Acta Universitatis Apulensis No. 21/2010
ISSN: 1582-5329 pp. 151-159

ON THE RELATION BETWEEN ORDERED SETS AND LORENTZ-MINKOWSKI DISTANCES IN REAL INNER PRODUCT SPACES

Oğuzhan Demirel, Emine Soytürk Seyrantepe

Abstract. Let X be a real inner product space of arbitrary finite or infinite dimension ≥ 2. In [Adv. Geom. 2003, suppl., S1–S12], Benz proved the following statement for $x, y \in X$ with $x < y$: The Lorentz-Minkowski distance between x and y is zero (i.e., $l(x, y) = 0$) if and only if $[x, y]$ is ordered. In [Appl. Sci. 10 (2008), 66–72], Demirel and Soytürk presented necessary and sufficient conditions for Lorentz-Minkowski distances $l(x, y) > 0$, $l(x, y) < 0$ and $l(x, y) = 0$ in n-dimensional real inner product spaces by the means of ordered sets and it's an orthonormal basis.

In this paper, we shall present necessary and sufficient conditions for Lorentz-Minkowski distances with the help of ordered sets in an arbitrary dimensional real inner product spaces. Furthermore, we prove that all the linear Lorentz transformations of X are continuous.

2000 *Mathematics Subject Classification*: 14P99, 46B20, 51F99, 51K99.

Keywords: Real inner product space, Lorentz-Minkowski distance, Lorentz transformation.

1. Introduction

Let X be a real inner product space of arbitrary finite or infinite dimension ≥ 2, i.e., a real vector space furnished with an inner product

$$g : X \times X \longrightarrow \mathbb{R}, \quad g(x, y) = xy$$

satisfying $xy = yx$, $x(y + z) = xy + xz$, $\alpha(xy) = (\alpha x)y$, $x^2 > 0$ (for all $x \neq 0$ in X) for all $x, y, z \in X$, $\alpha \in \mathbb{R}$. For a fixed $t \in X$ satisfying $t^2 = 1$, define

$$t^{\perp} := \{x \in X : \ tx = 0\}.$$

Then, clearly $t^{\perp} \oplus \mathbb{R}t = X$. For any $x \in X$, there are uniquely determined elements $\overline{x} = x - x_0 t \in t^{\perp}$ and $x_0 = tx \in \mathbb{R}$ with

$$x = \overline{x} + x_0 t.$$

Definition 1. The *Lorentz-Minkowski distance* of $x, y \in X$ defined by the expression

$$l(x,y) = (\overline{x} - \overline{y})^2 - (x_0 - y_0)^2.$$

Definition 2. If the mapping $\varphi : X \to X$ preserving the Lorentz-Minkowski distance for each $x, y \in X$, then φ is called *Lorentz transformation.*

Under all translations, Lorentz-Minkowski distances remain invariant and it might be noticed that the theory does not seriously depend on the chosen t, for more details we refer readers to [1].

Let p be an element of $t^{\perp}$ with $p^2 < 1$, and let $k \neq -1$ be a real number satisfying

$$k^2(1 - p^2) = 1.$$

Define

$$A_p(x) := x_0 p + (\overline{x}p)t.$$

for all $x \in X$. Let E denote the identity mapping of X and define

$$B_{p,k}(x) := E + kA_p + \frac{k^2}{k+1}A_p^2.$$

Since A_p is a linear mapping, $B_{p,k}$ is also linear. $B_{p,k}$ is called a Lorentz boost a proper one for $k \geq 1$, an improper one for $k \leq -1$. For the characterization of Lorentz boost, we refer readers to [3].

Theorem 1 (W. Benz [1]). *All Lorentz transformations λ of X are exactly given by*

$$\lambda(x) = (B_{p,k}w)(x) + d$$

with a boost $B_{p,k}$, an orthogonal and linear mapping w from X into X satisfying $w(t) = t$, and with an element d of X.

Notice that a Lorentz transformation λ of X need not be linear.

Theorem 2 (W. Benz [1]). *Let $B_{p,k}$ and $B_{q,K}$ be Lorentz boosts of X. Then $B_{p,k} \circ B_{q,K}$ must be a bijective Lorentz transformation of X fixing 0. Moreover,*

$$B_{p,k} \circ B_{q,K} = B_{r,m} \circ w,$$

where

$$m = \frac{1+pq}{\sqrt{1-p^2}\sqrt{1-q^2}}$$

and

$$p * q := r = \frac{p+q}{1+pq} + \frac{k}{k+1}\frac{(pq)p - p^2 q}{1+pq}.$$

2. Boundedness of linear Lorentz transformations

Definition 2. Let X and Y be normed linear spaces and let $T : X \longrightarrow Y$ be a linear transformation. T will be called a bounded linear transformation if there exist a real number $K \geq 0$ such that

$$\|T(x)\| \leq K\|x\|$$

holds for all $x \in X$.

If we take $\|T\| = \inf\{K\}$ in the above definition, we immediately obtain that

$$\|T(x)\| \leq \|T\|\|x\|.$$

The norm of the linear transformation T defined by the expression

$$\|T\| = \sup\left\{\frac{\|T(x)\|}{\|x\|} : \ x \in X - \{0\}\right\}.$$

There are numbers of alternate expressions for $\|T\|$ in the classical setting as follows:

$$\begin{aligned}
\|T\| &= \sup\{\|T(x)\| : \ \|x\| \leq 1\} \\
\|T\| &= \sup\{\|T(x)\| : \ \|x\| = 1\} \\
\|T\| &= \sup\left\{\frac{\|T(x)\|}{\|x\|} : \ 0 < \|x\| \leq 1\right\} \\
\|T\| &= \inf\{K : \ \|T(x)\| \leq K\|x\| \text{ for all } x \in X\}
\end{aligned}$$

The last statement is always valid, but the other statements is not if the underlying field is not equal to real or complex numbers field, see [6]. The following two theorems are well known and fundamental in functional analysis.

Theorem 3.*Let E and F be normed linear spaces and let $T : E \longrightarrow F$ be a linear transformation. The followings are equivalent:*

(i) *T is continuous at* 0,

(ii) *T is continuous,*

(iii) *There exists $c \geq 0$ such that $\|Tx\| \leq c\|x\|$ for all $x \in E$,*

(iv) $\sup\{\|Tx\| : \ x \in E, \ \|x\| \leq 1\} < \infty$.

Theorem 4. *Let $C(X, X)$ denote the all continuous linear transformations space. For all $T, G \in C(X, X)$ the followings hold:*

(i) *$T \circ G \in C(X, X)$,*

(ii) $\|T \circ G\| \leq \|T\|\|G\|$.

Theorem 5. *All Lorentz boosts of X are bounded.*

Proof. Let $B_{p,k}$ be a Lorentz boost of X. Clearly E is bounded and $\|E\| = 1$. For all $p \in t^{\perp}$ with $p^2 < 1$, A_p is bounded. In fact,

$$\begin{aligned}\|A_p(x)\|^2 =&(x_0 p + (\overline{x}p)t)^2\\ =&x_0^2 p^2 + (\overline{x}p)^2\\ =&x_0^2 p^2 + |\overline{x}p|^2\\ \leq&x_0^2 p^2 + \overline{x}^2 p^2\\ =&(x_0^2 + \overline{x}^2)\|p\|^2\end{aligned}$$

and we get $\|A_p(x)\| \leq \|p\|\|x\|$, i.e., A_p is a bounded transformation of X. Conversely,

$$\begin{aligned}\|p\|^2 =&p^2\\ =&\|p^2 t\|\\ =&\sqrt{(A_p(p))^2}\\ =&\|A_p(p)\|\\ \leq&\|A_p\|\|p\|,\end{aligned}$$

and this implies $\|A_p\| = \|p\|$. Clearly, A_p^2 is a bounded transformation of X and we get

$$\|A_p^2(x)\| \leq \|p\|^2\|x\|.$$

Conversely,

$$\begin{aligned}p^2\|p\| =&\|A_p^2(p)\|\\ \leq&\|A_p^2\|\|p\|,\end{aligned}$$

and then obtain

$$\|A_p^2\| = \|p\|^2.$$

Finally, for $k \geq 1$, we get

$$\begin{aligned}\|B_{p,k}\| =& \sup\left\{\frac{\|B_{p,k}(x)\|}{\|x\|} : \ 0 < \|x\| \leq 1\right\}\\ \leq& 1 + k\|p\| + \frac{k^2}{k+1}\|p\|^2\\ =& k(\|p\| + 1).\end{aligned}$$

A simple calculation shows that $\|B_{p,k}\| \leq 2+|k|(\|p\|+1)$ holds for $k \leq -1$. Obviously all the Lorentz boosts are bounded.

Corollary 1. *All the linear Lorentz transformations are continuous.*

3. On the relation between ordered sets and Lorentz-Minkowski Distances in real inner product spaces

Let X be a real inner product space of arbitrary finite or infinite dimension ≥ 2 and take $x, y \in X$. Define a relation on X by

$$x \leq y \Leftrightarrow l(x, y) \leq 0 \text{ and } x_0 \leq y_0$$

Observe that an element of X that need not be comparable to another element of X, for example neither $e \leq 0$ nor $0 \leq e$ if we take e from $t^{\perp}$. For the properties of "$\leq$", we refer readers to [2]. For the two elements of $x, y \in X$ satisfying $x < y$ ($x \leq y$, $x \neq y$) and define

$$[x, y] = \{z \in X : \ x \leq z \leq y\}.$$

$[x, y]$ is called ordered if and only if,

$$u \leq v \text{ or } v \leq u$$

is true for all $u, v \in [x, y]$.

W. Benz proved the following result:

Theorem 6 (W. Benz [2]). *Let $x, y \in X$ with $x < y$, then $l(x, y) = 0$ if and only if $[x, y]$ is ordered.*

In this section, we present necessary and sufficient conditions for Lorentz-Minkowski distances by the means of ordered sets in a real inner product space of arbitrary finite or infinite dimension ≥ 2.

Theorem 7. *Let X be a real inner product space of dimension ≥ 2 and x, y be elements of X with $x \neq y$ and $x_0 \leq y_0$. Then the followings are equivalent:*

(i) $l(x,y) > 0$,

(ii) *There exists at least one* $s \in X - \{x,y\}$ *such that* $[x,s]$, $[y,s]$ *are ordered while* $[x,y]$ *is not ordered.*

Proof. By the terminology of "$[x,y]$ is not ordered", we mean that $x \leq y$ and $[x,y] = \phi$ or $x \nleq y$. Since all Lorentz-Minkowski distances remains invariant under translations, see [1], instead of considering x and y, we may prove the theorem with respect to 0 and $y - x$.
$(i) \Rightarrow (ii)$. Let us put

$$z := y - x \text{ and } u := \overline{z} + \|\overline{z}\|t.$$

Obviously, $\|\overline{y} - \overline{x}\| > y_0 - x_0$, i.e., $\|\overline{z}\| > |z_0|$ and $l(0,u) = 0$. Since $u_0 = \|\overline{z}\| > 0$ we get $[0,u]$ is ordered. In addition to this, $[z,u]$ is not ordered since $l(z,u) = -((y_0 - x_0) - \|\overline{y} - \overline{x}\|)^2 < 0$. Now define

$$w := \frac{1}{2\|\overline{z}\|}(z_0 + \|\overline{z}\|)u.$$

It is easy to see that $l(0,w) = 0$ and $w_0 = \frac{1}{2}(z_0 + \|\overline{z}\|) > 0$, and thus, we get $[0,w]$ is ordered. Now, we have

$$l(z,w) = \left(1 - \frac{1}{2\|\overline{z}\|}(z_0 + \|\overline{z}\|)\right)^2 \|\overline{z}\|^2 - \left(z_0 - \frac{1}{2}(z_0 + \|\overline{z}\|)\right)^2 = 0$$

and

$$z_0 \leq \frac{z_0 + \|\overline{z}\|}{2} = w_0.$$

Therefore, we immediately obtain that $[z,w]$ is ordered.
$(ii) \Rightarrow (i)$. Assume that $[x,s]$, $[y,s]$ are ordered while $[x,y]$ is not ordered. In this way, we get

$$\begin{aligned} l(x,y) =& l(-x,-y) \\ =& l(s-x, s-y) \\ =& 2\left((-(\overline{s} - \overline{x})(\overline{s} - \overline{y})) + (s_0 - x_0)(s_0 - y_0)\right) \\ >& 0. \end{aligned}$$

Notice that

$$\begin{aligned} (\overline{s} - \overline{x})(\overline{s} - \overline{y}) &\leq |(\overline{s} - \overline{x})(\overline{s} - \overline{y})| \\ &\leq \|\overline{s} - \overline{x}\| \|\overline{s} - \overline{y}\| \\ &= (s_0 - x_0)(s_0 - y_0), \end{aligned}$$

by Cauchy-Schwarz inequality, i.e., we get $(s_0 - x_0)(s_0 - y_0) - (\overline{s} - \overline{x})(\overline{s} - \overline{y}) \geq 0$.

The following theorem can be easily proved when using $-y, -x$ instead of x, y in previous theorem.

Theorem 8. *Let X be a real inner product space of dimension ≥ 2 and x, y be elements of X with $x \neq y$ and $x_0 \leq y_0$. Then followings are equivalent:*

(i) $l(x, y) > 0$,

(ii) There exists at least one $k \in X - \{x, y\}$ such that $[k, x]$, $[k, y]$ are ordered while $[x, y]$ is not ordered.

Theorem 9. *Let X be a real inner product space of dimension ≥ 2 and x, y be elements of X with $x \neq y$ and $x_0 \leq y_0$. Then followings are equivalent:*

(i) $l(x, y) = 0$,

(ii) There exists at least $m, s \in X - \{x, y\}$ such that the $[m, s]$ is ordered and $x, y \in [m, s]$.

Proof. $(i) \Rightarrow (ii)$. Let us set

$$s := \eta\,(y - x) + x$$

for a real number $\eta > 1$. Obviously, we get $l\,(x, s) = 0$ and $0 < y_0 - x_0 < \eta\,(y_0 - x_0)$, i.e., $x_0 < \eta\,(y_0 - x_0) + x_0 = s_0$, i.e., $[x, s]$ is ordered. Likewise, $l\,(y, s) = 0$ and $y_0 - x_0 < \eta\,(y_0 - x_0)$, i.e., $y_0 < \eta\,(y_0 - x_0) + x_0 = s_0$, i.e., $[y, s]$ is ordered.
Now, define

$$m := \lambda\,(y - x) + x$$

for a real number $\lambda < 0$. It is easy to see that $l\,(m, x) = l\,(m, y) = 0$ and $m_0 = \lambda\,(y_0 - x_0) + x_0$ since $\lambda\,(y_0 - x_0) < 0$, i.e., $[m, x]$, $[m, y]$ are ordered sets. Finally, $[m, s]$ is ordered.
$(ii) \Rightarrow (i)$. Demirel and Soytürk, in [5], proved this result for finite dimensional real inner product spaces and it follows verbatimly same as in the proof of them.

Theorem 10. *Let X be a real inner product space of dimension ≥ 2 and x, y be elements of X with $x \neq y$ and $x_0 \leq y_0$. Then followings are equivalent.*

(i) $l(x, y) < 0$,

(ii) There exists at least $s \in X$ such that $[x, s]$, $[s, y]$ are ordered but $[x, y]$ is not ordered.

Proof. $(i) \Rightarrow (ii)$. Let us set

$$z := y - x \text{ and } u := \overline{z} + \|\overline{z}\|\, t.$$

Clearly, $[0,u]$ is ordered since $l\,(0,u) = 0$ and $0 \leq \|\overline{z}\| = u_0$, but $[z,u]$ is not ordered since $l\,(z,u) = -\left(z_0 - \|\overline{z}\|\right)^2 < 0$. Put

$$w := \frac{1}{2\,\|\overline{z}\|}\left(z_0 + \|\overline{z}\|\right) u,$$

and this yields $l\,(0,w) = 0$ and $w_0 = \frac{1}{2}\left(z_0 + \|\overline{z}\|\right) > 0$, i.e., $[0,w]$ is ordered. Finally, we get $l\,(z,w) = 0$, $w_0 = \frac{1}{2}\left(z_0 + \|\overline{z}\|\right) < z_0$ and this implies $[w,z]$ is ordered.
$(ii) \Rightarrow (i)$. Using the Cauchy-Schwarz inequality,

$$\begin{aligned} -\left(\overline{s} - \overline{x}\right)\left(\overline{s} - \overline{y}\right) &\leq \left|\left(\overline{s} - \overline{x}\right)\left(\overline{s} - \overline{y}\right)\right| \\ &\leq \|\overline{s} - \overline{x}\|\,\|\overline{s} - \overline{y}\| \\ &= \left(s_0 - x_0\right)\left(s_0 - y_0\right) \end{aligned}$$

we get

$$-\left(s_0 - x_0\right)\left(y_0 - s_0\right) - \left(\overline{s} - \overline{x}\right)\left(\overline{s} - \overline{y}\right) < 0,$$

i.e.,

$$(s_0 - x_0)(s_0 - y_0) - (\overline{s} - \overline{x})(\overline{s} - \overline{y}) < 0$$

and this inequality yields

$$\begin{aligned} l(x,y) =& l(s-x, s-y) \\ =& 2(-(\overline{s} - \overline{x})(\overline{s} - \overline{y}) + (s_0 - x_0)(s_0 - y_0)) \\ <& 0. \end{aligned}$$

References

[1] Benz, W., *Lorentz-Minkowski distances in Hilbert spaces*, Geom. Dedicata 81, 2000, 219–230.

[2] Benz, W., *On Lorentz-Minkowski geometry in real inner-product spaces*, Advances in Geometry, 2003 (Special Issue), S1-S12.

[3] Benz, W., Schwaiger, J., *A characterization of Lorentz boosts*, Aequation*es Math. 72, 2006, 288–298.

[4] Benz, W., *Classical geometries in modern contexts*, Birkhäauser, Basel, 2005.

[5] Demirel, O. and Soytürk, E., *On the ordered sets in n-dimensional real inner product spaces*, Applied Sciences, Vol. 10 , 2008, 66–72.

[6] Narici, L., Beckenstein, E. and Bachman, G., *Functional analysis and valuation theory*, Pure and Applied Mathematics Series, Vol.5, 1971.

Oğuzhan DEMİREL
Department of Mathematics
Afyon Kocatepe University
Faculty of Science and Arts, ANS Campus
03200 Afyonkarahisar Turkey
email:*odemirel@aku.edu.tr*

Emine SOYTÜRK SEYRANTEPE
Department of Mathematics
Afyon Kocatepe University
Faculty of Science and Arts, ANS Campus
03200 Afyonkarahisar Turkey
email:*soyturk@aku.edu.tr*