ABOUT BIHARMONIC PROBLEM VIA A SPECTRAL APPROACH AND DECOMPOSITION TECHNIQUES

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Abstract. In this work, we describe a spectral method coupled with a variational decomposition technique for solving a biharmonic equations. We construct new spaces. Using one approximation by a spectral method we can bring the resolution of Dirichlet problem for Δ^2 to a finite number of Dirichlet approach for -Δ. Some new theoretic spectral approaches are given, numerical solutions and illustrations are established to prove our theoretic study.

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1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^d$ $(d = 1, 2, 3$ in practice) of smooth boundary $Fr(\Omega)$. We consider the following problem:

$$(DP) \qquad \begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = g_1 & \text{on } Fr(\Omega)\,; \\ \frac{\partial u}{\partial \eta} = g_2 & \text{on } Fr(\Omega)\,, \end{cases}$$

where $\Delta^2 = \Delta(\Delta)$ is the biharmonic operator. The reduction of boundary problems to equivalent problems is doing by several theoretic methods. We are interested in this work to one direct method based on the Green formula. This approach is used with success, by several authors, to resolve mathematical modelling problems.

The solution of biharmonic problem (DP) is studied by some authors and by several methods, so by finite difference methods, finite element methods and duality ...

J.W. McLaurin ([5], 1974) has given a decomposition technique for the problem (DP). He has proved that a solution of this problem by finite difference methods is equivalent to resolve a sequence of Dirichlet problems for the operator Δ.

R. Glowinski and O. Pironneau ([4], 1977) have, only, established this decomposition method using the finite element methods correspondent to same problem on a domain of $\mathbb{R}^2$. They have proved that a solution of this problem by this method is equivalent, also, to resolve a sequence of Dirichlet problems for the operator Δ and have given the error estimate:

$$\|u - u_h\|_{H^1(\Omega)} + \|\Delta u + \varphi_h\|_{L^2(\Omega)} = \bigcirc(h^{k-1}).$$

Our work consists to give a spectral approach to this problem and estimate the error theoretically on new space. We illustrate our numerical approach by numerical tests on some examples.

Indeed, if we resolve the biharmonic problem by spectral method based on the direct variational formulation, then we can give as advantage of this method the following :

i) we store one convergent approximation of the solution u in norm $\|.\|_{H^1(\Omega)}$ in place of the norm $\|.\|_{H^2(\Omega)}$;

ii) We obtain a convergent approximation φ_N of $-\Delta u$.

However, we will think that, in spite of the simplicity of the space V_N, the practice problem is to compute (u_N, φ_N).

The basic idea in this case, consists to introduce a space $\mathcal{M}_N \subset V_N$ of multipliers such that

$$V_N = V_N^0 \oplus \mathcal{M}_N$$

where

$$V_N^0 = \{v_N \in V_N \ : \ v_N = 0 \text{ on } Fr(\Omega)\}$$

In this paper, we prove that a resolution of Dirichlet problem for biharmonic operator by Galerkin spectral method is equivalent to resolve a sequence of Dirichlet problems for operator Δ by the same method, and to resolve well conditioned linear system. We prove some results of convergence which are effecient and improve previously obtained results. We illustrate these results by numerical trials.

2. A Continuous Problem

Let Ω be a bounded domain of $\mathbb{R}^d$ ($d = 1, 2, 3$ in practice) with smooth boundary $Fr(\Omega)$. Define the following problem:

$$(DP) \qquad \begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = g_1 & \text{on } Fr(\Omega), \\ \frac{\partial u}{\partial \eta} = g_2 & \text{on } Fr(\Omega), \end{cases}$$

where g_1, g_2 and f are given functions. This problem is equivalent to find (u, w) such that

$$(EP) \qquad \begin{cases} -\Delta w = f & \text{in } \Omega, \\ -\Delta u = w & \text{on } \Omega, \\ u = g_1 & \text{on } Fr(\Omega), \\ \frac{\partial u}{\partial \eta} = g_2 & \text{on } Fr(\Omega). \end{cases}$$

We prove that if $f \in L^2(\Omega)$, $g_1 \in H^{\frac{3}{2}}(Fr(\Omega))$ and $g_2 \in H^{\frac{1}{2}}(Fr(\Omega))$ then (DP) has one and only one solution in $H^2(\Omega)$.

Theorem 1. *[4] For all reals $0 \leq \nu \leq \mu$ we have*

$$\|u\|_{H^\mu(\Omega)} \leq cN^{2(\mu-\nu)} \|u\|_{H^\nu(\Omega)}, \qquad \forall u \in V_N^d(\Omega).$$

Theorem 2. *[5] Define the space W^0 as follows:*

$$W^0 = \left\{(v,\, \psi) \in H_0^1(\Omega) \times L^2(\Omega),\ \forall \mu \in H^1(\Omega),\ S((v,\, \psi),\, \mu) = 0\right\},$$

where

$$S((v,\, \psi),\, \mu) = \int_\Omega \nabla v \nabla \mu \, dx \; - \int_\Omega \psi \mu \, dx.$$

Then, a mapping $:(v,\, \psi) \in W^0 \to \|\psi\|_{L^2(\Omega)}$ *is a norm on W^0 equivalent to the norm:*

$$(v,\, \psi) \in W^0 \to (|v|^2_{H^1(\Omega)} + \|\psi\|^2_{L^2(\Omega)})^{1/2}.$$

Further

$$W^0 = \left\{(v,\, \psi) \in H_0^2(\Omega) \times L^2(\Omega), \quad -\Delta v = \psi\right\}\ .$$

We have

Theorem 3. *[6] Let $\lambda \in H^{-\frac{1}{2}}(Fr(\Omega))$. The problem*

$$\begin{cases} \Delta^2 u = 0 \ \ in\ \Omega, \\ \\ -\Delta u = \lambda \ \ \text{on} \ \ Fr(\Omega), \\ \\ u = 0 \quad \text{on} \ \ Fr(\Omega), \end{cases} \tag{1}$$

has one and only one solution in $H^2(\Omega) \cap H_0^1(\Omega)$, and the linear operator A defined by

$$A\lambda = -\frac{\partial u}{\partial \eta} \ \text{on}\ Fr(\Omega)$$

is an isomorphism from $H^{-\frac{1}{2}}(Fr(\Omega))$ into $H^{\frac{1}{2}}(Fr(\Omega))$. Also, the bilinear form $a(.,.)$ defined by

$$a(\lambda,\, \mu) =< A\lambda,\ \mu >$$

where $< .,\, . >$ is the bilinear form of duality between $H^{-\frac{1}{2}}(Fr(\Omega))$ and $H^{\frac{1}{2}}(Fr(\Omega))$, is continuous, symmetric and $H^{-\frac{1}{2}}(Fr(\Omega))$-elliptic. Namely there exist an $\alpha > 0$ such that

$$a(\mu,\, \mu) \geq \alpha \left\|\mu\right\|^2_{H^{-\frac{1}{2}}(Fr(\Omega))}, \qquad \forall \mu \in H^{-\frac{1}{2}}(Fr(\Omega)).$$

We will reduce the (DP) problem to a variational equation in $H^{-\frac{1}{2}}(Fr(\Omega))$. Indeed, let w_0 and u_0 be solutions of

$$\begin{cases} -\Delta w_0 = f \quad \ \text{in}\ \Omega, \\ \\ w_0 = 0 \qquad \text{on}\ Fr(\Omega); \end{cases} \tag{2}$$

$$\begin{cases} -\Delta u_0 = w_0 \quad \text{in}\ \Omega, \\ \\ u_0 = g_1 \quad \text{on}\ Fr(\Omega). \end{cases} \tag{3}$$

We have then

Theorem 4. *[4] Let u be a solution of (DP). The trace λ of $-\Delta u$ on $Fr(\Omega)$ is an unique solution of variational equation*

$$< A\lambda, \, \mu > = < \frac{\partial u_0}{\partial \eta} - g_2, \, \mu > \ , \quad \forall \mu \in H^{-\frac{1}{2}}(Fr(\Omega)), \ \ \lambda \in H^{-\frac{1}{2}}(Fr(\Omega)).$$

Remark 1. *To determine $\frac{\partial u_0}{\partial \eta}$ we resolve two Dirichlet problems (2), (3), and we resolve more again two Dirichlet problems to obtain the couple (u, w).*

3. Main Results

3.1. Approximation of (DP) by Spectral Method

We consider the following finite dimension space:

$$V_N = Span\{L_0, L_1, ..., L_N\}$$

where $L_k(x)$ are Legendre polynomials. Let

$$V_N^0 = \left\{ v_N \in V_N \ : \ v_{N_{|Fr(\Omega)}} = 0 \right\}$$

and let $\mathcal{M}_N \subset V_N$ be such that $V_N = V_N^0 \oplus \mathcal{M}_N$. We introduce the following spaces :

$$W_N = \left\{ \begin{array}{l} (v_N, \, \psi_N) \in V_N \times V_N \, , \ v_{N_{|Fr(\Omega)}} = g_1, \qquad \text{and} \\ \\ \int_\Omega \nabla v_N \nabla \mu_N \, dx = \int_\Omega \psi_N \mu_N \, dx + \int_{Fr(\Omega)} g_2 \mu_N \, d\sigma, \quad \forall \mu_N \in V_N \end{array} \right\};$$

$$W_N^0 = \left\{ (v_N, \, \psi_N) \in V_N^0 \times V_N \, , \ \int_\Omega \nabla v_N \nabla \mu_N \, dx \ - \int_\Omega \psi_N \mu_N \, dx = 0 \, , \ \ \forall \mu_N \in V_N \right\}.$$

3.1.1. Choice of V_N^0 and Appropriate Basis for Galerkin Method

If we have a nonhomogenous Dirichlet problem associated to the operator (Δ) and posed on the space V_N, we can define one transformation for which this problem will be equivalent to one posed homogenous problem on V_N^0.

The Galerkin approach consists plainly to replace the test functions space by polynomials space.

The effectiveness of numerical method to resolve the linear system $Au = F$ wich has been given in the abstract form will be subordinate to :

i) the way from which space $V_N^0(\Omega)$ approaches the space V;

ii) steepness and the simpleness calculus of coefficients a_{ij} and F_j;

iii) steepness to resolve a linear system $Au = F$.

To satisfy the 1st criterion i), we will consider the space $V_N^0(\Omega)$ of enough large dimension.

To satisfy the 2nd and 3rd critera ii) and iii), it will require obtain one sufficiently deep matrix A such that the linear system $Au = F$ does not enough at cost (in time and required space machine), and such that there are not coefficients a_{ij} to compute.

What is to be done in the choice of a basis of $V_N^0(\Omega)$ such that the linear system to resolve will be possible ?

To answer this question, we need the following lemma :

Lemma 1. *[6] Put*

$$c_k = \frac{1}{\sqrt{4k+6}}, \qquad \phi_k(x) = c_k(L_k(x) - L_{k+2}(x))$$

$$a_{jk} = (\phi_k'(x),\, \phi_j'(x)) \quad , \qquad b_{jk} = (\phi_k(x),\, \phi_j(x)), \quad k,\, j = 0, 1, ..., N-2.$$

Then,

$$a_{jk} = \begin{cases} 1 & \text{if} \quad k = j, \\ 0 & \text{if} \quad k \neq j, \end{cases}$$

$$b_{kj} = b_{jk} = \begin{cases} c_k\, c_j\, (\frac{2}{2j+1} + \frac{2}{2j+5}), & \text{if} \quad k = j, \\ -c_k\, c_j, & \text{if} \quad k = j+2, \\ 0 & \text{else}, \end{cases}$$

and

$$V_N^0(I) = Span\,\{\phi_0(x),\, \phi_1(x), ..., \phi_{N-2}(x)\}\,,$$

where $(.,.)$ *is an* L^2*-inner product.*

3.1.2. Choice of $\mathcal{M}_N$

We select $\mathcal{M}_N \subset V_N$ such that $V_N = V_N^0 \oplus \mathcal{M}_N$, where

$$V_N^0 = Span\,\{\phi_0,\, \phi_1, ..., \phi_{N-2}\}\,.$$

Let ϕ_{-1}, ϕ_{N-1} be two functions such that $\phi_{-1},\ \phi_{N-1} \in V_N$ and ϕ_{-1}, ϕ_0, $\phi_1, ..., \phi_{N-2},\ \phi_{N-1}$ are linearly independent.

Proposition 1. *For $d = 1$, the space M_N is given by*

$$\mathcal{M}_N = Span\left\{\phi_{-1}(x),\ \phi_{N-1}(x)\right\}.$$

For $d = 2$, the space M_N is given by

$$\mathcal{M}_N = Span\left\{\begin{array}{ll} \phi_{-1}(x)\phi_j(y),\ \phi_{N-1}(x)\phi_j(y), & -1 \le j \le N-1 \\ \\ \phi_i(x)\phi_{-1}(y),\ \phi_i(x)\phi_{N-1}(y), & 0 \le i \le N-2 \end{array}\right\}.$$

For $d = 3$, the space M_N is given by

$$\mathcal{M}_N = Span\left\{\begin{array}{ll} \phi_i(x)\phi_j(y)\phi_{-1}(z),\ \phi_i(x)\phi_j(y)\phi_{N-1}(z), & -1 \le i,\ j \le N-1; \\ \\ \phi_i(x)\phi_{-1}(y)\phi_k(z),\ \phi_i(x)\phi_{N-1}(y)\phi_k(z), & -1 \le i \le N-1, \\ & \quad 0 \le k \le N-2; \\ \\ \phi_{-1}(x)\phi_j(y)\phi_k(z),\ \phi_{N-1}(x)\phi_j(y)\phi_k(z), & 0 \le j,\ k \le N-2 \end{array}\right\}.$$

Remark 2. *The choice of functions ϕ_{-1}, ϕ_{N-1} is not unique, then the choice of space M_N is not unique.*

Now, we give some choices of ϕ_{-1}, ϕ_{N-1}:

Remark 3. *i)* $\phi_{-1} = L_0,\ \phi_{N-1} = L_1$
ii) $\phi_{-1} = L_0,\ \phi_{N-1} = L_{N-1} - L_1$
iii) $\phi_{-1} = L_1 - L_0,\ \phi_{N-1} = L_{N-1}$.

Numerical tests are proofs to the good choice.

We approache then (DP) by

$$(DP)_N \qquad \left\{\begin{array}{l} \text{Find } (u_N, \varphi_N) \in W_N \quad \text{such that} \\ \\ j_N(u_N,\ \varphi_N) \le j_N(v_N,\ \psi_N),\ \forall (v_N,\ \psi_N) \in W_N, \end{array}\right.$$

where

$$j_N(v_N,\ \psi_N) = \frac{1}{2}\int_\Omega |\psi_N|^2\, dx - \int_\Omega f v_N\, dx.$$

This problem has one and only one solution.

4. Convergence Results

Consider the following problem:

$$(DP)_0 \qquad \begin{cases} \Delta^2 u = g & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad Fr(\Omega), \\ \frac{\partial u}{\partial \eta} = 0 & \text{on} \quad Fr(\Omega), \end{cases}$$

which is equivalent to the following optimization problem:

$$(DP)_1 \quad \begin{cases} \text{Find } \ u \in H_0^2(\Omega) \ \text{ such that} \\ J_0(u) \leq J_0(v), \quad \forall v \in H_0^2(\Omega), \end{cases}$$

where

$$J_0(v) = \frac{1}{2}\int_\Omega |\Delta v|^2 \, dx \; - \int_\Omega gvdx.$$

Now, we prove some results of convergence.

Lemma 2. *Let u be a solution of problem $(DP)_0$. Then there exist a constant $c > 0$ independent of N such that*

$$|u - u_N|_{H^1(\Omega)} + \|\Delta u + \varphi_N\|_{L^2(\Omega)} \leq c(\inf_{(v_N,\, \psi_N) \in W_N^0} (|u - v_N|_{H^1(\Omega)} + \|\Delta u + \psi_N\|_{L^2(\Omega)})$$

$$+ \inf_{\mu_N \in V_N} \|\Delta u + \mu_N\|_{H^1(\Omega)}).$$

Proof. Let u be a solution of problem $(DP)_0$, then

$$-\int_\Omega \nabla v \nabla\left(\Delta u\right) dx = \int_\Omega \Delta u \Delta v dx = \int_\Omega gvdx, \quad \forall v \in \mathcal{D}(\Omega)$$

but $\mathcal{D}(\Omega)$ is dense in $H_0^1(\Omega)$, then

$$-\int_\Omega \nabla v \nabla(\Delta u) dx = \int_\Omega gvdx, \quad \forall v \in H_0^1(\Omega).$$

We have

$$S((v,\, \psi),\, \mu) = \int_\Omega \nabla v \nabla \mu dx \; - \int_\Omega \psi \mu dx,$$

and

$$S((v,\,\psi),-\Delta u)=-\int_\Omega \nabla v\nabla(\Delta u)dx\ +\int_\Omega \psi\Delta u dx=\int_\Omega g v dx\ +\int_\Omega \psi\Delta u dx.$$

Let $(v_N,\,\psi_N)\in W_N^0$ and $\mu_N\in V_N$, then

$$\begin{aligned}
S((u_N-v_N,\varphi_N-\psi_N),\Delta u+\mu_N) &= \textstyle\int_\Omega \nabla(u_N-v_N)\nabla(\Delta u+\mu_N)dx \\
&\quad -\textstyle\int_\Omega(\varphi_N-\psi_N)(\Delta u+\mu_N)dx \\
&= S((u_N-v_N,\,\varphi_N-\psi_N),\,\Delta u)+S((u_N-v_N,\,\varphi_N-\psi_N),\,\mu_N) \\
&= -\textstyle\int_\Omega g(u_N-v_N)dx\ -\int_\Omega(\varphi_N-\psi_N)\Delta u dx \\
&\quad +\textstyle\int_\Omega \nabla(u_N-v_N)\nabla\mu_N dx\ -\int_\Omega(\varphi_N-\psi_N)\mu_N dx \\
&= -\textstyle\int_\Omega(\varphi_N-\psi_N)\Delta u dx\ -\int_\Omega g u_N dx\ +\int_\Omega g v_N dx \\
&\quad +\textstyle\int_\Omega \nabla u_N\nabla\mu_N dx-\int_\Omega \nabla v_N\nabla\mu_N dx\ -\int_\Omega \varphi_N\mu_N dx+\int_\Omega \psi_N\mu_N dx.
\end{aligned}$$

Or

$$\int_\Omega \nabla u_N\nabla\mu_N dx=\int_\Omega \varphi_N\mu_N dx\ \text{ and }\ \int_\Omega \varphi_N\psi_N dx=\int_\Omega g v_N dx,$$

then

$$\int_\Omega(\varphi_N-\psi_N)(\Delta u+\varphi_N)dx=-S((u_N-v_N,\,\varphi_N-\psi_N),\,\Delta u+\mu_N)$$

By continuity of $S\left((.,.)\,,.\right)$, we have

$$\begin{aligned}
\left|\textstyle\int_\Omega(\varphi_N-\psi_N)(\Delta u+\varphi_N)dx\right| &\le c(|u_N-v_N|_{H^1(\Omega)}\left\|\Delta u+\mu_N\right\|_{H^1(\Omega)} \\
&\quad +\left\|\varphi_N-\psi_N\right\|_{L^2(\Omega)}\left\|\Delta u+\mu_N\right\|_{H^1(\Omega)})
\end{aligned}$$

Using Theorem 2, it holds,

$$|v|_{H^1(\Omega)}\le c(\Omega)\left\|\psi\right\|_{L^2(\Omega)}$$

and

$$\left|\int_\Omega(\varphi_N-\psi_N)(\Delta u+\varphi_N)dx\right| \le c(c(\Omega)\left\|\varphi_N-\psi_N\right\|_{L^2(\Omega)}\left\|\Delta u+\mu_N\right\|_{H^1(\Omega)} + \left\|\varphi_N-\psi_N\right\|_{L^2(\Omega)}\left\|\Delta u+\mu_N\right\|_{H^1(\Omega)}) \le d(\Omega)\left\|\varphi_N-\psi_N\right\|_{L^2(\Omega)}\left\|\Delta u+\mu_N\right\|_{H^1(\Omega)},$$

where $d(\Omega)=\max(cc(\Omega),\, c)$. Thus

$$\left\|\varphi_N-\psi_N\right\|^2_{L^2(\Omega)} = \int_\Omega(\varphi_N-\psi_N)(\Delta u+\varphi_N)dx - \int_\Omega(\varphi_N-\psi_N)(\Delta u+\psi_N)dx$$

$$\left\|\varphi_N-\psi_N\right\|^2_{L^2(\Omega)} \le d(\Omega)\left\|\varphi_N-\psi_N\right\|_{L^2(\Omega)}\left\|\Delta u+\mu_N\right\|_{H^1(\Omega)} + \left\|\varphi_N-\psi_N\right\|_{L^2(\Omega)}\left\|\Delta u+\psi_N\right\|_{L^2(\Omega)}.$$

It holds that

$$\left\|\varphi_N-\psi_N\right\|_{L^2(\Omega)} \le d(\Omega)\left\|\Delta u+\mu_N\right\|_{H^1(\Omega)} + \left\|\Delta u+\psi_N\right\|_{L^2(\Omega)} \tag{4}$$

and

$$|u-u_N|_{H^1(\Omega)} + \left\|\Delta u+\varphi_N\right\|_{L^2(\Omega)} \le |u-v_N|_{H^1(\Omega)} + |v_N-u_N|_{H^1(\Omega)} + \left\|\Delta u+\psi_N\right\|_{L^2(\Omega)} + \left\|\varphi_N-\psi_N\right\|_{L^2(\Omega)}$$
$$\le |u-v_N|_{H^1(\Omega)} + \left\|\Delta u+\psi_N\right\|_{L^2(\Omega)} + (1+c(\Omega))\left\|\varphi_N-\psi_N\right\|_{L^2(\Omega)}.$$

Using inequality (4), we obtain

$$|u-u_N|_{H^1(\Omega)} + \left\|\Delta u+\varphi_N\right\|_{L^2(\Omega)} \le (|u-v_N|_{H^1(\Omega)} + \left\|\Delta u+\psi_N\right\|_{L^2(\Omega)} + (1+c(\Omega)))\cdot(d(\Omega)\left\|\Delta u+\mu_N\right\|_{H^1(\Omega)} + \left\|\Delta u+\psi_N\right\|_{L^2(\Omega)}),$$

hence

$$|u-u_N|_{H^1(\Omega)} + \left\|\Delta u+\varphi_N\right\|_{L^2(\Omega)} \le |u-v_N|_{H^1(\Omega)} + (2+c(\Omega))\left\|\Delta u+\psi_N\right\|_{L^2(\Omega)} + (1+c(\Omega))d(\Omega)\left\|\Delta u+\mu_N\right\|_{H^1(\Omega)}$$
$$\le c^*(|u-v_N|_{H^1(\Omega)} + \left\|\Delta u+\psi_N\right\|_{L^2(\Omega)} + \left\|\Delta u+\mu_N\right\|_{H^1(\Omega)}),$$

where

$$c^* = \max(2 + c(\Omega), (1 + c(\Omega))d(\Omega)).$$

Therefore, it holds

$$|u - u_N|_{H^1(\Omega)} + \|\Delta u + \varphi_N\|_{L^2(\Omega)} \leq c^*\Big(\inf_{(v_N,\psi_N)\in W_N^0}(|u - v_N|_{H^1(\Omega)} + \|\Delta u + \psi_N\|_{L^2(\Omega)})$$

$$+ \inf_{\mu_N\in V_N} \|\Delta u + \mu_N\|_{H^1(\Omega)}\Big).$$

Lemma 3. *Let $u \in H^2(\Omega)$ be a solution of optimization problem $(DP)_0$. If $u \in H^{k+2}(\Omega)$, $k \geq 2$, then there exist a constant $c > 0$ independent of N such that*

$$|u - u_N|_{H^1(\Omega)} + \|\Delta u + \varphi_N\|_{L^2(\Omega)} \leq cN^{1-k}(|u|_{H^{k+2}(\Omega)} + |\Delta u|_{H^k(\Omega)})$$

Proof. Let $(v_N, \psi_N) \in W_N^0$ and $\mu_N \in V_N$. Put

$$\nu_N = \mu_N + \psi_N, \ \ \nu_N \in V_N.$$

Then, we have

$$S((v_N, \psi_N), \nu_N) = 0, \tag{5}$$

$$\int_\Omega \Delta u \nu_N \, dx = -\int_\Omega \nabla u \nabla \nu_N \, dx \ + \int_{Fr(\Omega)} \frac{\partial u}{\partial \eta}\nu_N \, d\sigma$$

Or $\dfrac{\partial u}{\partial \eta} |_{Fr(\Omega)}= 0$, so one has

$$\int_\Omega \Delta u \nu_N dx = -\int_\Omega \nabla u \nabla \nu_N dx. \tag{6}$$

By (5) and (6), we have

$$\int_\Omega (\Delta u + \psi_N)\nu_N \, dx = \int_\Omega \nabla(v_N - u)\nabla \nu_N \, dx$$

$$\left|\int_\Omega (\Delta u + \psi_N)\nu_N \, dx\right| \leq |u - v_N|_{H^1(\Omega)} |\nu_N|_{H^1(\Omega)} .$$

By Theorem 1, we obtain

$$|\nu_N|_{H^1(\Omega)} \leq cN^2 \|\nu_N\|_{L^2(\Omega)}$$

so

$$\left| \int_{\Omega} (\Delta u + \psi_N)\nu_N \, dx \right| \leq cN^2 \left| u - v_N \right|_{H^1(\Omega)} \left\| \nu_N \right\|_{L^2(\Omega)}$$

And

$$\begin{aligned} \left\| \nu_N \right\|^2_{L^2(\Omega)} &= \textstyle\int_{\Omega} (\mu_N - \Delta u)\nu_N \, dx + \int_{\Omega} (\Delta u + \psi_N)\nu_N \, dx \\ &\leq \left\| \Delta u - \mu_N \right\|_{L^2(\Omega)} \left\| \nu_N \right\|_{L^2(\Omega)} + cN^2 \left| u - v_N \right|_{H^1(\Omega)} \left\| \nu_N \right\|_{L^2(\Omega)} \\ &\Rightarrow \left\| \nu_N \right\|_{L^2(\Omega)} \leq \left\| \Delta u - \mu_N \right\|_{L^2(\Omega)} + cN^2 \left| u - v_N \right|_{H^1(\Omega).} \end{aligned}$$

Hence

$$\left\| \Delta u + \psi_N \right\|_{L^2(\Omega)} = \left\| \Delta u + \nu_N - \mu_N \right\|_{L^2(\Omega)}$$

$$\leq \left\| \Delta u - \mu_N \right\|_{L^2(\Omega)} + \left\| \nu_N \right\|_{L^2(\Omega)} \leq 2 \left\| \Delta u - \mu_N \right\|_{L^2(\Omega)} + cN^2 \left| u - v_N \right|_{H^1(\Omega)} .$$

Then

$$\left| u - v_N \right|_{H^1(\Omega)} + \left\| \Delta u + \psi_N \right\|_{L^2(\Omega)} \leq (1 + cN^2) \left| u - v_N \right|_{H^1(\Omega)} + 2 \left\| \Delta u - \mu_N \right\|_{L^2(\Omega)} ,$$

$$\forall \, (v_N, \psi_N) \in W_N^0 , \, \forall \mu_N \in V_N.$$

Thus

$$\inf_{(v_N,\psi_N) \in W_N^0} (\left| u - v_N \right|_{H^1(\Omega)} + \left\| \Delta u + \psi_N \right\|_{L^2(\Omega)}) \leq (1 + cN^2) \inf_{v_N \in V_N^0} \left| u - v_N \right|_{H^1(\Omega)}$$

$$+2 \inf_{\mu_N \in V_N} \left\| \Delta u - \mu_N \right\|_{L^2(\Omega)} .$$

Lemma 2 implies

$$\left| u - u_N \right|_{H^1(\Omega)} + \left\| \Delta u + \varphi_N \right\|_{L^2(\Omega)} \leq c((1 + cN^2) \inf_{v_N \in V_N^0} \left| u - v_N \right|_{H^1(\Omega)}$$

$$+ \inf_{\mu_N \in V_N} \left\| \Delta u - \mu_N \right\|_{H^1(\Omega)}).$$

Further, we have

$$\inf_{v_N \in V_N^0} \left| u - v_N \right|_{H^1(\Omega)} \leq cN^{-1-k} \left| u \right|_{H^{k+2}(\Omega)}$$

$$\inf_{\mu_N \in V_N} \left\| \Delta u + \mu_N \right\|_{H^1(\Omega)} \leq cN^{1-k} \left| \Delta u \right|_{H^k(\Omega)} ,$$

then, it holds

$$|u-u_N|_{H^1(\Omega)} + \|\Delta u+\varphi_N\|_{L^2(\Omega)} \le c((1+cN^2)cN^{-1-k}\,|u|_{H^{k+2}(\Omega)} + cN^{1-k}\,|\Delta u|_{H^k(\Omega)})$$

$$\le cN^{1-k}(|u|_{H^{k+2}(\Omega)} + |\Delta u|_{H^k(\Omega)}).$$

Theorem 5. *Le $u \in H^2(\Omega)$ be a solution of a problem (DP), then for all integer $k \ge 2$, if $u \in H^{k+2}(\Omega)$ we have*

$$\|u-u_N\|_{H^1(\Omega)} + \|\Delta u+\varphi_N\|_{L^2(\Omega)} \le cN^{1-k}\,\|u\|_{H^{k+2}(\Omega)}$$

If $u \in H^{k+1}(\Omega),\ k \ge 3,$ then

$$\|u-u_N\|_{L^2(\Omega)} + \frac{1}{N^2}\,\|\Delta u+\varphi_N\|_{L^2(\Omega)} \le cN^{-1-k}\,\|u\|_{H^{k+1}(\Omega)}$$

Proof. We will reduce the problem $(PD)_N$ to one variational problem in $\mathcal{M}_N$. Indeed, let $a_N(.,.) : \mathcal{M}_N \times \mathcal{M}_N \to \mathbb{R}$ be a bilinear form defined for $\lambda_N \in \mathcal{M}_N$ by

$$\int_\Omega \nabla w_N \nabla v_N dx = 0\ ,\ \ \forall v_N \in V_N^0\ \ ,\quad w_N - \lambda_N \in V_N^0; \tag{7}$$

$$\int_\Omega \nabla u_N \nabla v_N dx = \int_\Omega w_N v_N dx\ ,\ \ \forall v_N \in V_N^0\ ,\ u_N \in V_N^0; \tag{8}$$

$$a_N(\lambda_N,\, \mu_N) = \int_\Omega w_N \mu_N dx - \int_\Omega \nabla u_N \nabla \mu_N dx, \quad \forall \mu_N \in \mathcal{M}_N. \tag{9}$$

Lemma 4. *A bilinear form $a_N(.,.)$ is positive definite and symmetric on $\mathcal{M}_N \times \mathcal{M}_N$. Moreover*

$$a_N(\lambda_{1N}, \lambda_{2N}) = \int_\Omega w_{1N} w_{2N} dx,\ \ \forall\, \lambda_{1N}, \lambda_{2N} \in \mathcal{M}_N,$$

where w_{1N}, w_{2N} are solutions of (7) associated to $\lambda_{1N},\ \lambda_{2N}$.

Proof. By definition one has

$$a_N(\lambda_{1N},\, \lambda_{2N}) = \int_\Omega w_{1N} \lambda_{2N}\, dx\ - \int_\Omega \nabla u_{1N} \nabla \lambda_{2N}\, dx, \quad \forall \mu_N \in \mathcal{M}_N,$$

where w_{1N}, w_{2N} are solutions of (7) and (8) respectively.

Put $\lambda_{2N} = (\lambda_{2N} - w_{2N}) + w_{2N}$, then

$$a_N(\lambda_{1N},\, \lambda_{2N}) = \int_\Omega w_{1N} w_{2N}\, dx \; - \int_\Omega \nabla u_{1N} \nabla w_{2N}\, dx + \int_\Omega \nabla u_{1N} \nabla (w_{2N} - \lambda_{2N})\, dx$$
$$- \int_\Omega w_{1N} \nabla (w_{2N} - \lambda_{2N})\, dx$$

One has $u_{1N} \in V_N^0$, using (7) one has

$$\int_\Omega \nabla u_{1N} \nabla (w_{2N} - \lambda_{2N})\, dx = \int_\Omega w_{1N} \nabla (w_{2N} - \lambda_{2N})\, dx.$$

Then

$$a_N(\lambda_{1N},\, \lambda_{2N}) = \int_\Omega w_{1N} w_{2N}\, dx, \quad \forall\, \lambda_{1N},\; \lambda_{2N} \in \mathcal{M}_N.$$

It is obvious that $a_N(.,.)$ is symmetric and coercive.

Theorem 6. *Let $(u_N,\, w_N)$ be a solution of $(DP)_N$ and λ_N the component of w_N in $\mathcal{M}_N$. Then λ_N is an unique solution of the linear variational problem:*

$$a_N(\lambda_N,\, \mu_N) = \int_\Omega \nabla u_{0N} \nabla \mu_N\, dx \; - \int_\Omega w_{0N} \mu_N\, dx \; - \int_{Fr(\Omega)} g_{2N} \mu_N\, d\sigma, \qquad \forall \mu_N \in \mathcal{M}_N \;,\quad \lambda_N \in \mathcal{M}_N \tag{10}$$

where w_{0N} is a solution of

$$\int_\Omega \nabla w_{0N}\, \nabla v_N dx = \int_\Omega f\, v_N\, dx \;,\;\; \forall v_N \in V_N^0 \;,\;\; w_{0N} \in V_N^0 \tag{11}$$

and u_{0N} is a solution of

$$\begin{cases} \int_\Omega \nabla u_{0N}\, \nabla v_N dx = \int_\Omega w_{0N}\; v_N\, dx \;,\;\; \forall v_N \in V_N^0 \;, \\ u_{0N} = g_1 \qquad \text{on} \qquad Fr(\Omega). \end{cases} \tag{12}$$

Proof. We have

$$a_N(\lambda_N,\, \mu_N) = \int_\Omega \bar{w}_N \mu_N\, dx - \int_\Omega \nabla \bar{u}_N \nabla \mu_N\, dx$$

$$= \int_\Omega (w_N - w_{0N}) \mu_N\, dx - \int_\Omega \nabla (u_N - u_{0N}) \nabla \mu_N\, dx$$

$$= \int_\Omega \nabla u_{0N} \nabla \mu_N dx \; - \int_\Omega w_{0N} \mu_N dx - \left(\int_\Omega \nabla u_N \nabla \mu_N\, dx - \int_\Omega w_N \mu_N\, dx\right).$$

But $(u_N, w_N) \in W_N$, hence

$$\int_\Omega \nabla u_N \nabla \mu_N \, dx - \int_\Omega w_N \mu_N \, dx = \int_{Fr(\Omega)} g_{2N} \mu_N d\sigma, \qquad \forall \mu_N \in \mathcal{M}_N$$

Therefore

$$a_N(\lambda_N, \, \mu_N) = \int_\Omega \nabla u_{0N} \nabla \mu_N dx - \int_\Omega w_{0N} \mu_N dx - \int_{Fr(\Omega)} g_{2N} \mu_N d\sigma, \quad \forall \mu_N \in \mathcal{M}_N.$$

Existence and uniqueness of a solution of (10) is a direct consequence from Lax-Milgram Lemma. A system is symmetric because a form $a_N(.,.)$ is symmetric.

Remak 4. *The problem* (10) *is equivalent to a system from which associated matrix is positive definite and symmetric.*

Now we prove a problem (10) is equivalent to a system from which associated matrix is positive definite.

5. Numerical Approach

5.1. Construction of Linear System Associated to the Variational Problem (10)

Put

$$l_N(\mu_N) = \int_\Omega \nabla u_{0N} \nabla \mu_N \, dx \; - \int_\Omega w_{0N} \mu_N \, dx \; - \int_{Fr(\Omega)} g_2 \mu_N \, d\sigma, \quad \forall \mu_N \in \mathcal{M}_N$$

Then solve a problem (10) is equivalent to resolve the following system :

$$a_N(\lambda_N, \, \mu_N) = l_N(\mu_N), \quad \forall \mu_N \in \mathcal{M}_N, \;\; \lambda_N \in \mathcal{M}_N \; . \tag{13}$$

Define the set

$$I_d = \begin{cases} \{-1, \, N-1\} & \text{if } d=1 \\ \left\{ \begin{array}{ll} (-1, \, j), \, (N-1, \, j) \, , & -1 \leq j \leq N-1 \\ (i, \, -1), \; (i, \, N-1) \; , & 0 \leq i \leq N-2 \end{array} \right\}, & \text{if } d=2 \\ \left\{ \begin{array}{l} (i, \, j, \, -1), \, (i, \, j, \, N-1) \, , \, -1 \leq i, \, j \leq N-1, \\ (i, \, -1, \, k), \; (i, \, N-1, \, k) \;\; , \; -1 \leq i \leq N-1, 0 \leq k \leq N-2 \\ (-1, \, j, \, k), \; (N-1, \, j, \, k), \quad 0 \leq j, \, k \leq N-2 \end{array} \right\}, & \text{if } d=3, \end{cases}$$

and

$$\phi_{i^d} = \phi_{i_1}\phi_{i_2} \cdot ... \cdot \phi_{i_d} \qquad d = 1,\, 2,\, 3$$

where $i^d = (i_1,\, i_2, ...,\, i_d)$. So (13) is equivalent to

$$\begin{cases} a_N(\lambda_N, \phi_{i^d}) = l_N(\phi_{i^d})\,, \;\; i^d \in I_d \\ \\ \lambda_N = \sum\limits_{j^d \in I_d} \lambda_{j^d}\phi_{j^d}. \end{cases} \tag{14}$$

Namely

$$\sum_{j^d \in I_d} \lambda_{j^d} a_N(\phi_{j^d}, \phi_{k^d}) = l_N(\phi_{k^d}), \qquad k^d \in I_d \tag{15}$$

and

$$p = Card(I_d) = \dim(\mathcal{M}_N) = \begin{cases} 2, & \text{if} \quad d = 1, \\ \\ 4N, & \text{if} \quad d = 2, \\ \\ 6N^2 + 2, & \text{if} \quad d = 3. \end{cases}$$

5.2. Computation of the Matrix A_N

Denote $\mathcal{B}_N$ the basis of $\mathcal{M}_N$ and $w_{j^dN},\, u_{j^dN}$ solutions respectively of

$\int_\Omega \nabla w_{j^dN} \nabla v_N dx = 0, \quad \forall v_N \in V_N^0, \qquad w_{j^dN} \in V_N\,,\; w_{j^dN} - \phi_{j^d} \in V_N^0\,;$

$\int_\Omega \nabla u_{j^dN} \nabla v_N dx = \int_\Omega w_{j^dN} v_N dx, \quad \forall v_N \in V_N^0\,, \qquad u_{j^dN} \in V_N^0,$

further we have

$$a_{i^dj^d} = a_N(\phi_{j^d}, \phi_{i^d}) = \int_\Omega w_{j^dN}\phi_{i^d} dx - \int_\Omega \nabla u_{j^dN} \nabla \phi_{i^d} dx, \qquad i^d,\, j^d \in I_d.$$

It results that a computation of a matrix A_N requires the computation of $2p$ Dirichlet problems for an operator Δ.

We have

Theorem 7. *For all integer $N \geq 1$, we have*

$$\alpha \frac{1}{N} \|\gamma_0 \lambda_N\|^2_{L^2(Fr(\Omega))} \leq a_N(\lambda_N,\, \lambda_N) \leq \beta \|\gamma_0 \lambda_N\|^2_{L^2(Fr(\Omega))}\,, \quad \forall \lambda_N \in \mathcal{M}_N,$$

where α and β are independents of N and λ_N.

Proof. Let $\lambda_N \in \mathcal{M}_N$, we have

$$a_N(\lambda_N,\, \lambda_N) = \int_\Omega w_N^2\, dx,$$

where w_N is a solution of (7). Let $\tilde{w}_N \in H^1(\Omega)$ be a solution of

$$\begin{cases} \Delta \tilde{w}_N = 0 \quad \text{in } \Omega, \\ \\ \tilde{w}_N = \lambda_N \quad \text{on } Fr\,(\Omega)\,, \end{cases}$$

then one has

$$\sqrt{a_N(\lambda_N,\, \lambda_N)} = \|w_N\|_{L^2(\Omega)}$$

$$\leq \|w_N - \tilde{w}_N\|_{L^2(\Omega)} + \|\tilde{w}_N\|_{L^2(\Omega)} = \|w_N - \tilde{w}_N\|_{L^2(\Omega)} + \sqrt{< A\gamma_0\lambda_N,\, \gamma_0\lambda_N >}.$$

Put $|A| = \|A\|_{\mathcal{L}c(L^2(Fr(\Omega)),L^2(Fr(\Omega)))}$, it holds that

$$|< A\lambda,\, \mu >| \leq |A|\, \|\lambda\|_{L^2(Fr(\Omega))} \|\mu\|_{L^2(Fr(\Omega))}\,, \quad \forall\, \lambda,\, \mu \in L^2(Fr(\Omega))$$

$$\Rightarrow \sqrt{a_N(\lambda_N,\, \lambda_N)} \leq \|w_N - \tilde{w}_N\|_{L^2(\Omega)} + \sqrt{|A|}\, \|\gamma_0\lambda_N\|_{L^2(Fr(\Omega))}\,.$$

and we have

$$\|w_N - \tilde{w}_N\|_{L^2(\Omega)} \leq cN^{-\frac{3}{2}}\, \|\tilde{w}_N\|_{H^{\frac{3}{2}}(\Omega)}\,,$$

and

$$\|\tilde{w}_N\|_{H^{\frac{3}{2}}(\Omega)} \leq c\, \|\lambda_N\|_{H^1(Fr(\Omega))}$$

$$\|\lambda_N\|_{H^1(Fr(\Omega))} \leq cN\, \|\lambda_N\|_{L^2(Fr(\Omega))}\,.$$

Namely

$$\sqrt{a_N(\lambda_N,\, \lambda_N)} \leq c\, \|\gamma_0\lambda_N\|_{L^2(Fr(\Omega))}\,.$$

And

$$c\frac{1}{N}\, \|\gamma_0 v_N\|^2_{L^2(Fr(\Omega))} \leq \|v_N\|^2_{L^2(\Omega)}\,, \quad \forall v_N \in V_N.$$

It holds that

$$a_N(\lambda_N,\, \lambda_N) = \|w_N\|^2_{L^2(\Omega)} \geq c\frac{1}{N}\, \|\gamma_0 w_N\|^2_{L^2(Fr(\Omega))} = c\frac{1}{N}\, \|\gamma_0\lambda_N\|^2_{L^2(Fr(\Omega))}\,.$$

Proposition 2. (*Condition Number of the Matrix A_N*) *We have*

$$cond(A) = O(N),$$

where A is a square matrix associated to the problem (10).

Proof. Let A be a square inversible matrix $N \times N$. We have

$$cond(A) = \|A\| \, . \left\|A^{-1}\right\|,$$

where $\|.\|$ is a matrix norm associated to canonical euclidean norm of $\mathbb{R}^N$. If A is positive definite and symmetric then

$$cond(A) = \frac{\lambda_{\max}}{\lambda_{\min}},$$

where $\lambda_{\max}$ is the largest eigenvalue and $\lambda_{\min}$ is the smallest eigenvalue of A.

By Theorem 7, there exist two constants α and $\beta > 0$ such that

$$\alpha\frac{1}{N}\left\|\gamma_0\lambda_N\right\|^2_{L^2(Fr(\Omega))} \leq a_N(\lambda_N,\, \lambda_N) \leq \beta\left\|\gamma_0\lambda_N\right\|^2_{L^2(Fr(\Omega))}, \quad \forall\lambda_N \in \mathcal{M}_N.$$

Put

$$p_{\max} = \sup_{\lambda_N\in\mathcal{M}_N-\{0\}} \frac{a_N(\lambda_N,\lambda_N)}{\|\gamma_0\lambda_N\|^2_{L^2(Fr(\Omega))}}$$

$$p_{\min} = \inf_{\lambda_N\in\mathcal{M}_N-\{0\}} \frac{a_N(\lambda_N,\lambda_N)}{\|\gamma_0\lambda_N\|^2_{L^2(Fr(\Omega))}},$$

then we have

$$p_{\max} \leq \beta \quad \text{and} \quad p_{\min} \geq \alpha\frac{1}{N},$$

where $p_{\max}$ is the largest eigenvale and $p_{\min}$ is the smallest eigenvalue of A_N. Therefore, we have

$$\frac{1}{p_{\min}} \leq \frac{N}{\alpha} \Rightarrow \frac{p_{\max}}{p_{\min}} \leq cN \Rightarrow cond(A) = \bigcirc(N)$$

5.3. Algorithm (Conjugate Gradient Method Applied to Variational Problem (10)

We introduce the isomorphism $r_N : \mathcal{M}_N \to \mathbb{R}^p$ defined as follows :

$$\mu_N \in \mathcal{M}_N\,,\ \mu_N = \sum_{i=1}^{p}\mu_i\varphi_i,$$

$$r_N = \{\mu_1,\, \mu_2, ...,\, \mu_P\}\,, \quad \forall \mu_N \in \mathcal{M}_N.$$

Then

$$a_N(\lambda_N,\, \mu_N) = (A_N r_N \lambda_N,\, r_N \mu_N)_N, \quad \forall \lambda_N,\ \mu_N \in \mathcal{M}_N$$

and

$$\int_\Omega \nabla u_{0N} \nabla \mu_N \, dx - \int_\Omega w_{0N} \mu_N dx - \int_{Fr(\Omega)} g_{2N} \mu_N d\sigma = (b_N,\, r_N \mu_N)_N, \ \ \forall \mu_N \in \mathcal{M}_N,$$

where $(.,.)_N$ is the inner euclidien scalar of $\mathbb{R}^p$ and $\|.\|_N$ is the associated norm.

Algorithm :

Step 1: $k = 0$, let $\lambda_N^0 \in \mathcal{M}_N$ be an arbitrary initial data.

$$g_N^0 = A_N r_N \lambda_N^0 - b_N \ , \quad d_N^0 = g_N^0$$

Step 2 : If $\|g_N^0\|_N \leq \varepsilon$ stop, else put

$$\rho_n = \tfrac{(d_N^n, g_N^n)_N}{(A_N d_N^n, d_N^n)_N}$$

$$r_N \lambda_N^{n+1} = r_N \lambda_N^n - \rho_n d_N^n$$

$$g_N^{n+1} = g_N^n - \rho_n A_N d_N^n$$

and

$$\beta_n = \tfrac{(g_N^{n+1},\ g_N^{n+1})_N}{(g_N^n,\, g_N^n)_N}$$

$$d_N^{n+1} = g_N^{n+1} + \beta_n d_N^{n+1}$$

Step 3 : $k \leftarrow k+1$ and return to step 2.

Numerical Results :

Example 1. *Consider the following problem :*

$$\begin{cases} \Delta^2 u = -128\pi^4 \cos(4\pi x) \ \text{ in } \Omega = \left]-1,+1\right[^2, \\ \\ u = 0 \quad \text{ on } \ Fr(\Omega), \\ \\ \dfrac{\partial u}{\partial \eta} = 0 \ \text{ on } \ Fr(\Omega). \end{cases}$$

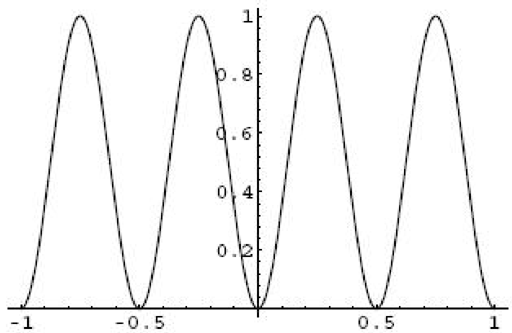

Figure 1: exact solution $u(x) = (\sin(2\pi x))^2$

This problem has one and only one solution : $u(x, y) = (\sin(2\pi x))^2$.

We have:

If we select the 3 rd choice of the space $\mathcal{M}_N$ with

$$\begin{cases} \phi_{-1} = L_1 - L_0, \\ \\ \phi_{N-1} = L_{N-1}, \end{cases}$$

then we have

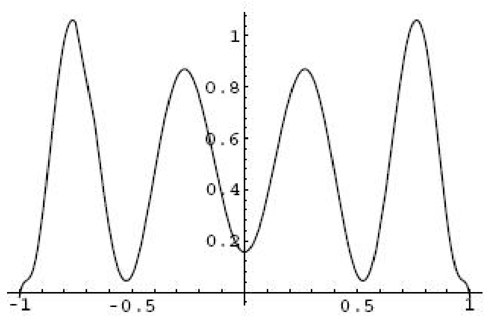

Figure 2: approach solution for N=12

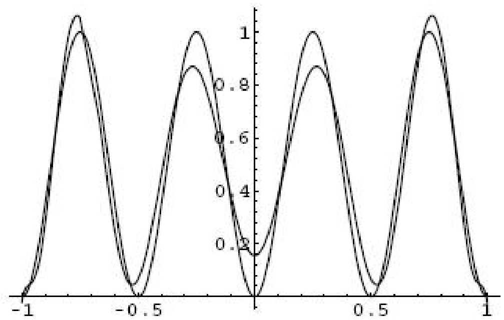

Figure 3: Comparison between exact and approach solutions for N=12

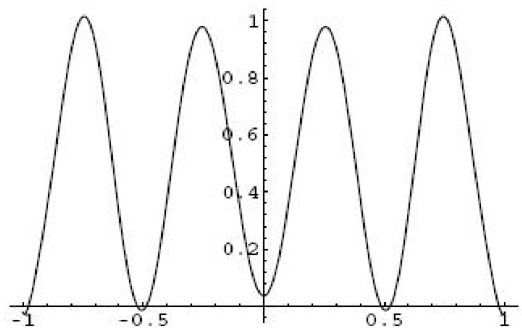

Figure 4: approach solution for N=14

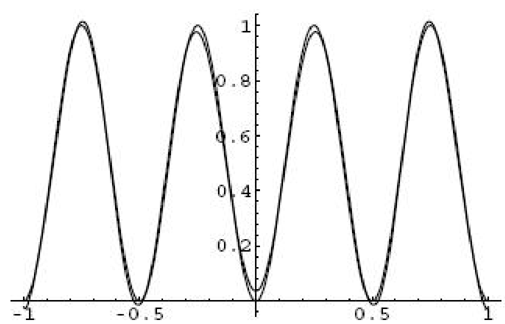

Figure 5: Comparison between exact and approach solutions for N=14

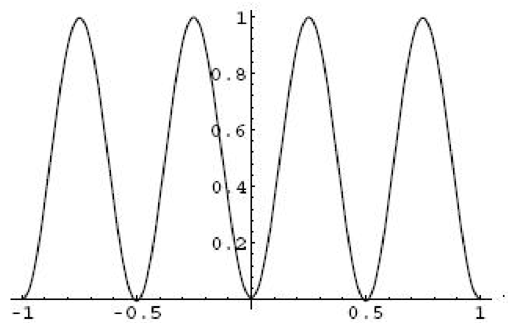

Figure 6: approach solution for N=16

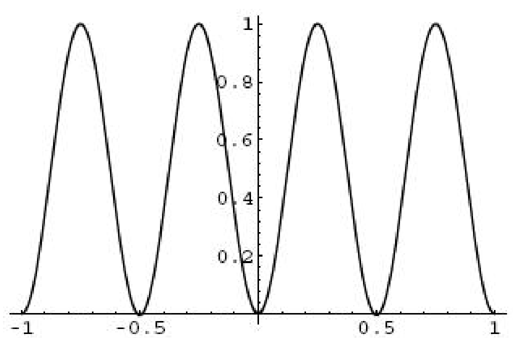

Figure 7: Comparison between exact and approach solutions for N=16

Example 2. *Consider the following problem :*

$$\begin{cases} \Delta^2 u = 24(1-x^2)^2 + 24(1-y^2)^2 + 32(3x^2-1)(3y^2-1) \ \ in\ \Omega = \left]-1,+1\right[^2, \\ u = 0 \quad \ on\ Fr(\Omega), \\ \dfrac{\partial u}{\partial \eta} = 0\ on\ Fr(\Omega). \end{cases}$$

This problem has one and only one solution: $u(x,\, y) = (1-x^2)^2(1-y^2)^2$. We have

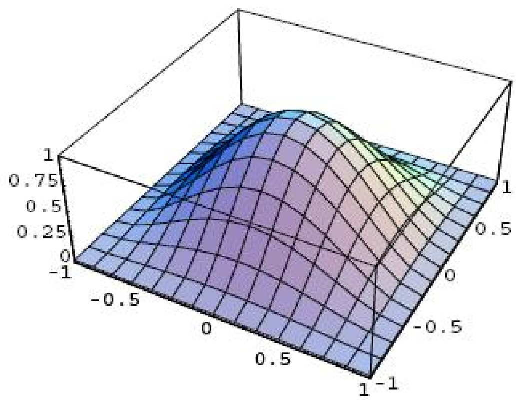

Figure 8: exact solution

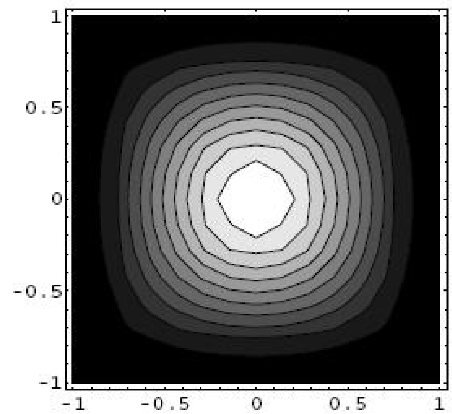

Figure 9: exact solution with contour

If we select the 1st choice of the space $\mathcal{M}_N$ with

$$\begin{cases} \phi_{-1} = L_0, \\ \phi_{N-1} = L_1, \end{cases}$$

then we have

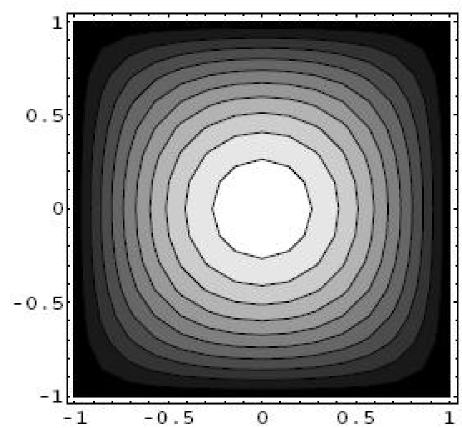

Figure 10: approach solution with line of contour for N=2

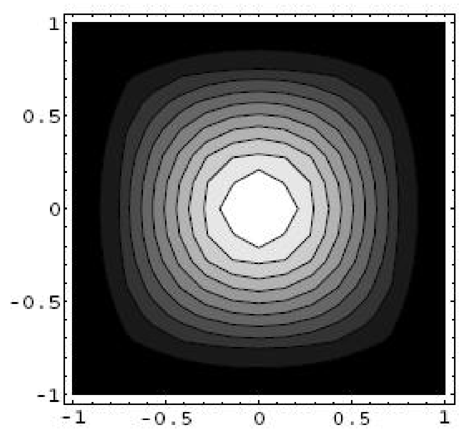

Figure 11: approach solution with lines of contour for N=4

6. Conclusion

We will say that developped methods are globally interesting when we dispose beforehand a good and well program to resolve the Dirichlet approach problem. One generalization of this problem is given in ([2]). An other more intersting problem is to study the evolution problem and the estimate of error for this last.

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