\beta(j+1-2\alpha)\right]a_j\leq 2\beta(1-\alpha)\,. \tag{7}$$

2. Main results

Let consider the integral operator $I_{\lambda,\gamma}:A\to A$, where $1<\lambda<\infty$ and $\gamma=1,2,\ldots$, defined by

$$f(z)=I_{\lambda,\gamma}(F(z))=\lambda\int_0^1 u^{\lambda-\gamma-1}F(u^\gamma z)du. \tag{8}$$

Remark 2.1 *For $F(z)=z+\sum\limits_{j=2}^{\infty}a_jz^j$, from (8) we obtain*

$$f(z)=I_{\lambda,\gamma}(F(z))=z+\sum_{j=2}^{\infty}\frac{\lambda}{\lambda+(j-1)\gamma}a_jz^j.$$

Also, we notice that $0<\dfrac{\lambda}{\lambda+(j-1)\gamma}<1$, *where* $1<\lambda<\infty$, $j\geq 2$, $\gamma=1,2,\ldots$.

Remark 2.2 *It is easy to prove, by using Theorem 1.1 and Remark 2.1, that for* $F(z) \in T$ *and* $f(z) = I_{\lambda,\gamma}(F(z))$*, we have* $f(z) \in T$*, where* $I_{\lambda,\gamma}$ *is the integral operator defined by (8).*

By using the previously remark and the Theorem 1.2, we obtain the following result:

Theorem 2.1 *Let* $F(z)$ *be in the class* $T^*(\alpha)$*,* $\alpha \in [0,1)$*,* $F(z) = z - \sum_{j=2}^{\infty} a_j z^j$*,* $a_j \geq 0$*,* $j \geq 2$*. Then* $f(z) = I_{\lambda,\gamma}(F(z)) \in T^*(\alpha)$*, where* $I_{\lambda,\gamma}$ *is the integral operator defined by (8).*

Proof. From Remark 2.2 we obtain $f(z) = I_{\lambda,\gamma}(F(z)) \in T$.

From (2) we have $\sum_{j=2}^{\infty} \frac{j-\alpha}{1-\alpha} a_j \leq 1$ and $f(z) = z - \sum_{j=2}^{\infty} b_j z^j$, where $b_j = \frac{\lambda}{\lambda + (j-1)\gamma} a_j$. By using the fact that $0 < \frac{\lambda}{\lambda + (j-1)\gamma} < 1$, where $1 < \lambda < \infty$, $j \geq 2$, $\gamma = 1, 2, \ldots$, we obtain $\frac{j-\alpha}{1-\alpha} b_j < \frac{j-\alpha}{1-\alpha} a_j$ and thus $\sum_{j=2}^{\infty} \frac{j-\alpha}{1-\alpha} b_j \leq 1$. This mean (see Theorem 1.2) that $f(z) = I_{\lambda,\gamma}(F(z)) \in T^*(\alpha)$.

Similarly (by using Remark 2.2 and the Theorems 1.3 and 1.4) we obtain:

Theorem 2.2 *Let* $F(z)$ *be in the class* $T^c(\alpha)$*,* $\alpha \in [0,1)$*,* $F(z) = z - \sum_{j=2}^{\infty} a_j z^j$*,* $a_j \geq 0$*,* $j \geq 2$*. Then* $f(z) = I_{\lambda,\gamma}(F(z)) \in T^c(\alpha)$*, where* $I_{\lambda,\gamma}$ *is the integral operator defined by (8).*

Theorem 2.3 *Let* $F(z)$ *be in the class* $C^*(\alpha, \beta)$*,* $\alpha \in [0,1)$*,* $\beta \in (0,1]$*,* $F(z) = z - \sum_{j=2}^{\infty} a_j z^j$*,* $a_j \geq 0$*,* $j \geq 2$*. Then* $f(z) = I_{\lambda,\gamma}(F(z)) \in C^*(\alpha, \beta)$*, where* $I_{\lambda,\gamma}$ *is the integral operator defined by (8).*

Theorem 2.4 *Let* $F(z)$ *be in the class* $T_n(\alpha, \beta)$*,* $\alpha \in [0,1)$*,* $\beta \in (0,1]$*,* $n \in \mathbb{N}$*,* $F(z) = z - \sum_{j=2}^{\infty} a_j z^j$*,* $a_j \geq 0$*,* $j \geq 2$*. Then* $f(z) = I_{\lambda,\gamma}(F(z)) \in T_n(\alpha, \beta)$*, where* $I_{\gamma,\gamma}$ *is the integral operator defined by (8).*

Remark 2.3 *By choosing* $\beta = 1$*, respectively* $n = 0$*, in the above theorem, we obtain the similarly results for the classes* $T_n(\alpha)$ *and* $S^*(\alpha, \beta)$*.*

Remark 2.4 *If we consider* $\gamma = 1$ *and* $\lambda = c + \delta$*, where* $0 < c < \infty$ *and* $1 \leq \delta < \infty$*, we obtain the results from [1].*

References

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