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THE GAUSS MAP OF HESSIAN MANIFOLDS IN SPACES OF CONSTANT HESSIAN SECTIONAL CURVATURE VIA KOZSUL FORMS

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Abstract. This paper considers the Gauss map of a Hessian manifold in a classical point of view. We analyse a Hessian manifold of constant curvature as an Einsteinian manifold in terms of Kozsul forms and obtain new results on it.

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Introduction

The immersions of a Riemannian $n-$manifold M into a Euclidean space have a great research field almost in all branches of mathematics and physics and has been extensively studied. Gauss map x is one of the special type of these. In general, it assigns to a point p of M the n-plane through the origin of E^N and parallel to the tangent plane of $x(M)$ at $x(p)$ and is a map of M into the Grassmann manifold $G_{n,N} = O(N)/O(n) \times O(N-n)$. According to Obata [1], the Gauss map of an immersion x into S^N is meant a map of M into the Grassmann manifold $\mathrm{G}_{n+1,N+1}$ which assigns to each point p of M the great $n-$ sphere tangent to $x(M)$ at $x(p)$, or the $(n+1)-$ plane spanned by the tangent space of $x(M)$ at $x(p)$ and the normal to S^N at $x(p)$ in E^{N+1}. More generally, with an immersion x of M into a simply connected complete $N-$ space V of constant curvature there is associated a map which assigns to each point p of M the totally geodesic $n-$ subspace tangent to $x(M)$ at $x(p)$. Such a map is called the (generalized) Gauss map. Thus the Gauss map in our generalized sense is a map: $M \to Q$, where Q stands for the space of all the totally geodesic $n-$ subspaces in V.

The purpose of his paper; first to obtain a relationship among the Ricci form of the immersed manifold and the second and third fundamental forms of the immersion, and then to give a geometric interpretation of the third fundamental form in this case by using the notion of Gauss map. Also he showed that the Gauss map associated with a minimal immersion is conformal if and only if the manifold is Einsteinian.

On the other hand, the geometry of Hessian manifold is new and fruitful area for scientists because of having close analogy with Kaehlerian manifolds, affine differential geometry and statistics.[3 − 7] Inspite of its importance a little is known about its geometric structures.

In the present work, first we give the basic notations for Hessian manifolds then construct the Gauss map of Hessian manifolds in a classical point of view. Also giving the relations between Kaehlerian and Hessian structures we obtain new results in terms of Kozsul forms[2].

1. Hessian structures [3]

Definition 1.1. *A Riemannian metric g on a flat manifold (M, D) is called a Hessian metric if g can be locally expressed by*

$$g = Dd\varphi$$

that is

$$g_{ij} = \frac{\partial^2 \varphi}{\partial x^i \partial x^j}$$

where $\{x^1, ..., x^n\}$ is an affine coordiante system withrespect to D. Then the pair (D, g) is called a Hessian structure on M, and φ is said to be a potential of (D, g). A manifold M with a Hessian structure (D, g) is called a Hessian manifold and is denoted by (M, D, g).

A Riemannian metric on a flat manifold is a Hessian metric if it can be locally expressed by the Hessian with respect to an affine coordinate system. On the other hand, a Riemannian metric on a complex manifold is said to be Kaehlerian metric if it can be locally given by the complex Hessian with respect to a holomorphic coordinate system.

The tangent bundle over a Hessian manifold admits a Kaehlerian metric induced by the Hessian metric.

Definition 1.2. *A Riemannian metric g on a complex manifold is said to be a Hermitian metric if*

$$g_{ij} = g_{\overline{ij}} = 0$$

We denote the Hermitian metric by

$$g = \sum g_{i\overline{j}} dz^i dz^{\overline{j}}.$$

Definition 1.3. *A Hermitian metric g on a complex manifold (M, J) is said to be Kaehlerian metric if g can be locally expressed by the complex Hessian of a function φ*

$$g_{i\bar{j}} = \frac{\partial^2 \varphi}{\partial z^i \partial z^{\bar{j}}}$$

where $\left\{z^1, ..., z^n\right\}$ is a holomorphic coordinate system. The pair (J, g) is called a Kaehlerian structure on M. A complex manifold M with a Kaehlerian structure (J, g) is said to be a Kaehlerian manifold and is denoted by (M, J, g).

Proposition 1.1. *Let g be a Hermitian metric on a complex manifold M. Then the following conditions are equivalent*
(1) *g is Kaehlerian metric*
(2) *The Kaehlerian form ρ is closed.*

Let us introduce the Kaehlerian form for (J, g).

Definition 1.4. *For a Hermitian metric g we set*

$$\rho(X, Y) = g(JX, Y)$$

Then the skew-symmetric bilinear form ρ is called a Kaehlerian form for (J, g) and using a holomorphic coordiante system we have

$$\rho = \sqrt{-1} \sum g_{i\bar{j}} dz^i \wedge d\bar{z}^j$$

Proposition 1.2. *Let (M, D) be a flat manifold and g a Riemannian metric on M. Then the following conditions are equivalent*
(1) *g is a Hessian metric on (M, D).*
(2) *g^T is a Kaehlerian metric on (TM, J_D).*

2. Hessian curvature tensors

Definition 2.1. *Let (D, g) be a Hessian structure and let $\gamma = \nabla - D$ be the difference tensor between the Levi-Civita connection ∇ for g and D. a tensor field Q of type $(1, 3)$ defined by the covariant differential*

$$Q = D_\gamma$$

of γ is said to be the Hessian curvature tensor for (D, g). The components Q^i_{jkl} of Q with respect to an affine coordinate system $\left\{x^1, ..., x^n\right\}$ are given by

$$Q^i_{jkl} = \frac{\partial \gamma^i_{jl}}{\partial x^k}, [3].$$

[3] .

Proposition 2.1. *Let* $g_{ij} = \frac{\partial^2 \varphi}{\partial x^i \partial x^j}$. *Then we have*
(1) $Q^i_{jkl} = \frac{1}{2} \frac{\partial^4 \varphi}{\partial x^i \partial x^j \partial x^k \partial x^l} - \frac{1}{2} g^{rs} \frac{\partial^3 \varphi}{\partial x^i \partial x^k \partial x^r} \frac{\partial^3 \varphi}{\partial x^j \partial x^l \partial x^s}$.
(2) $Q_{ijkl} = Q_{kjil} = Q_{ilkj} = Q_{jilk}$. [3] .

Proposition 2.2. *Let R be the Riemannian curvature tensor for g. Then*

$$R_{ijkl} = \frac{1}{2} \left(Q_{ijkl} - Q_{jikl} \right), [3] .$$

Proposition 2.3. *Let R^T be the Riemannian curvature tensor on the Kaehlerian manifold $\left(TM, J, g^T\right)$. Then we have*

$$R^T_{i\bar{j}k\bar{l}} = \frac{1}{2} Q_{ijkl} \circ \pi, [3] .$$

3.Hessian sectional curvature

Definition 3.1. *Let Q be a Hessian curvature tensor on a Hessian manifold (M, D, g). We define an endomorphism $\widehat{Q}$ on the space of symmetric contravariant tensor fields of degree 2 by*

$$\widehat{Q} \left(\xi \right)^{ik} = Q^{ik}_{jl} \xi^{jl}.$$

The endomorphism $\widehat{Q}$ is symmetric with respect to the inner prouct $\langle , \rangle$ induced by the Hessian metric g. In fact, by Proposition 2.1 we have

$$\langle \widehat{Q} \left(\xi \right), \eta \rangle = \langle \xi, \widehat{Q}(\eta) \rangle, \ \ [3] .$$

Definition 3.2. *Let $\xi_x \neq 0$ be a symmetric contravariant tensor field of degree 2. We put*

$$q(\xi_x) = \frac{\langle \widehat{Q} \left(\xi_x \right), \xi_x \rangle}{\langle \xi_x, \xi_x \rangle},$$

and call it the Hessian sectional curvature for ξ_x, [3] .

Definition 3.3. *If $q(\xi_x)$ is a constant c for all symmetric contravariant tensor field $\xi_x \neq 0$ of degree 2 and for all $x \in M$, then (M, D, g) is said to be a Hessian manifold of constant Hessian sectional curvature c,* [3] .

Proposition 3.1. *The Hessian sectional curvature of* (M, D, g) *is a constant* c *if and only if*

$$Q_{ijkl} = \frac{c}{2}\left(g_{ij}g_{kl} + g_{il}g_{kj}\right), [3].$$

Proposition 3.2. *The following condition* (1) *and* (2) *are equivalent*
(1) *The Hessian sectional curvature of* (M, D, g) *is a constant* c.
(2) *The holomorphic sectional curvature of* $\left(TM, J, g^T\right)$ *is a constant* $-c$, [3].

Corollary 3.1. *Suppose that a Hessian manifold* (M, D, g) *is a space of constant Hessian sectional curvature* c. *Then the Riemannian manifold* (M, g) *is a space form of constant sectional curvature* $-\frac{c}{4}$. [3].

Now let us consider a Hessian domain $(\Omega, D, g = Dd\varphi)$ in $\mathbb{R}^n$ of constant Hessian sectional curvature c as indicated [3].

Proposition 3.3. *The following Hessian domains are examples of spaces of constant Hessian sectional curvature 0.*
(1) *Euclidean space* $\left(\mathbb{R}^n, D, g = Dd\left(1/2\sum\limits_{i=1}^{n}\left(x^i\right)^2\right)\right)$.
(2) $\left(\mathbb{R}^n, D, g = Dd\left(\sum\limits_{i=1}^{n} e^{x^i}\right)\right)$.

Proposition 3.4. *Let* c *be positive real number and let*

$$\Omega = \left\{\left(x^1, ..., x^n\right) \in \mathbb{R}^n \,\middle|\, x^n > \frac{c}{2}\sum_{i=1}^{n-1}\left(x^i\right)^2\right\},$$

and let φ *be a smooth function on* Ω *defined by*

$$\varphi = -\frac{1}{c}\log\left\{x^n - \frac{1}{2}\sum_{i=1}^{n-1}\left(x^i\right)^2\right\}.$$

Then $\left(\Omega, D,\ g = D^2\varphi\right)$ *is a simply connected Hessian manifold of positive constant Hessian sectional curvature* c.

Hence the following theorem can be proved as a consequence of the properties above.

It is really surprising that (Ω, g) is isometric to hyperbolic space form $\left(H\left(-\frac{c}{4}\right), g\right)$ of constant sectional curvature $-c/4$;

$$\begin{aligned} H &= \left\{\left(\xi^1, ..., \xi^{n-1}, \xi^n\right) \in \mathbb{R}^n \,|\, \xi^n > 0\right\}, \\ g &= \frac{1}{\left(\xi^n\right)^2}\left\{\sum_{i=1}^{n}\left(d\xi^i\right)^2 + \frac{4}{c}\left(d\xi^n\right)^2\right\}. \end{aligned}$$

Proposition 3.5. *Let φ be a smooth function on $\mathbb{R}^n$ defined by*

$$\varphi = -\frac{1}{c}\log\left(\sum_{A=1}^{n} e^{-cx^A} + 1\right),$$

where c is a negative constant. Then $\left(\mathbb{R}^n, \widetilde{D},\ g = \widetilde{D}^2\varphi\right)$ *is a simply connected Hessian manifold of negative constant Hessian sectional curvature c. The Riemannian manifold* $(\mathbb{R}^n, g)$ *is isometric a domain of the sphere* $\sum_{i=1}^{n+1} \xi_A^2 = -\frac{4}{c}$ *defined by* $\xi_A > 0$ *for all* $A.$ [3] .

From now on let V denote one of the following simply-connected complete Hessian manifold of dimension N.

According to the fact that the tangent bundle over a Hessian manifold admits a Kaehlerian metric and using Proposition 1.2. , the bundle $F(V)$ on V can be identified one of the following due to the type of V.

$i)$ The complex Euclidean space $\mathbb{C}^N, (c = 0)$,

$ii)$ The complex hyperbolic space $\mathbb{H}^N(\mathbb{C}), (-c < 0)$,

$iii)$ The complex projective space $\mathbb{P}^N(\mathbb{C}), (-c > 0)$.

The bundle $F(V)$ of the orthonormal frames V can be identified with the group $G(n)$ as follows according to the type of V

$i)$ The unitary group $U(N+1)$,

$ii)$ The special unitary group $SU(1, N)$,

$iii)$ The projective unitary group $PU(N+1)$, [8],[9], [10] .

Let $F(V)$ be a complex $N-$ dimensional Kaehlerian manifold with complex structure J. We can choose a local field of orthonormal frames $e_1, ..., e_N, e_1^* = Je_1, ..., e_N^* = Je_N$ in $F(V)$ with respect to the frame field of $F(V)$ chosen above, let $w^1, ..., w^N, w^{1*}, ..., w^{N*}$. be the field of dual frames .

Let $w = \left(w_j^i\right)$ $i, j = 1, ..., 2N$ be the connection form of M. Then we have

$$w_b^a = w_{b^*}^{a^*} \quad , \ w_{b^*}^a = -w_b^{a^*}, \quad w_b^a = -w_a^b, \quad w_{b^*}^a = w_{a^*}^b$$

where $a, b = 1, ..., N$. We denote by $\Omega = \left(\Omega_j^i\right)$ the curvature form and write

$$\Omega_j^i = \frac{1}{2}\sum_{k,l} R_{jkl}^i w^k \wedge w^l.$$

We set

$$\begin{aligned} \xi_a &= \frac{1}{2}(e_a - ie_{a^*}) \qquad & \xi_{\overline{a}} = \frac{1}{2}(e_a - ie_{a^*}), \\ \theta^q &= w^a + iw^{a^*} \qquad , & \theta^{\overline{a}} = w^a - iw^{a^*} \end{aligned}$$

Then $\{\xi_a\}$ form a complex basis of $T_X^{1,0}(F(V))$ and $\{\xi_{\overline{a}}\}$ form a complex basis of $T_X^{1,0}(F(V))$. Then Kaehlerian metric g^T is given by

$$g^T = \sum_a \theta^a \otimes \overline{\theta}^a$$

Moreover we set

$$\begin{aligned} \theta_b^a &= w_b^a + i w_b^{a^*} \qquad , \qquad \theta_{\overline{b}}^{\overline{a}} = w_b^a - i w_b^{a^*} \\ \Psi_b^a &= \Omega_b^a + i\Omega_b^{a^*} \qquad , \quad \Psi_b^a = \Omega_b^a - i\Omega_b^{a^*} \end{aligned}$$

Then we obtain

$$\begin{aligned} d\theta^a &= -\sum \theta_b^a \wedge \theta^b \qquad , \qquad \theta_b^a + \theta_{\overline{a}}^{\overline{b}} = 0 \\ d\theta_b^a &= -\sum \theta_c^a \wedge \theta_b^c + \Psi_b^a \quad , \qquad \Psi_b^a = \sum_{c,d} K_{bc\overline{d}}^a \theta^c \wedge \theta^{\overline{d}} \\ K_{bc\overline{d}}^a &= \frac{1}{2}\left[R_{bcd}^a + R_{b^*c^*d^*}^a + i\left(R_{bcd^*}^a - R_{b^*cd}^a\right)\right]. \end{aligned} \tag{1}$$

We know that $F(V)$ is constant holomorphic sectional curvature c if and only if

$$K_{bc\overline{d}}^a = \frac{c}{4}\left(\delta_{ac}\delta_{bd} + \delta_{ab}\delta_{cd}\right)$$

or

$$\psi_b^a = \frac{c}{4}\left(\theta^q \wedge \theta^{\overline{b}} + \delta_{ab}\sum \theta^c \wedge \theta^\tau\right) \tag{2}$$

[11].

Let M be a Hessian manifold isometrically immersed into the space V by a mapping $x : M \to V$, $F(M)$ denote the bundle M and B the set of elements $b = (p, e_1, ..., e_N, e_1^*, ..., e_N^*)$ that $b = (p, e_1, ..., e_N, e_1^*, ..., e_n^*) \in F(M)$ and $(x(p), e_1, ..., e_N, e_1^*, ..., e_N^*) \in F(V)$ where e_i, e_i^* ,$1 \leq i \leq N$ are identified with $dx(e_i)$.

Then $\Phi : B \to M$ can be viewed as a principal bundle $U(n) \times U(N-n)$ and $\widetilde{x} : B \to F(V) = G(N)$ is the natural immersion defined by $\widetilde{x}(b) = (x(p), e_1, ..., e_N, e_1^*, ..., e_N^*)$.

We know take a complex coordinate system $\{z^1, ..., z^n\}$ in M. We set

$$Z_a = \frac{\partial}{\partial z^a}, \qquad Z_{\overline{a}} = \overline{Z}_a = \frac{\partial}{\partial \overline{z}^a}, \qquad a = 1, ..., n$$

We extend a Hermitian metric g to a complex symmetric bilinear form in $T_x^c(M)$. We put

$$g_{AB} = g(Z_A, Z_B) \qquad A, B = 1, ..., N, \overline{1}, ..., \overline{N}$$

Then we have

$$g_{ab} = g_{\overline{a}\overline{b}} = 0$$

and $g_{a\bar{b}}$ is a Hermitian matrix.

We write

$$ds^2 = 2\sum_{a,b} g_{a\bar{b}} dz^a d\overline{z}^b$$

for the metric g.

Let w_b^a and w^a be the 1-forms on B induced form θ_b^a and θ^a by the map $\widetilde{x}$. Then we have

$$w_r = 0 \tag{3}$$

and the Hessian metric ds^2 on M is given by

$$ds^2 = \sum_i (w_i)^2$$

where from now on we agree on the following ranges of indices

$$1 \leq i, j, k, l, ... \leq n, \qquad n+1 \leq r, s, t, ... \leq N$$

Furthermore from (2) we obtain

$$\begin{aligned} w_{ir} &= \sum_j A_{r_{ij}} w_j, \qquad A_{r_{ij}} = A_{r_{ji}}. \\ dw_i &= \sum_j w_j \wedge w_{ji}. \\ dw_{ij} &= \sum w_{ik} \wedge w_{kj} - \sum_r w_{ir} \wedge w_{jr} - \frac{c}{4} w_i \wedge w_j \end{aligned}$$

The curvature forms Ω_{ij} of M can be written as

$$\Omega_{ij} = \frac{1}{2} \sum_{k,l} K_{ijkl} w_k \wedge w_{l.}$$

We obtain

$$K_{ijkl} = -\frac{c}{4} \left(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} \right) - \sum_r \left(A_{rik}A_{rjl} - A_{ril}A_{rjk} \right). \tag{4}$$

Obviously K_{ijkl} give the components of the curvature tensor of M.

At a point $b = (p, e_1, ..., e_n, e_1^*, ..., e_n^*)$ in B by forming the form

$$II = \sum w_{ir} w_{ilr} = \sum_{r,i,j} A_{rij} w_i w_{jlr.}$$

We know that II is independent of the choice of the point to over p and is a normal vector valued quadratic differential form on M. II is called second fundamental form of the immersion x where vanishing defines a totally geodesic immersion. The normal vector

$$N = \sum II(e_i, e_i) = \sum A_r e_r$$

where

$$A_r = \sum A_{r_{ii}}$$

is independent of the frame and is called the mean curvature vector of the immersion x. If N vanishes identically then x is said to be minimal. Let $X = \sum X_r e_r$ be a normal vector of $x(M)$ at $x(p)$. Then the quadratic differential form defined by

$$II_X = \langle II, X \rangle = \sum A_{rij} X_r w_i w_j$$

is called the second fundamental form of the immersion x in the direction X.

Since N is uniquely determined by the immersion, the form

$$II_N = \sum A_r A_{rij} w_i w_j$$

has a special meaning related to the immersion x. It is easy to see that $II_N = 0$ if and only if $N = 0$. Thus the immersion is minimal if and only if II_N vanishes identically.

If the form II_N is proportional to the Hessian metric ds^2 on M, that is, if

$$II_N = \rho ds^2 = \rho \sum w_i w_i$$

the immersion is said to be pseudo-umbilical. Here we have

$$\rho = \frac{1}{n} \sum_r A_r^2 = \frac{1}{n} \|N\|^2 .$$

Then considering the quadratic differential form

$$\Psi = \sum K_{jk} w_j w_k$$

called the Ricci form of M, where we have put

$$K_{jk} = \sum K_{ijki}.$$

The Ricci form is independent of the choice of the frame and therefore is a quadratic differential form on M. We have from (4)

$$K_{jk} = -(n-1)\frac{c}{4}\delta_{jk} - \sum A_{rik} A_{rji} + \sum A_\tau A_{rjk},$$

and hence

$$\Psi = -(n-1)\frac{c}{4}\sum w_i w_i - \sum (w_{ir})^2 + II_N \tag{5}$$

We shall next consider the meaning of the term $\sum\limits_{i,r} (w_{ir})^2$.

4. The Gauss map

Let us denote the set of all the totally geodesic $n-$ spaces in V with Q. Then $F(V)$ acts on Q transitively. Take a point in Q. The isotropy subgroup at p is identified with $G(n) \times U(N-n)$ where $G(n)$ is viewed as acting on the totally geodesic $n-$ space V_0 representing the point p in Q and $U(N-n)$ on the totally geodesic $(N-n)-$ space orthogonal to V_0 at the point of intersection which is kept.

Therefore Q is identified with a homogeneous space

$$Q = F(V)/G(n) \times U(N-n)$$

By using Maurer-Cartan forms θ_b^a of $F(V)$ we introduce a quadratic differential form $d\sum^2$ on Q :

In the case $F(V) = HP^n$, $d\sum^2$ is the standart pseudo-Hessian metric with respect to which Q is a pseudo-Hessian symmetric space.

In the case $F(V) = \mathbb{P}^n$, $d\sum^2$ coincides with the quadratic differential form induced from the standart Hessian metric on $G_{n,N}$ by the projection.

Taking the immersion $x : M \to V$ we associate the (generalized) Gauss map $f : M \to Q$ where $f(p), p \in M$ is totally geodesic $n-$space tangent to $x(M)$ and $x(p)$ and consider the following scheme

$$\begin{array}{ccc} B & \xrightarrow{\quad F \quad} & F(V) = G(N) \\ \downarrow \Psi & & \downarrow \pi \\ M & \xrightarrow{\quad f \quad} & Q = F(V)/G(n) \times U(N-n) \end{array}$$

where π is the natural projection and F is the natural identification of a frame in B with an element of $F(V)$.

The quadratic differential form III induced from $d\sum^2$ on Q by the Gauss map f is written as

$$III = f^* d\sum{}^2 = \sum (w_{ir})^2 = A_{rij} A_{rik} w_j w_k \tag{6}$$

The Gauss map is a constant map if and only if III vanishes identically i.e $w_{ir} = 0$ and therefore if and only if the immersion x is totally geodesic.

Considering (5) and (6) together we obtain

$$\Psi - II_N + III = -\frac{(n-1)\,c}{4} ds^2 \tag{7}$$

and then we get the following

Theorem 4.1. *Suppose that a Hessian manifold M is isometrically immersed into a simply-connected complete space of constant curvature $-\frac{c}{4}$. Then the relation (7) holds among Ricci form Ψ on M, the second fundamental form II_N in the direction of the mean curvature vector, and the third fundamental form of III of the immersion.*

Supposing M is Einsteinian and then form (7) II_N is proportional to ds^2 if and only if III is. Thus we obtain

Theorem 4.2. *Let x be an isometric immersion of an Einstein space into a V. Then x is pseudo-umbilical if and only if Gauss map is conformal.*

Definition 4.1. *Let v be the volume element of g. We define a closed 1-form α and a symmetric bilinear form β by*

$$\begin{aligned} D_X v &= \alpha\,(x)\,v \\ \beta &= D_{\alpha}. \end{aligned}$$

The forms α and β are called the first and the second Kozsul form for a Hessian structure (D, g), respectively, [3] .

Proposition 4.1. *We have*

$$\beta_{ij} = \frac{\partial \alpha_i}{\partial x^j} = \frac{1}{2}\frac{\partial^2 \log \det\,[g_{kl}]}{\partial x^i \partial x^j} = Q^r_{rij} = Q^r_{ijr}.$$

By this fact we can easily see the relationship between Hessian curvature tensor and second Kozsul form, [3] .

Definition 4.2. *If a Hessian structure (D, g) satisfies the condition*

$$\beta = \lambda g, \qquad \lambda = \frac{\beta^i_i}{n}$$

then the Hessian structure is said to be Einstein-Hessian, [3] .

Taking into account of Theorem 4.2 with Definition 4.2. We conclude the following

Corollary 4.1. *If the second Kozsul form of a Hessian manifold holds the relation*

$$\beta = \frac{(n+1)\,c}{2} g$$

and x be an isometric immersion of W into V. Then x is a pseudo-umbilical if and only if Gauss map is conformal.

Corollary 4.2. *If the Hessian sectional curvature of (M, D, g) is a constant c, then the Hessian structure (D, g) is Einstein-Hessian and*

$$\beta = \frac{(n+1)\,c}{2} g.$$

Proof. The above assertion follows from Proposition 4.1. and Proposition 3.1.

Corollary 4.3. *If the second Kozsul form of a Hessian manifold holds the relation*

$$\beta = \frac{(n+1)\,c}{2} g$$

and x be an isometric immersion of W into V. Then x is pseudo-umbilical if and only if Gauss map is conformal.

Suppose that II_N is proportional to ds^2. Then Ψ is proportional to ds^2 if and only if III is. In particular if II_N vanishes identically, Ψ is proportional to ds^2 if and only if III is. In this case if dim > 2 the proportional factor of Ψ is constant and the same holds for III. Hence we get the following theorems.

Theorem 4.3. *Let x be a pseudo-umbilical immersion of a Hessian manifold M into a V. Then the Gauss map is conformal if and only if the second Kozsul form of M holds the relation $\beta = \frac{(n+1)c}{2} g$. In the case* $\dim M > 2$, *the Gauss map is homothetic if and only if the second Kozsul form of M holds the same relation $\beta = \frac{(n+1)c}{2} g$.*

Theorem 4.4. *Let x be an isometric immersion of a Hessian manifold into a V. Then x is pseudo-umbilical if and only if the second Kozsul form of M holds the relation $\beta = \frac{(n+1)c}{2} g$.*

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