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MAJORIZATION FOR CERTAIN CLASSES OF ANALYTIC FUNCTIONS

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Abstract. In the present paper, we investigate the majorization properties for certain classes of multivalent analytic functions defined by fractional derivatives. Moreover, we point out some new or known consequences of our main result.

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1. Introduction

Let f and g be analytic in the open unit disk

$$\Delta = \{z : z \in \mathbb{C} \ , |z| < 1\} . \tag{1}$$

We say that f is majorized by g in Δ (see [3]) and write

$$f(z) \ll g(z) \qquad (z \in \Delta), \tag{2}$$

if there exists a function φ, analytic in Δ such that

$$|\varphi(z)| \leq 1 \quad \text{and} \quad f(z) = \varphi(z)g(z) \quad (z \in \Delta). \tag{3}$$

It may be noted here that (2) is closely related to the concept of quasi-subordination between analytic functions.

For two functions f and g, analytic in Δ, we say that the function f is subordinate to g in Δ, and write

$$f(z) \prec g(z),$$

if there exists a Schwarz function ω, which is analytic in Δ with

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \quad (z \in \Delta)$$

such that

$$f(z) = g\big(\omega(z)\big) \qquad (z \in \Delta).$$

Indeed, it is known that

$$f(z) \prec g(z) \Longrightarrow f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta).$$

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \qquad (p \in \mathbb{N} := \{1, 2, 3, ...\}), \tag{4}$$

which are analytic in the open unit disk Δ. For simplicity, we write $\mathcal{A}_1 =: \mathcal{A}$.

For two functions $f_j \in \mathcal{A}_p$ $(j = 1, 2)$ given by

$$f_j(z) = z^p + \sum_{k=p+1}^{\infty} a_{k,j} z^k \qquad (j = 1, 2;\ p \in \mathbb{N}), \tag{5}$$

we define the Hadamard product (or convolution) of f_1 and f_2 by

$$(f_1 * f_2)(z) = z^p + \sum_{k=p+1}^{\infty} a_{k,1} a_{k,2} z^k = (f_2 * f_1)(z).$$

In the following, we recall the definitions of fractional integral and fractional derivative.

Definition 1. (See [6]; see also [8]) *The fractional integral of order $\lambda(\lambda > 0)$ is defined, for a function f, analytic in a simply-connected of the complex plane containing origin by*

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\xi)}{(z-\xi)^{1-\lambda}} d\xi, \tag{6}$$

where the multiplicity of $(z-\xi)^{\lambda-1}$ is removed by requiring $\log(z-\xi)$ to be real when $z - \xi > 0$.

Definition 2. (See [6]; see also [8]) *Under the hypothesis of Definition 1, the fractional derivative of f of order $\lambda(\lambda \geq 0)$*

$$D_z^{\lambda} f(z) = \begin{cases} \frac{1}{\Gamma(1-\lambda)} \int_0^z \frac{f(\xi)}{(z-\xi)^{\lambda}} d\xi & (0 \leq \lambda < 1), \\ \frac{d^n}{dz^n} D_z^{\lambda - n} f(z) & (n \leq \lambda < n+1; n \in \mathbb{N} \cup \{0\}), \end{cases} \tag{7}$$

where the multiplicity of $(z-\xi)^{\lambda-1}$ *is removed by requiring* $\log(z-\xi)$ *to be real when* $z-\xi>0$.

Recently, Patel and Mishra [7] defined the extended fractional differ-integral operator

$$\Omega_z^{(\lambda,p)} : \mathcal{A}_p \to \mathcal{A}_p$$

for a function $f \in \mathcal{A}_p$ and for a real number $\lambda(-\infty<\lambda<p+1)$ by

$$\Omega_z^{(\lambda,p)} f(z) = \frac{\Gamma(p+1-\lambda)}{\Gamma(p+1)} z^{\lambda} D_z^{\lambda} f(z), \tag{8}$$

where $D_z^{\lambda} f(z)$ is the fractional integral of f of order λ if $0 \leq \lambda < p+1$. It is easy to see from (8) that for a function f of the form (1), we have

$$\Omega_z^{(\lambda,p)} f(z) = z^p + \sum_{n=p+1}^{\infty} \frac{\Gamma(n+p+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(n+p+1-\lambda)} a_n z^n \qquad (z \in \Delta)$$

and

$$z(\Omega_z^{(\lambda,p)} f(z))' = (p-\lambda)\Omega_z^{(\lambda+1,p)} f(z) + \lambda \Omega_z^{(\lambda,p)} f(z) \qquad (-\infty<\lambda<p;\ z\in\Delta). \tag{9}$$

Definition 3. *A function* $f(z) \in \mathcal{A}_p$ *is said to be in the class* $S_p^{\lambda,j}[A,B;\gamma]$ *of* p*-valent functions of complex order* $\gamma \neq 0$ *in* Δ *if and only if*

$$\left\{1+\frac{1}{\gamma}\left(\frac{z\left(\Omega_z^{(\lambda,p)} f(z)\right)^{(j+1)}}{\left(\Omega_z^{(\lambda,p)} f(z)\right)^{(j)}} - p + j\right)\right\} \prec \frac{1+Az}{1+Bz} \tag{10}$$

$$(z\in\Delta;\ -1\leq B<A\leq 1;\ p\in\mathbb{N};\ j\in\mathbb{N}_0:=\mathbb{N}\cup\{0\};\ \gamma\in\mathbb{C}-\{0\};\ -\infty<\lambda<p).$$

Clearly, we have the following relationships:

1. $S_p^{\lambda,j}[1,-1;\gamma] = S_p^{\lambda,j}(\gamma)$;
2. $S_1^{0,0}[1,-1;\gamma] = S(\gamma) \quad (\gamma\in\mathbb{C}-\{0\})$;
3. $S_1^{1,0}[1,-1;\gamma] = K(\gamma) \quad (\gamma\in\mathbb{C}-\{0\})$;
4. $S_1^{0,0}[1,-1;1-\alpha] = S^*(\alpha)$ for $0\leq\alpha<1$.

The class $S_p^{\lambda,j}(\gamma)$ was introduced by Goyal and Goswami [2]. The classes $S(\gamma)$ and $K(\gamma)$ are said to be classes of starlike and convex of complex order $\gamma \neq 0$ in Δ which were considered by Naser and Aouf [4] and Wiatrowski [9], and $S^*(\alpha)$ denote the class of starlike functions of order α in Δ.

An majorization problem for the class $S(\gamma)$ has recently been investigated by Altinas *et al.* [1]. Also, majorization problems for the class $S^* = S^*(0)$ have been investigated by MacGregor [3]. In the present paper, we investigate a majorization problem for the class $S_p^{\lambda,j}[A, B; \gamma]$.

2. Majorization problem for the class $S_p^{\lambda,j}[A, B; \gamma]$

We begin by proving the following result.

Theorem 1. *Let the function $f \in \mathcal{A}_p$ and suppose that $g \in S_p^{\lambda,j}[A, B; \gamma]$. If $(\Omega_z^{\lambda,p} f(z))^{(j)}$ is majorized by $(\Omega_z^{\lambda,p} g(z))^{(j)}$ in Δ, then*

$$\left| \left(\Omega_z^{\lambda+1,p} f(z)\right)^{(j)} \right| \leq \left| \left(\Omega_z^{\lambda+1,p} g(z)\right)^{(j)} \right| \qquad (|z| \leq r_0), \tag{11}$$

where $r_0 = r_0(p, \gamma, \lambda, A, B)$ is smallest the positive root of the equation

$$r^3|\gamma(A-B)-(p-\lambda)B| - [(p-\lambda)+2|B|]r^2 - [|\gamma(A-B)-(p-\lambda)B|+2]r + (p-\lambda) = 0, \tag{12}$$
$$(-1 \leq B < A \leq 1;\ p \in \mathbb{N};\ \gamma \in \mathbb{C} - \{0\}).$$

Proof. Since $g \in S_p^{\lambda,j}[A, B; \gamma]$, we find from (10) that

$$1 + \frac{1}{\gamma}\left(\frac{z\left(\Omega_z^{(\lambda,p)} g(z)\right)^{(j+1)}}{\left(\Omega_z^{(\lambda,p)} g(z)\right)^{(j)}} - p + j \right) = \frac{1 + Aw(z)}{1 + Bw(z)}, \tag{13}$$

where ω is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \quad \text{and} \quad |\,\omega(z)\,| < 1 \quad (z \in \mathbb{U}).$$

From (13), we get

$$\frac{z\left(\Omega_z^{(\lambda,p)} g(z)\right)^{(j+1)}}{\left(\Omega_z^{(\lambda,p)} g(z)\right)^{(j)}} = \frac{(p-j) + [(\gamma(A-B) + (p-j)B]w(z)}{1 + Bw(z)}. \tag{14}$$

By noting that

$$z(\Omega_z^{(\lambda,p)}g(z))^{(j+1)} = (p-\lambda)\Big(\Omega_z^{(\lambda+1,p)}g(z)\Big)^{(j)} + (\lambda-j)\Big(\Omega_z^{(\lambda,p)}g(z)\Big)^{(j)}, \tag{15}$$

by virtue of (14) and (15), we get

$$\left|\Big(\Omega_z^{(\lambda,p)}g(z)\Big)^{(j)}\right| \leq \frac{(p-\lambda)[1+|B||z|]}{(p-\lambda)-|\gamma(A-B)-(p-\lambda)B||z|}\left|\Big(\Omega_z^{(\lambda+1,p)}g(z)\Big)^{(j)}\right|. \tag{16}$$

Next, since $\Big(\Omega_z^{(\lambda,p)}f(z)\Big)^{(j)}$ is majorized by $\Big(\Omega_z^{(\lambda,p)}g(z)\Big)^{(j)}$ in the unit disk Δ, from (3), we have

$$\Big(\Omega_z^{(\lambda,p)}f(z)\Big)^{(j)} = \varphi(z)\Big(\Omega_z^{(\lambda,p)}g(z)\Big)^{(j)}$$

Differentiating it with respect to z and multiplying by z, we get

$$z\Big(\Omega_z^{(\lambda,p)}f(z)\Big)^{(j+1)} = z\varphi'(z)\Big(\Omega_z^{(\lambda,p)}g(z)\Big)^{(j)} + z\varphi(z)\Big(\Omega_z^{(\lambda,p)}g(z)\Big)^{(j+1)}. \tag{17}$$

Using (15), in the above equation, it yields

$$\Big(\Omega_z^{(\lambda+1,p)}f(z)\Big)^{(j)} = \frac{z\varphi'(z)}{(p-\lambda)}\Big(\Omega_z^{(\lambda,p)}g(z)\Big)^{(j)} + \varphi(z)\Big(\Omega_z^{(\lambda+1,p)}g(z)\Big)^{(j)} \tag{18}$$

Thus, by noting that $\varphi \in \mathcal{P}$ satisfies the inequality (see, e.g. Nehari [5])

$$|\varphi'(z)| \leq \frac{1-|\varphi(z)|^2}{1-|z|^2} \qquad (z \in \Delta) \tag{19}$$

and making use of (16) and (19) in (18), we get

$$\left|\Big(\Omega_z^{(\lambda+1,p)}f(z)\Big)^{(j)}\right| \\ \leq \left(|\varphi(z)| + \frac{1-|\varphi(z)|^2}{1-|z|^2}\frac{|z|(1+|B||z|)}{[(p-\lambda)-|\gamma(A-B)-(p-\lambda)B||z|]}\right)\left|\Big(\Omega_z^{(\lambda+1,p)}g(z)\Big)^{(j)}\right|, \tag{20}$$

which upon setting

$$|z| = r \quad \text{and} \quad |\varphi(z)| = \rho \quad (0 \leq \rho \leq 1)$$

leads us to the inequality

$$\left|\Big(\Omega_z^{\lambda+1,p}f(z)\Big)^{(j)}\right| \leq \frac{\Phi(\rho)}{(1-r^2)(p-\lambda)-|\gamma(A-B)-(p-\lambda)B|r}\left|\Big(\Omega_z^{\lambda+1,p}g(z)\Big)^{(j)}\right|,$$

where

$$\Phi(\rho) = -r(1+|B|r)\rho^2+(1-r^2)[(p-\lambda)-|\gamma(A-B)+(p-\lambda)B|r)]\rho+r(1+|B|r) \quad (21)$$

takes its maximum value at $\rho = 1$, with $r_0 = r_0(p, \gamma, \lambda, A, B)$, where $r_0(p, \gamma, \lambda, A, B)$ is smallest the positive root of the equation (12). Furthermore, if $0 \leq \rho \leq r_0(p, \gamma, \lambda, A, B)$, then the function $\psi(\rho)$ defined by

$$\psi(\rho) = -\sigma(1+|B|\sigma)\rho^2+(1-\sigma^2)[(p-\lambda)-|\gamma(A-B)+(p-\lambda)B|\sigma)]\rho+\sigma(1+|B|\sigma) \quad (22)$$

is a increasing function on the interval $0 \leq \rho \leq 1$, so that

$$\begin{gathered}\psi(\rho) \leq \psi(1) = (1-\sigma^2)[(p-\lambda)-|\gamma(A-B)+(p-\lambda)B|\sigma)] \\ (0 \leq \rho \leq 1;\ 0 \leq \sigma \leq r_0(p, \gamma, \lambda, A, B)).\end{gathered} \quad (23)$$

Hence upon setting $\rho = 1$, in (22), we conclude that (11) of Theorem (1) holds true for

$$|z| \leq r_0 = r_0(p, \gamma, \lambda, A, B),$$

where $r_0(p, \gamma, \lambda, A, B)$ is the smallest positive root of the equation (12). This completes the proof of the Theorem 1. □

Setting $A = 1$ and $B = -1$ in Theorem 1, we get the following result.

Corollary 1. (See [2]) *Let the function* $f \in \mathcal{A}_p$ *and suppose that* $g \in S_p^{\lambda,j}(\gamma)$. *If* $\left(\Omega_z^{\lambda,p} f(z)\right)^{(j)}$ *is majorized by* $\left(\Omega_z^{\lambda,p} g(z)\right)^{(j)}$ *in* Δ, *then*

$$\left|\left(\Omega_z^{\lambda+1,p} f(z)\right)^{(j)}\right| \leq \left|\left(\Omega_z^{\lambda+1,p} g(z)\right)^{(j)}\right| \quad for\ |z| \leq r_1, \quad (24)$$

where

$$r_1 = r_1(p, \gamma, \lambda) = \frac{k - \sqrt{k^2 - 4(p-\lambda)|2\gamma - p + \lambda|}}{2|2\gamma - p + \lambda|}, \quad (25)$$

$$(k = 2 + p - \lambda + |2\gamma - p + \lambda|;\ p \in \mathbb{N};\ \gamma \in \mathbb{C} - \{0\}).$$

Putting $A = 1, B = -1$, and $j = 0$ in Theorem 1, we obtain the following result.

Corollary 2. *Let the function* $f \in \mathcal{A}$ *and suppose that* $g \in S_p^{\lambda,0}(\gamma)$. *If* $\Omega_z^{\lambda,p} f(z)$ *is majorized by* $(\Omega_z^{\lambda,p} g(z))$ *in* Δ, *then*

$$\left|\left(\Omega_z^{\lambda+1,p} f(z)\right)\right| \leq \left|\left(\Omega_z^{\lambda+1,p} g(z)\right)\right| \quad for\ |z| \leq r_1,$$

where r_1 *is given by* (25).

Further putting $\lambda = 0, j = 0$ in Corollary 2, we get

Corollary 3. (See [1]) *Let the function* $f \in \mathcal{A}$ *and suppose that* $g \in S_p^{\lambda,0}(\gamma)$. *If* $\Omega_z^{\lambda,p} f(z)$ *is majorized by* $\Omega_z^{\lambda,p} g(z)$ *in* Δ, *then Let the function* $f(z) \in \mathcal{A}$ *be analytic in th open unit disk* Δ *and suppose that* $g \in S(\gamma)$. *If* $f(z)$ *is majorized by* $g(z)$ *in* Δ, *then*

$$|f'(z)| \leq |g'(z)| \qquad (|z| \leq r_2)$$

where

$$r_2 = r_2(\gamma) = \frac{3 + |2\gamma - 1| - \sqrt{9 + 2|2\gamma - 1| + |2\gamma - 1|^2}}{2|2\gamma - 1|}.$$

For $\gamma = 1$, Corollary 3 reduces to the following result:

Corollary 4. (See [3]) *Let the function* $f \in \mathcal{A}$ *be an analytic and univalent in the open unit disk* Δ *and suppose that* $g \in S^* = S^*(0)$. *If* $f(z)$ *is majorized by* $g(z)$ *in* Δ, *then*

$$|f'(z)| \leq |g'(z)| \qquad (|z| \leq 2 - \sqrt{3}).$$

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References

[1] O. Altinas, O. Ozkan and H. M. Srivastava, *Majorization by starlike functions of complex order*, Complex Var. 46 (2001), 207–218.

[2] S. P. Goyal and P. Goswami, *Majorization for certain classes of analytic functions defined by fractional derivatives*, Appl. Math. Lett. (2009), in press.

[3] T. H. MacGreogor, *Majorization by univalent functions*, Duke Math. J. 34 (1967), 95–102.

[4] M. A. Naser and M. K. Aouf, *Starlike function of complex order*, J. Natur. Sci. Math. 25 (1985), 1–12.

[5] Z. Nehari, *Conformal mapping* ,MacGra-Hill Book Company, New York, Toronto and London (1955).

[6] S. Owa, *On distortion theorems* I, Kyunpook Math. J. 18 (1978), 53–59.

[7] J. Patel and A. K. Mishra, *On certain subclasses of multivalent functions associated with an extended fractional differintegral operator*, J. Math. Anal. Appl. 332 (2007), 109–122.

[8] H. M. Srivastava, S. Owa (Eds.), *Univalent Functions, Fractional Calculus, and Their Applications*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, 1989.

[9] P. Wiatrowski, *On the coefficients of some family of holomorphic functions*, Zeszyry Nauk. Uniw. Lddz. Nauk. Mat.-Przyrod., 39 (1970), 75–85.

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