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# ABOUT THE EXISTENCE AND UNIQUENESS OF SOLUTION TO FRACTIONAL BURGERS EQUATION

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Abstract. In this work, we study local and global solutions of an evolution problem governed by fractional Bürgers equations. We have generalized Bürgers equation with a fractional degree of Laplacian in the main part and an algebraic degree in nonlinear part. Such equations intervene, naturally, in continuum mechanics area. Our results prove existence, uniqueness and regularity of solutions of Cauchy's problem for Fractional Bürgers equation. These problems arise in a variety of engineering analysis and design situations.

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## 1.Introduction

The one-dimensional Bürgers equation

$$\frac{\partial}{\partial t}u(t,\, x) = \frac{\partial^2}{\partial x^2}u(t,x) + u(t,x)\frac{\partial u}{\partial x} \tag{1.1}$$

was proposed by Bürgers ([3], 1948) in 1948 as a model for turbulent phenomena of viscous fluids. Since then, Bürgers equation has been investigated in many fields of application, such as traffic flows and formation of large clusters in the universe.

In order to model solutions of Navier-Stokes equations, several authors have studied Bürgers equations with random initial conditions, including white and stable noises.

Equation (1.1) can be solved in closed form (in terms of the initial conditions) by using the Hopf-Cole substitution, which reduces it to a heat equation.

Bürgers equations involving in their linear parts fractional powers $\Delta_\alpha := -(-\Delta)^{\alpha/2}$ of the Laplacian, $\alpha \in (0, 2]$, have been investigated in connection with certain models of hydrodynamical phenomena ; see Shlesinger and al. ([12], 1995), Funaki and al. ([4], 1995) and Biler and al. ([2], 1998). In Biler and al. ([2]), Biller, Funaki and Woyczynski studied existence, uniqueness, regularity and asymptotic behavior of solutions to the multidimensional fractal Bürgers-type equation

$$\frac{\partial}{\partial t}u(t,x) = \nu\Delta_\alpha u(t,x) - a\nabla u^r(t,x) \tag{1.2}$$

where $x \in \mathbb{R}^d$, $d \geq 1$, $\alpha \in (0,2]$, $r \geq 1$, and $a \in \mathbb{R}^d$. For $\alpha > 3/2$ and $d = 1$ they prove existence of a unique regular weak solution to (1.2) for initial conditions in $H^1(\mathbb{R})$.

Bürgers equations in financial mathematics arise in connection with the behavior of the risk premium of the market portfolio of risky assets under Black-Scholes assumptions.

In this work, we study local and global solutions of an evolution problem governed by equations of fractional Bürgers kind. Namely, we study the time-fractional Bürgers equation. We have generalized Bürgers equation with fractional degree of Laplacian in the main part and algebraic degree in nonlinear part. Such equations intervene, naturally, in continuum mechanics area and engineering mechanics. These problems arise in a variety of engineering analysis and design situations.

Our results prove existence, uniqueness and regularity of solutions of Cauchy's problem to the following time-Bürgers equation :

$$u_t = u_{xx} - \frac{1}{2}(u^2)_x + f(x,\, t),$$

where

$$x \in I \subset \mathbb{R},\ t \geq 0,\ u : I \times \mathbb{R}^+ \to \mathbb{R}.$$

## 2. Mains Results

### 2.1. A Direct Approach to Weak Solutions

We study existence and uniqueness solutions of Cauchy's problem. We generalize the following Bürgers equation

$$u_t = u_{xx} - \frac{1}{2}(u^2)_x + f(x,t)$$

for one fractional degree of Laplacian in a main part and one algebraic degree in nonlinear part. We introduce and develop the following generalization:

$$u_t = -D^\alpha u - \frac{1}{2}(u^2)_x + f(x,t),\ 0 < \alpha \leq 2, \tag{2.1}$$

with

$$u(x,0) = u_0(x), \tag{2.2}$$

and $D^\alpha \equiv (-\partial^2/\partial x^2)^{\frac{\alpha}{2}}$.

Using a priori elementary estimates, we prove results for Cauchy's problem (2.1). In particular, this will prove a role of operator $-D^\alpha$ and its power relative to the nonlinear term $uu_x$.

Define $D^\alpha$ as ([9] and [10]):

$$(D^\alpha v)(x) = \frac{1}{\Gamma(-\alpha)} \int_0^x (x-z)^{-\alpha-1} v(z) dz, \tag{2.3}$$

where $\Gamma(\alpha) = \int_0^\infty z^{\alpha-1} e^{-z} dz$ denotes Euler's Gamma function.

We look for weak solutions of problem (2.1) with initial data $u(x,0) = u_0(x)$ in $V_2$ such that

$$V_2 = L^\infty(]0, T[\ ; L^2(I)) \cap L^2(]0, T[\ ; H^1(I))$$

satisfying the identity

$$\begin{aligned} &\int u(x,t)\phi(x,t)dx - \int_0^t \int u(x,t)\phi_t(x,t)dxdt + \\ &\int_0^t \int D^{\frac{\alpha}{2}} u(x,t) D^{\frac{\alpha}{2}} \phi(x,t) dxdt - \int_0^t \int \frac{1}{2} u^2(x,t)\phi_x(x,t)dxdt \\ &= \int u_0(x)\phi(x,0)dx + \int_0^t \int f(x,t)dxdt, \\ &\qquad \text{for } \ t \in ]0, T[ \ \text{ and } \ \phi(x,t) \in H^1(I \times ]0, T[). \end{aligned} \tag{2.4}$$

In order to simplify our construction, suppose $u(t) \in H^1(I)$ for $t \in ]0, T[$ instead of $u(t) \in H^{\frac{\alpha}{2}}(I)$ for $t \in ]0, T[$ which will can be waiting for an ordinary generalization of a definition of a weak solution of one parabolic (see [7]).

Suppose, also, initial condition $u_0(x) \in H^1(I)$.

**Theorem 1.** ([2]) Let $\frac{3}{2} < \alpha \leq 2$, $T > 0$, and $u_0(x) \in H^1(I)$. Then Cauchy's problem (2.1)-(2.2) has an unique weak solution $u \in V_2$. Moreover, $u$ satisfies the following regularity properties:

$$u \in L^\infty(]0, T[\ ; H^1(I)) \cap L^2(]0, T[\ ; H^{1+\frac{\alpha}{2}}(I)) \tag{2.5}$$

and

$$u_t \in L^\infty(]0, T[\ ; L^2(I)) \cap L^2(]0, T[\ ; H^{\frac{\alpha}{2}}(I)) \tag{2.6}$$

*Proof.* Suppose $u$ is a weak solution of (2.1)-(2.2) and let $S_n$ be a truncation operator such that $u_n = S_n u$ then we can consider the following approximate problem:

$$(u_n)_t = -D^\alpha u_n - \frac{1}{2}(u_n^2)_x + f(x,t), \quad 0 < \alpha \leq 2 \tag{2.7}$$

with initial data $u_{n_{|t=0}} = S_n u_0$.

Let us multiply (2.7) by $u_n$, then

$$\frac{d}{dt}\int u_n^2(x,t) + \int (D^{\frac{\alpha}{2}} u_n(x,t))^2 + \int u_n(x,t)(u_n)_x(x,t)u_n(x,t) = \int f(x,t)u_n(x,t)$$

which implies

$$\frac{d}{dt}\int u_n^2(x,t) + \int (D^{\frac{\alpha}{2}} u_n(x,t))^2 + \int u_n^2(x,t)(u_n)_x(x,t) = \int f(x,t)u_n(x,t).$$

One has

$$\int u_n^2(x,t)(u_n)_x(x,t) = (\frac{1}{3}u_n^3(x,t))\Big|_I = CS(t) = C_1 \left|u_n\right|_2.$$

Then it holds that

$$\frac{d}{dt}\left|u_n\right|_2^2 + \left|D^{\frac{\alpha}{2}} u_n(t)\right|_2^2 \leq (\left|f\right|_2 + C_1)\left|u_n\right|_2. \tag{2.8}$$

Likewise, upon differentiation in formula (2.7) according to $x$ and multiply by $(u_n)_x$, we obtain

$$\tfrac{d}{dt}\int u_n(x,t)(u_n)_x(x,t) + \int (D^{\frac{\alpha}{2}} u_n(x,t))(u_n)_x(x,t)$$
$$+\int \tfrac{1}{2}(u_n^2(x,t))_x(u_n)_x(x,t) = \int f(x,t)(u_n)_x(x,t),$$

which implies

$$\tfrac{1}{2}\tfrac{d}{dt}\int u_n^2(x,t) + \int (D^{\frac{\alpha}{2}} u_n(x,t))(u_n)_x(x,t)$$
$$+\int \tfrac{1}{2}(u_n^2(x,t))_x(u_n)_x(x,t) = \int f(x,t)(u_n)_x(x,t)$$

It holds that

$$\frac{d}{dt}\left|(u_n)_x\right|_2^2 + 2\left|D^{1+\frac{\alpha}{2}} u_n(t)\right|_2^2 \leq \left|(u_n)_x\right|_3^3 + 2\left|f\right|_2\left|u_n\right|_2 \tag{2.9}$$

because

$$-\int \frac{1}{2}(u_n^2)_x u_x = \int u_n\ (u_n)_x (u_n)_{xx} = \frac{1}{2}\int u_n\ ((u_n)_x^2)_x = -\frac{1}{2}\int (u_n)_x^3.$$

Now, a part of right member of (2.9) can be approximated by

$$|(u_n)_x|_3^3 \leq \|u_n\|_{1,3}^3 \leq C\,\|u_n\|_{1+\alpha/2}^{7/(2+\alpha)}\,|u_n|_2^{3-7/(2+\alpha)} \leq \|u_n\|_{1+\alpha/2}^2 + C\,|u_n|_2^m\,,$$

for any $m > 0$.

Assumption $\alpha > 3/2$ has been used in the interpolation of $W^{1,3}$ of norm of $u$ by norms of its fractional derivative $7/(2+\alpha)$. Indeed, that one follows from ([6], 1982; p. 99). Devising this with (2.7), (2.8) and (2.9) we obtain

$$\frac{d}{dt}\,\|u_n\|_1^2 + \|u_n\|_{1+\alpha/2}^2 \leq C(|f|_2\,|u_n|_2 + |u_n|_2^2 + |u_n|_2^m + C_1)$$

and by (2.8) one has

$$\frac{d}{dt}\,|u_n|_2^2 \leq\ (|f|_2 + C_1)\,|u_n|_2\ \Rightarrow |u_n(t)|_2 \leq M + |(u_n)_0|_2\,,\ \ \forall\ t\in[0,T],$$

hence, we obtain

$$\|u_n(t)\|_1^2 + \int_0^t \|u_n(t)\|_{1+\alpha/2}^2\,ds \leq C = C(T,\ f,\ \|(u_n)_0\|_1). \tag{2.10}$$

Now, approximate a derivative according to the time of a solution. Multiply (2.1) by $(u_n)_t$

$$\tfrac{d}{dt}\int u_n(x,t)(u_n)_t(x,t) + \int (D^{\frac{\alpha}{2}}u_n(x,t))(u_n)_t(x,t)$$

$$+\int u_n(x,t)(u_n)_x(x,t)(u_n)_t(x,t) = \int f(x,t)(u_n)_t(x,t).$$

Hence

$$\tfrac{1}{2}\tfrac{d}{dt}\int (u_n^2(x,t))_t + \tfrac{1}{2}\int D^{\frac{\alpha}{2}}(u_n^2(x,t))_t +$$

$$\int u_n(x,t)(u_n)_x(x,t)(u_n)_t(x,t) = \int f(x,t)(u_n)_t(x,t).$$

After some calculations, we obtain

$$\frac{d}{dt}\,|(u_n)_t|_2^2 + \left|D^{\frac{\alpha}{2}}(u_n)_t\right|_2^2 = -\int (u_n)_x(u_n)_t^2 + 2\int f(x,t)(u_n)_t(x,t) \tag{2.11}$$

since

$$-\int (u_n\ (u_n)_x)_t (u_n)_t = -\int\ (u_n)_x (u_n)_t^2 - \frac{1}{2}\int u_n\ ((u_n)_t^2)_x - \frac{1}{2}\int (u_n)_x (u_n)_t^2.$$

Now, approximate a right member (2.7) by

$$\frac{1}{2}\int |(u_n)_x|\,(u_n)_t^2 \le C\,\|(u_n)_t\|_{\alpha/2}^{1/\alpha}\ |(u_n)_t|_2^{2-1/\alpha}\,|(u_n)_x|_2$$

$$\le \frac{1}{2}\,\|(u_n)_t\|_{1+\alpha/2}^2 + C\,|(u_n)_t|_2^2$$

and

$$\int f(x,t)(u_n)_t(x,t)\ \le |f|_2\,|(u_n)_t|_2\,.$$

A classical Gronwall inequality gives

$$|(u_n)_t(t)|_2^2 + \int_0^t \|(u_n)_t(s)\|_{\alpha/2}^2\ ds \le C(T) \tag{2.12}$$

It holds, from (2.10) and (2.12), that a solution $u_n$ is bounded. Then it is sufficient in order to apply approximation Galerkin's procedure. Hence, we can extract a subsequence which converges to a limit $u$ in $L^\infty(]0,\,T[\ ;\,H^1(I))\cap L^2(]0,\,T[\ ;\,H^{1+\frac{\alpha}{2}}(I))$. To finish, it remains to know if $u$ is a solution of problem?

Since injection of $H^1(I)$ into $L^2(I)$ is compact, we can apply Ascoli theorem and conclude a strongly convergence of $(u_n)_{n\in\mathbb{N}}$ to $u$ in $L^2(]0,\,T[\ ;\,L^2(I))$.

In order to conclude, it is enough to prove that $(u_n)^2$ converges strongly to $u^2$ in $L^1(]0,\,T[\ ;\,L^2(I))$. Remark that

$$\left\|(u_n)^2 - u^2\right\|_{L^1(]0,T[\,;\,L^2(I))} \le \|u_n - u\|_{L^1(]0,T[\,;\,L^4(I))}\,\big(\|u_n\|_{L^1(]0,T[\,;\,L^4(I))}$$

$$+\,\|u\|_{L^1(]0,T[\,;\,L^4(I))}\big),$$

it is enough to prove that $u_n - u$ converges strongly in $L^1(]0,\,T[\ ;\,L^4(I))$. This last result holds by Gagliardo-Nirenberg's inequality ([1], [2])

$$\|u_n - u\|_{L^2(]0,T[;\,L^4(I))} \le C\,\|u_n - u\|_{L^2(]0,T[\,;\,L^4(I))}^{1-\frac{1}{4}}\,\|\nabla(u_n - u)\|_{L^2(]0,T[\,;\,L^4(I))}^{\frac{1}{4}}$$

$$\le C\,\|u_n - u\|_{L^2(]0,T[\,;\,L^4(I))}^{1-\frac{1}{4}}$$

and to prove that $D^\alpha u_n$ converges strongly to $D^\alpha u$ in $L^1(]0,\,T[\ ;\,L^2(I))$.

In the same way, we remark that

$$\|D^\alpha u_n - D^\alpha u\|_{L^1(]0,T[\,;L^2(I))} \leq$$

$$\left\| \frac{\partial^2 u_n}{\partial x^2} - \frac{\partial^2 u}{\partial x^2} \right\|_{L^1(]0,T[\,;\,H^{1+\frac{\alpha}{2}}(I))} \left( \left\| \frac{\partial^2 u_n}{\partial x^2} \right\|_{L^1(]0,T[\,;\,H^{1+\frac{\alpha}{2}}(I))} + \left\| \frac{\partial^2 u}{\partial x^2} \right\|_{L^1(]0,T[\,;\,H^{1+\frac{\alpha}{2}}(I))} \right)$$

and since the term $\partial^2/\partial x^2$ is linear, approach problem converges weakly to a limit point, then the existence holds.

Now we prove uniqueness solution. Consider two weak solutions $u$ and $v$ of (2.1). Then their difference $w = u - v$ satisfies

$$\begin{aligned} \tfrac{d}{dt} |w|_2^2 + 2 \left| D^{\frac{\alpha}{2}} w(t) \right|_2^2 &= 2 \textstyle\int (v v_x - u\ u_x) w \\ &= -2 \textstyle\int (v w w_x - w^2\ u_x) = 2 \int w^2 (v_x/2 -\ u_x). \end{aligned} \tag{2.13}$$

A right member of (2.13) can be limited and we obtain

$$\begin{aligned} |w|_4^2 \,|v_x - 2\ u_x|_2 &\leq C\, \|w\|_{\alpha/2}^{1/\alpha}\ \ |w|_2^{2-1/\alpha} \left(|u_x|_2 + |v_x|_2\right) \\ &\leq \tfrac{1}{2}\, \|w\|_{1+\alpha/2}^2 + C\,|w|_2^2\,. \end{aligned}$$

From (2.10), a factor $(|u_x|_2 + |v_x|_2)$ is bounded. By Gronwall's lemma it holds that $w(t) \equiv 0$ on $[0,\, T]$

**2.2. Parabolic Reguralization**

• For $\alpha > 3/2$, a diffusion operator $D^\alpha$ is strong in order to control a nonlinear part $\frac{1}{2}(u^2)_x$, furthermore Cauchy problem (2.1) has one and only one solution.

• For $\alpha \leq 3/2$, we cannot wait to prove uniqueness of weak solution according to the time for initial data (2.2). We shall use an other technical to obtain weak solutions. The construction will be done by a parabolic regularization method. Namely, we study the problem

$$\begin{aligned} u_t &= -D^\alpha u - \tfrac{1}{2}(u^2)_x + \varepsilon u_{xx}\ \ + f(x,t) \\ u(x,0) &= u_0(x), \end{aligned} \tag{2.14}$$

with $u_\varepsilon = u$, $\varepsilon > 0$ (see, for example, ([1], 1979) ; ([8], 1969)).

In particular, solutions of Bürgers equation can be obtained as limits of solutions of

$$u_t = -\frac{1}{2}(u^2)_x + f(u) + \varepsilon u_{xx}\ \ \text{as } \varepsilon \to 0.$$

**Theorem 2.** ([2]) Let $0 < \alpha \leq 2$. Let $u_\varepsilon = u$, $(\varepsilon > 0)$, be a solution of Cauchy problem (2.14), with $u_0 \in L^1 \cap H^1$, $(u_0)_x \in L^1$. Then, for all $t \geq 0$, we have

$$|u(t)|_2 \leq C_2, \quad |u(t)|_1 \leq C_1, \qquad |u_x(t)|_1 \leq C_{x,1}.$$

*Proof.* The existence of solutions for regularized equation (2.14) is standard as previously. Denote the operator $A = -D^\alpha - D^2$. Then for every $v \in D(A)$ (domain of $A$)([1]), one has

$$\int (Av)sgn(v(x)) \leq 0 \tag{2.15}$$

In fact,

$$\int (Av)sgn(v(x)) = -\int (D^\alpha v + D^2 v)sgn(v(x))$$

and

$$\int (Av)sgn(v(x)) = \lim_{(s\to 0)} s^{-1} \int (e^{sA}v(x) - v(x))sgn(v(x))$$

$$\leq \lim_{(s\to 0)} \sup s^{-1} (\int (\left|e^{sA}v(x)\right| - \int |v(x))|) \leq 0$$

Then, let us multiply (2.14) by $sgn(u)$ and integrate on $I$, we obtain

$$\tfrac{d}{dt} \int |u(x,t)| = -\int (D^\alpha u + \varepsilon u_{xx})sgn(u) - \int \tfrac{1}{2}(u^2)_x sgn(u)$$

$$+\int f(x,t)sgn(u) \leq -\int \tfrac{1}{2}(u^2)_x sgn(u) + \int f(x,t)sgn(u).$$

What allows to conclude that $\frac{d}{dt}|u(t)|_1$ is bounded. By integration, on $[0,\, t]$, of this last quantity we prove that $|u(t)|_1$ is bounded.

Introduce a function $sgn_\eta$ called function of sign ([5]) of increasing regularization, $\eta > 0$, such that $sgn_\eta \to sgn$ as $\eta \to 0$. For such regularization we have

$$\int (u^2)_x sgn_\eta u = \left[(u^2)sgn_\eta u\right]_I - \int u^2 (sgn'_\eta u)u_x$$

$$= trace((u^2)sgn_\eta u)\big|_I - \int u^2(sgn'_\eta u)u_x = M_{Tr} - \int u^2(sgn'_\eta u)u_x.$$

Therefore $u_x$ is bounded in $H^1$ for every $\varepsilon > 0$. We see that the integral $\int u^2(sgn'_\eta u)u_x$ converges to 0 as $\eta \to 0$.

Then by multiplying (2.14) by $sgn(u_x)$ we obtain

$$\frac{d}{dt}|u_x|_1 \leq -\int (u\ u_x)_x sgn(u_x) + \int f(x,t)sgn(u_x).$$

Still, let us approach $sgn$ by functions $sgn_\eta$, we transform the integral

$$\int (u\ u_x)_x sgn(u_x) = trace((u\ u_x) sgn_\eta u_x)|_I - \int u\ u_{xx} u_x (sgn'_\eta(u_x)).$$

We see that the integral $\int u\ u_{xx} u_x sgn'_\eta(u_x)$ converges to 0 as $\eta \to 0$. We can now pass to a limit on $\varepsilon$ when $\varepsilon \to 0$ in regularized equation (2.14).

In what follows, by a weak solution of (2.1) we hear $u \in L^\infty((0,T)\,;\,L^2(I))$ and satisfying the following equation:

$$\int u(x,t)\phi(x,t) - \int_0^t \int u(x,t)\phi_t(x,t) + \int_0^t \int (uD^\alpha\phi - \tfrac{1}{2}u^2\phi_x$$

$$= \int u_0(x)\phi(x,0) + \int_0^t \int f(x,t) \text{ for } t \in (0,T),$$

$\phi \in C^\infty(I \times [0,T])$ with compact support. Let us note that we do not assume $u(t) \in H^{\alpha/2}$.

**Corolary 1.** Let $0 < \alpha < 2$, $u_0 \in L^1 \cap H^1$ with $(u_0)_x \in L^1$, there is a weak solution $u$ of (2.1) obtained as a limit of a sequence $u_\varepsilon$ such that

$$u \in L^\infty((0,\infty)\,;\,L^\infty(I)) \cap L^\infty((0,\infty)\,;\,H^{1/2-\delta}(I)),$$

for every $\delta > 0$. Furthermore, $u \in L^\infty((0,\infty)\,;\,BV(I))$ with

$$\|u(t)\|_{BV(I)} \leq |(u_0)_x|_1$$

*Proof.* From injection $W^{1,1} \subset H^{1/2-\delta}$ we conclude that a subsequence $u_\varepsilon$ converges weakly to a limit function $u$ in $L^\infty((0,\infty)\,;\,H^{1/2-\delta}(I))$. A sequence $u_\varepsilon$ is bounded in $L^\infty$ holds from obvious inequality $|u|_\infty \leq |u_x|_1$. The strong convergence in $L^\infty((0,\infty)\,;\,H^{1/2-\delta}(I))$ is a consequence of Aubin-Lions's lemma ([8], p. 57).

**Remark 1.**

• If $\alpha > 1/2$ then weak solutions of (2.14) (constructed by a parabolic regularization method) remain in $H^1(\mathbb{R})$ for $t \in [0,T)$ for some $T > 0$. Moreover, if $\|u_0\|_1$ is enough small, then these regulary solutions are global in the time.

• If $\alpha < 1$ these weak solutions, defined on time-finite interval, introduce shocks.

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