A NEW KIND OF ERROR BOUNDS FOR SOME QUADRATURE RULES

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Abstract. A new kind of error bounds for some quadrature rules are derived. Few real examples which illustrate the applicability of these error bounds are given.

2000 *Mathematics Subject Classification*: 26D15, 41A55.

1.Introduction

In recent years a number of authors have considered error analyses for known and new quadrature rules. For example see [1]-[8]. They have obtained many new results on this topic - new quadrature rules and new kind of estimations of errors.

In this paper we give a new approach to the problem of approximate quadratures. A new kind of error bounds for some quadrature rules are derived. We start from the well-known trapezoidal quadrature rule and derive a new quadrature rule. We also establish error bounds for this new rule and show that this new rule can be much better than the trapezoidal rule. The main general idea is to find a quadrature formula for a class of practical problems. In the present case we have in mind approximate calculations of some special functions: Dawson integral ($\int_0^x \exp(t^2)dt$), Exponential integral ($\int_x^\infty \exp(-t)/tdt = \gamma + \ln x + \int_0^x (\exp(t) - 1)/tdt$, where $\gamma = 0.5772156649$ is the Euler constant), Sine integral ($\int_0^x \sin t/tdt$), Cosine integral ($\int_x^\infty \cos t/tdt = \gamma + \ln x + \int_0^x (\cos t - 1)/tdt$), Hyperbolic cosine integral, Hyperbolic sine integral, Error function, Fresnel integrals, etc.

In Section 2 we give a general procedure for deriving of such a rule. In Section 3 we give a particular rule and obtain the error bounds. In Section 4 we demonstrate the above assertion of the effectiveness of the new rule. We apply this rule to obtain approximate values of only few of the mentioned special integrals (Dawson integral, Exponential integral) which is sufficient to demonstrate the effectiveness of the new rule. We also demonstrate the applicability of this new rule to arbitrary functions.

2. A general procedure

We consider the trapezoidal quadrature rule

$$\int_a^b f(t)dt = \frac{b-a}{2}\left[f(a)+f(b)\right]+R(f), \tag{1}$$

where

$$R(f) = -\int_a^b \left(t-\frac{a+b}{2}\right) f'(t)dt \tag{2}$$

or

$$R(f) = \int_a^b \left[\frac{1}{2}\left(t-\frac{a+b}{2}\right)^2 - \frac{(b-a)^2}{8}\right] f''(t)dt. \tag{3}$$

Of course, the functions

$$K_1(t) = -\left(t-\frac{a+b}{2}\right) \tag{4}$$

and

$$K_2(t) = \frac{1}{2}\left(t-\frac{a+b}{2}\right)^2 - \frac{(b-a)^2}{8} \tag{5}$$

are the Peano kernels.

We now define

$$R(x) = \int_a^x f(t)dt - \frac{x-a}{2}\left[f(a)+f(x)\right],\ x \in [a,b]. \tag{6}$$

Let $g(x)$ be an integrable function. Then we have

$$\begin{aligned} &\int_a^b R(x)g(x)dx \qquad (7)\\ &= \int_a^b g(x)\int_a^x f(t)dtdx - \int_a^b g(x)\frac{x-a}{2}\left[f(a)+f(x)\right]dx \end{aligned}$$

or

$$\begin{aligned} & \int_a^b R(x)g(x)dx \\ = \quad & \int_a^b f(t)\int_t^b g(x)dxdt - \int_a^b g(x)\frac{x-a}{2}\left[f(a)+f(x)\right]dx. \end{aligned} \tag{8}$$

From (1), (2) and (6) we see that we can write

$$\int_a^b R(x)g(x)dx = -\int_a^b g(x)dx\int_a^x \left(t-\frac{a+x}{2}\right)f'(t)dt$$

what can be written in the form

$$\int_a^b R(x)g(x)dx = -\int_a^b f'(t)dt\int_t^b g(x)\left(t-\frac{a+x}{2}\right)dx. \tag{9}$$

In a similar way, from (1), (3) and (6) we get

$$\begin{aligned} & \int_a^b R(x)g(x)dx \\ = \quad & \int_a^b g(x)dx\int_a^x \left[\frac{1}{2}\left(t-\frac{a+x}{2}\right)^2-\frac{(x-a)^2}{8}\right]f''(t)dt \end{aligned}$$

what can be written in the form

$$\begin{aligned} & \int_a^b R(x)g(x)dx \\ = \quad & \int_a^b f''(t)dt\int_t^b g(x)\left[\frac{1}{2}\left(t-\frac{a+x}{2}\right)^2-\frac{(x-a)^2}{8}\right]dx. \end{aligned} \tag{10}$$

From (8)-(10) we get the next two formulas

$$\begin{aligned} & \int_a^b f(t)\int_t^b g(x)dxdt - \int_a^b g(x)\frac{x-a}{2}\left[f(a)+f(x)\right]dx \\ = \quad & \int_a^b f''(t)dt\int_t^b g(x)\left[\frac{1}{2}\left(t-\frac{a+x}{2}\right)^2-\frac{(x-a)^2}{8}\right]dx \end{aligned} \tag{11}$$

and

$$\begin{aligned} & \int_a^b f(t)\int_t^b g(x)dxdt - \int_a^b g(x)\frac{x-a}{2}\left[f(a)+f(x)\right]dx \\ = \quad & -\int_a^b f'(t)dt\int_t^b g(x)\left(t-\frac{a+x}{2}\right)dx. \end{aligned} \tag{12}$$

We can choose the function $g(x)$ in different ways, depending of the underlying problem. The different choices give different rules. Here we choose $g(x) = 1$ since it is sufficient to show the effectiveness of the new rule. Note also that for all integrals mentioned in Section 1 we can exactly calculate the integral

$$\int_a^b g(x)xf(x)dx = \int_a^b xf(x)dx, \quad (g(x) = 1).$$

3. A particular rule and error bounds

If we choose $g(x) = 1$ in (11) then we have

$$\begin{aligned} &\int_a^b f(t)(b-t)dt - \frac{(b-a)^2}{4} f(a) - \int_a^b \frac{x-a}{2} f(x)dx \\ = &\int_a^b f''(t)dt \int_t^b \left[\frac{1}{2}\left(t - \frac{a+x}{2}\right)^2 - \frac{(x-a)^2}{8} \right] dx \end{aligned}$$

or

$$\begin{aligned} &(b+\frac{a}{2}) \int_a^b f(t)dt - \frac{3}{2}\int_a^b tf(t)dt - \frac{(b-a)^2}{4} f(a) \\ = &\frac{1}{24}\int_a^b \left[2(t-a)^3 - (b-a)^3 - 8\left(t - \frac{a+b}{2}\right)^3 \right] f''(t)dt. \end{aligned}$$

Hence, we get

$$\int_a^b f(t)dt = \frac{2}{2b+a}\left[\frac{3}{2}\int_a^b tf(t)dt + \frac{(b-a)^2}{4} f(a) \right] + \bar{R}$$

where

$$\bar{R} = \frac{1}{24}\frac{2}{2b+a}\int_a^b \left[2(t-a)^3 - (b-a)^3 - 8\left(t - \frac{a+b}{2}\right)^3 \right] f''(t)dt.$$

Now it is not difficult to find that

$$\left|\bar{R}\right| \leq \frac{1}{48}\left|\frac{2}{2b+a}\right| (b-a)^4 \left\|f''\right\|_\infty .$$

Of course, we supposed that $2b + a \neq 0$.

The usual estimations of errors for quadrature rules (see for example [2], [7]) are given by

$$|R| \leq C(b-a)^m \left\| f^{(n)} \right\|_\infty \tag{13}$$

where $m = n+1$. Note that for the above rule we have $m = n+2$. If a is close to b, $h = b-a$, then this rule is of order $O(h^4)$, while the usual estimation (for the trapezoidal rule) gives the order $O(h^3)$. Thus we can expect that this rule will give better results than the trapezoidal rule. This will be demonstrated in the next section.

We now formulate a formal result.

Theorem 1 *Let $f \in C^2(a,b)$ and $2b + a \neq 0$. Then*

$$\int_a^b f(t)dt = \frac{2}{2b+a}\left[\frac{3}{2}\int_a^b tf(t)dt + \frac{(b-a)^2}{4}f(a)\right] + \bar{R},$$

where

$$\left|\bar{R}\right| \leq \frac{1}{24}\left|\frac{1}{2b+a}\right|(b-a)^4 \left\| f'' \right\|_\infty .$$

If we choose $g(x) = 1$ in (12) then we have

$$\begin{aligned} & \int_a^b f(t)(b-t)dt - \frac{(b-a)^2}{4}f(a) - \int_a^b \frac{x-a}{2}f(x)dx \\ = & -\int_a^b f'(t)dt \int_t^b \left(t - \frac{a+x}{2}\right)dx \end{aligned}$$

or

$$\begin{aligned} & (b+\frac{a}{2})\int_a^b f(t)dt - \frac{3}{2}\int_a^b tf(t)dt - \frac{(b-a)^2}{4}f(a) \\ = & -\frac{1}{16}\int_a^b f'(t)\left[(t-a)^2 - 4\left(t - \frac{a+b}{2}\right)^2\right]dt. \end{aligned}$$

Hence, we get

$$\int_a^b f(t)dt = \frac{2}{2b+a}\left[\frac{3}{2}\int_a^b tf(t)dt + \frac{(b-a)^2}{4}f(a)\right] + \tilde{R}$$

where

$$\tilde{R} = -\frac{1}{4}\frac{2}{2b+a}\int_a^b f'(t)\left[(t-a)^2 - 4\left(t - \frac{a+b}{2}\right)^2\right]dt.$$

Now it is not difficult to find that

$$\left|\bar{R}\right| \leq \frac{2}{27}\left|\frac{2}{2b+a}\right|(b-a)^3\left\|f'\right\|_\infty.$$

Note that for the above rule we have $m = n+2$, where m and n are defined in (13). If a is close to b, $h = b - a$, then this rule is of order $O(h^3)$, while the usual estimation (for the trapezoidal rule) gives the order $O(h^2)$, when the estimation contains the first derivative $f'(x)$.

Finally, we formulate a corresponding formal result.

Theorem 2 *Let $f \in C^1(a,b)$ and $2b + a \neq 0$. Then*

$$\int_a^b f(t)dt = \frac{2}{2b+a}\left[\frac{3}{2}\int_a^b tf(t)dt + \frac{(b-a)^2}{4}f(a)\right] + \tilde{R},$$

where

$$\left|\tilde{R}\right| \leq \frac{4}{27}\left|\frac{1}{2b+a}\right|(b-a)^3\left\|f'\right\|_\infty.$$

4.Numerical results

Since this new rule is obtained using the trapezoidal rule it is natural to compare these two rules.

We now write the known compound trapezoidal formula

$$\int_a^b f(t)dt = \sum_{i=0}^{n-1}\frac{f(x_i)+f(x_{i+1})}{2}h + R_T, \tag{14}$$

where $x_{i+1} = x_i + h$, $h = (b-a)/n$, $i = 0, 1, 2, ..., n-1$.

We also write the compound formula

$$\int_a^b f(t)dt = \sum_{k=0}^{m-1}\frac{2}{2x_{i+1}+x_i}\left[\frac{3}{2}\int_{x_i}^{x_{i+1}} tf(t)dt + \frac{h^2}{4}f(x_i)\right] + R_N, \tag{15}$$

where $x_{i+1} = x_i + h$, $h = (b-a)/m$, $i = 0, 1, 2, ..., m-1$.

Example 1 *We choose* $f(t) = \exp(t^2)$ *and* $a = 0$, $b = 1$. *Then*

$$\int t \exp(t^2) dt = \frac{1}{2} \exp(t^2) + C$$

such that

$$\int_{x_i}^{x_{i+1}} t f(t) dt = \frac{1}{2}(\exp(x_{i+1}^2) - \exp(x_i^2)). \tag{16}$$

If we apply the trapezoidal rule (14) with the above data and $n = 1000$ *then we get the approximate result* $\int_0^1 \exp(t^2) dt \approx 1.46265219895$. *If we apply the rule (15) with (16) and* $m = 100$ *then we get* $\int_0^1 \exp(t^2) dt \approx 1.46265197603$. *Since the "exact" result is* $\int_0^1 \exp(t^2) dt = 1.46265172958$ *we see that the errors are* $4.69E - 07$ *and* $2.46E - 07$, *respectively.*

Remark 1 *Note that we have to calculate the functions values* $f(x_i)$ 1001 *times if we apply the trapezoidal rule to obtain the error of order* $E - 07$ *and we have to calculate the functions values* $f(x_i)$ 101 *times if we apply the new rule to obtain the error of the same order* $E - 07$. *Hence, we can say that the improved rule is "ten times better" than the trapezoidal rule. This leave no doubt which rule would be applied for this example.*

Example 2 *We choose* $f(t) = \frac{\exp(t)-1}{t}$, $f(0) = 1$ *and* $a = 0$, $b = 1$. *Then*

$$\int t \frac{\exp(t) - 1}{t} dt = \exp(t) - t + C$$

such that

$$\int_{x_i}^{x_{i+1}} t f(t) dt = \exp(x_{i+1}) - x_{i+1} - \exp(x_i) + x_i. \tag{17}$$

If we apply the trapezoidal rule (14) with the above data and $n = 1000$ *then we get the approximate result* $\int_0^1 f(t) dt \approx 1.31790219312$. *If we apply the rule (15) with (17) and* $m = 100$ *then we get* $\int_0^1 f(t) dt \approx 1.31790218314$. *Since the "exact" result is* $\int_0^1 f(t) dt = 1.317902151454404$ *we see that the errors are* $1.42E - 08$ *and* $2.42E - 08$, *respectively.*

Remark 2 *In the last example we got results similar to those of the first example. Thus the observations from Remark 1 are valid in this case, too. Now one can conclude that this new rule is effective only for examples of integrals given in Section 1. It is not so. For that purpose we give the next example.*

Example 3 *We choose* $f(t) = \sin(t)$ *and* $a = 10000$, $b = 10001$. *Then*

$$\int t\sin(t)dt = -t\cos(t) + \sin(t) + C$$

such that

$$\int_{x_i}^{x_{i+1}} tf(t)dt = -x_{i+1}\cos(x_{i+1}) + x_i\cos(x_i) + \sin(x_{i+1}) - \sin(x_i). \tag{18}$$

If we apply the trapezoidal rule (14) with the above data and $n = 1000$ *then we get the approximate result* $\int_a^b \sin(t)dt \approx -0.6948692101$.. *If we apply the rule (15) with (18) and* $m = 5$ *then we get* $\int_a^b \sin(t)dt \approx -0.6948692604$. *Since the exact result is* $\int_a^b \sin(t)dt = -\cos(1)+1 = -0.694869268$ *we see that the errors are* $5.81E-08$ *and* $7.5E-09$, *respectively. Hence, we obtain "200 times better result" if we apply the new rule.*

Remark 3 *In fact, the last example shows that the new quadrature rule depends on the interval of integration. This is a direct consequence of the estimations of errors given in the theorems 1 and 2.*

5.Conclusion remarks

The new quadrature rule is obtained as it is presented. We have to observe that we can seek similar rules, for example, by considering integrals of the form $\int_a^b P_2(t)f'(t)dt$ or generally of the form $\int_a^b P_k(t)f'(t)dt$, where $P_2(t)$ and $P_k(t)$ are polynomials of degree 2 and $k > 2$, respectively. Few examples with these kinds of integrals were done but didn't give significantly better results. Thus they are not mentioned in this paper. It is also interesting fact that the results can depend on the interval of integration and this can be used in many practical situations. Here we have emphasized one of them. Finally, the obtained rule is exact for polynomials of degree ≤ 1 but its "order" is $\sim O(h^4)$. This is also an interesting fact which deserves to be mentioned.

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