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THE UNIVALENCE OF A NEW INTEGRAL OPERATOR

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Abstract. We derive some criteria for univalence of a new integral operator for analytic functions in the open unit disk.

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1. Introduction

Let $\mathcal{A}$ be the class of the functions $f(z)$ which are analytic in the open unit disk

$$U = \{z \in \mathbb{C} :| z |< 1\} \text{ and } f(0) = f'(0) - 1 = 0.$$

We denote by S the subclass of $\mathcal{A}$ consisting of functions $f(z) \in \mathcal{A}$ which are univalent in U. Miller and Mocanu [4] have considered the integral operator M_α given by

$$M_\alpha(z) = \left\{ \frac{1}{\alpha} \int_0^z (f(u))^{\frac{1}{\alpha}} u^{-1} du \right\}^\alpha, \quad z \in U \tag{1.1}$$

for functions $f(z)$ belonging to the class $\mathcal{A}$ and for some α be complex numbers, $\alpha \neq 0$. It is well known that $M_\alpha(z) \in S$ for $f(z) \in S^*$ and $\alpha > 0$, where S^* denotes the subclass of S consisting of all starlike functions $f(z)$ in U.

In this paper, we introduce a new integral operator $J_{\gamma,n}$which is defined by

$$J_{\gamma,n}(z) = \left\{ \frac{1}{\gamma} \int_0^z u^{-n} \left(f\left(u\right)\right)^{\frac{1}{\gamma}+n-1} du \right\}^\gamma, \quad z \in U \tag{1.2}$$

for functions $f(z) \in \mathcal{A}, n \in N$ and for some complex numbers γ , $\gamma \neq 0$.

From (1.2), for $n = 1$ and $\gamma = \alpha$ we obtain the integral operator $M_\alpha(z)$.

If $\frac{1}{\gamma} = 1$ and $n \in N - \{0, 1\}$, from (1.2) we obtain the integral operator

$$K_n(z) = \int_0^z \left(\frac{f(u)}{u} \right)^n du, \quad z \in U \tag{1.3}$$

which is the case particular of the integral operator Kim-Merkes [2] ,for $\alpha = n$.

From(1.2), for $\frac{1}{\gamma} = 1, n = 1$ we obtain the integral operator Alexander define by

$$H(z) = \int_0^z \frac{f(u)}{u} du \tag{1.4}$$

If $n = 0$, from (1.2) we obtain the integral operator define by

$$G_\gamma(z) = \left\{ \frac{1}{\gamma} \int_0^z (f(u))^{\frac{1}{\gamma}-1} du \right\}^\gamma \tag{1.5}$$

In the present paper, we consider some sufficient conditions for the integral operator $J_{\gamma,n}$ to be in the class S.

2. Preliminary results

To discuss our problems for univalence of integral operator $J_{\gamma,n}$, we need the following lemmas.

Lemma 2.1 [7] *Let α be a complex number with Re $\alpha > 0$ and $f(z) \in A$.If $f(z)$ satisfies*

$$\frac{1 - |z|^{2 \operatorname{Re} \alpha}}{\operatorname{Re} \alpha} \left| \frac{z f''(z)}{f'(z)} \right| \le 1 \tag{2.1}$$

for all $z \in U$, then the function

$$F_\alpha(z) = \left\{ \alpha \int_0^z u^{\alpha-1} f'(u) du \right\}^{\frac{1}{\alpha}} \tag{2.2}$$

is in the class S.

Lemma 2.2 *(Schwarz[3]) Let $f(z)$ the function regular in the disk $U_R = \{z \in \mathbb{C} : |z| < R\}$ with $|f(z)| < M, M$ fixed. If $f(z)$ has in $z = 0$ one zero with multiply $\ge m$, then*

$$|f(z)| \le \frac{M}{R^m} |z|^m , z \in U_R \tag{2.3}$$

the equality (in the inequality (2.3) *for* $z \neq 0$*) can hold only if*

$$f(z) = e^{i\theta} \frac{M}{R^m} z^m,$$

where θ *is constant.*

Lemma 2.3 *(Caratheodory* [1] , [5]*) Let* f *be analytic function in* U*, with* $f(0) = 0$.

If f *satisfies*

$$\operatorname{Re} f(z) \leq M \tag{2.4}$$

for some $M > 0$*, then*

$$(1 - |z|)\, |f(z)| \leq 2M\, |z|\,, \quad z \in U \tag{2.5}$$

2. Main results

Theorem 3.1. *Let* γ *be a complex number,* $a = \operatorname{Re} \frac{1}{\gamma} > 0$*,* $n \in N$ *and* $f(z) \in A$*,* $f(z) = z + a_2 z^2 + ...$

If

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{(2a+1)^{\frac{2a+1}{2a}} |\gamma|}{2(1 + |\gamma|\, |n-1|)} \tag{3.1}$$

for all $z \in U$*, then the integral operator* $J_{\gamma,n}$ *define by (1.2) is in the class* S*.*

Proof. We observe that

$$J_{\gamma,n}(z) = \left\{ \frac{1}{\gamma} \int_0^z u^{\frac{1}{\gamma}-1} \left(\frac{f(u)}{u} \right)^{\frac{1}{\gamma}+n-1} du \right\}^{\gamma} \tag{3.2}$$

Let us consider the function

$$g(z) = \int_0^z \left(\frac{f(u)}{u} \right)^{\frac{1}{\gamma}+n-1} du. \tag{3.3}$$

The function g is regular in U.

We define the function $p(z) = \frac{zg''(z)}{g'(z)}$, $z \in U$ and we obtain

$$p(z) = \frac{zg''(z)}{g'(z)} = \left(\frac{1}{\gamma} + n - 1 \right) \left(\frac{zf'(z)}{f(z)} - 1 \right) \tag{3.4}$$

From (3.1) and (3.4) we have

$$|p(z)| \leq \frac{(2a+1)^{\frac{2a+1}{2a}}}{2} \tag{3.5}$$

for all $z \in U$.

The function p satisfies the condition $p(0) = 0$ and applying Lemma 2.2 we obtain

$$|p(z)| \leq \frac{(2a+1)^{\frac{2a+1}{2a}}}{2} |z|, \quad z \in U \tag{3.6}$$

From (3.6) we get

$$\frac{1-|z|^{2a}}{a} \left| \frac{zg''(z)}{g'(z)} \right| \leq \frac{(2a+1)^{\frac{2a+1}{2a}}}{2} \frac{\left(1-|z|^{2a}\right)}{a} |z| \tag{3.7}$$

for all $z \in U$.

Because

$$\max_{|z|\leq 1} \left\{ \frac{1-|z|^{2a}}{a} |z| \right\} = \frac{2}{(2a+1)^{\frac{2a+1}{2a}}}$$

from (3.7) we obtain

$$\frac{1-|z|^{2a}}{a} \left| \frac{zg''(z)}{g'(z)} \right| \leq 1 \tag{3.8}$$

for all $z \in U$.

From (3.8) and because $g'(z) = \left(\frac{f(z)}{z}\right)^{\frac{1}{\gamma}+n-1}$, by Lemma 2.1. we obtain that the integral operator $J_{\gamma,n}$ is in the class S.

Corallary 3.2. *Let γ be a complex number, $a = \operatorname{Re} \frac{1}{\gamma} > 0$ and $f(z) \in A,\ f(z) = z + a_2 z^2 + ...$*

If

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{(2a+1)^{\frac{2a+1}{2a}}}{2} |\gamma| \tag{3.9}$$

for all $z \in U$, then the integral operator M_γ given by (1.1) is in the class S.

Proof. For $n = 1$, from Theorem 3.1 we obtain that $M_\gamma(z)$ is in the class S

Corollary 3.3. *Let $n \in N - \{0,1\}$ and $f \in A,\ f(z) = z + a_2 z^2 + ...$*

If

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{3\sqrt{3}}{2\left(1+|n-1|\right)} \tag{3.10}$$

for all $z \in U$*, then the integral operator* K_n *define by (1.3) belongs to class* S.

Proof. We take $\frac{1}{\gamma} = 1$ in Theorem 3.1 and we get $K_n \in S$.

Corollary 3.4. *Let the function* $f(z) \in A,\ f(z) = z + a_2z^2 + ...$
If

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{3\sqrt{3}}{2}, \quad z \in U \tag{3.11}$$

then, the integral operator H *define by* (1.4) *is in the class* S.

Proof. In Theorem 3.1. we take $\frac{1}{\gamma} = 1$ and $n = 1$.

Corollary 3.5. *Let* γ *be a complex number* $a = \operatorname{Re}\frac{1}{\gamma} > 0$ *and* $f \in A$, $f(z) = z + a_2z^2 + a_3z^3 + ...$
If

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{(2a+1)}{2}^{\frac{2a+1}{2a}} \cdot \frac{|\gamma|}{1 + |\gamma|} \tag{3.12}$$

for all $z \in U$, *then the integral operator* G_γ *given by* (1.5) *is in the class* S.

Proof. For $n = 0$, from Theorem 3.1 we have $G_\gamma \in S$.

Theorem 3.6. *Let* γ *be a complex number,* $\operatorname{Re}\frac{1}{\gamma} > 0,\ f \in A$, $f(z) = z + a_2z^2 + ...$
If

$$\operatorname{Re}\left\{e^{i\theta}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} \leq \frac{|\gamma|\operatorname{Re}\frac{1}{\gamma}}{4\left(1 + |\gamma|\,|n-1|\right)},\ 0 < \operatorname{Re}\frac{1}{\gamma} < 1 \tag{3.13}$$

or

$$\operatorname{Re}\left\{e^{i\theta}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} \leq \frac{|\gamma|}{4\left(1 + |\gamma|\,|n-1|\right)},\ \operatorname{Re}\frac{1}{\gamma} \geq 1 \tag{3.14}$$

for all $z \in U,\ \theta \in [0, 2\pi]$ *and* $n \in N$*, then the integral operator* $J_{\gamma,n}$ *is in the class* S.

Proof. The integral operator $J_{\gamma,n}$ is the form (3.2). We consider the function $g(z)$ which is the form (3.3). We have

$$\frac{zg''(z)}{g'(z)} = \left(\frac{1}{\gamma} + n - 1\right)\left(\frac{zf'(z)}{f(z)} - 1\right) \tag{3.15}$$

Let us consider the function

$$\psi(z) = e^{i\theta}\left(\frac{zf'(z)}{f(z)} - 1\right), \quad z \in U,\ \theta \in [0, 2\pi] \tag{3.16}$$

and we observe that $\psi(0) = 0$.

By (3.13) and Lemma 2.3. for $\operatorname{Re}\frac{1}{\gamma}\in(0,1)$
we obtain

$$|\psi(z)|\leq\frac{|z|\,|\gamma|\operatorname{Re}\frac{1}{\gamma}}{2(1-|z|)(1+|\gamma|\,|n-1|)},\quad z\in U,\ n\in N \tag{3.17}$$

From (3.14) and Lemma 2.3, for $\operatorname{Re}\frac{1}{\gamma}\in[1,\infty)$ we have

$$|\psi(z)|\leq\frac{|z|\,|\gamma|}{2(1-|z|)(1+|\gamma|\,|n-1|)},\quad z\in U,\ n\in N \tag{3.18}$$

From (3.15) and (3.17) we get

$$\frac{1-|z|^{2\operatorname{Re}\frac{1}{\gamma}}}{\operatorname{Re}\frac{1}{\gamma}}\left|\frac{zg''(z)}{g'(z)}\right|\leq\frac{\left(1-|z|^{2\operatorname{Re}\frac{1}{\gamma}}\right)|z|}{2(1-|z|)},\quad z\in U,\quad \operatorname{Re}\frac{1}{\gamma}\in(0,1) \tag{3.19}$$

Because $1-|z|^{2\operatorname{Re}\frac{1}{\gamma}}\leq 1-|z|^2$ for $\operatorname{Re}\frac{1}{\gamma}\in(0,1)$, $z\in U$, from (3.19) we have

$$\frac{1-|z|^{2\operatorname{Re}\frac{1}{\gamma}}}{\operatorname{Re}\frac{1}{\gamma}}\left|\frac{zg''(z)}{g'(z)}\right|\leq 1 \tag{3.20}$$

for all $z\in U$, $\operatorname{Re}\frac{1}{\gamma}\in(0,1)$.

For $\operatorname{Re}\frac{1}{\gamma}\in[1,\infty)$ we have $\frac{1-|z|^{2\operatorname{Re}\frac{1}{\gamma}}}{\operatorname{Re}\frac{1}{\gamma}}\leq 1-|z|^2$, $z\in U$ and from (3.15) and (3.18) we obtain

$$\frac{1-|z|^{2\operatorname{Re}\frac{1}{\gamma}}}{\operatorname{Re}\frac{1}{\gamma}}\left|\frac{zg''(z)}{g'(z)}\right|\leq 1 \tag{3.21}$$

for all $z\in U$, $\operatorname{Re}\frac{1}{\gamma}\in[1,\infty)$.

Using (3.20) and (3.21) by Lemma 2.1. it results that $J_{\gamma,n}$ given by (1.2) is in the class S.

Corollary 3.7. *Let* γ *be a complex number,* $\operatorname{Re}\frac{1}{\gamma}>0$, $f\in A$, $f(z)=z+a_2z^2+\ldots$

If

$$\operatorname{Re}\left\{e^{i\theta}\left(\frac{zf'(z)}{f(z)}-1\right)\right\}\leq\frac{|\gamma|\operatorname{Re}\frac{1}{\gamma}}{4},\quad \operatorname{Re}\frac{1}{\gamma}\in(0,1) \tag{3.22}$$

or

$$\operatorname{Re}\left\{e^{i\theta}\left(\frac{zf'(z)}{f(z)}-1\right)\right\}\leq\frac{|\gamma|}{4},\quad \operatorname{Re}\frac{1}{\gamma}\in[1,\infty) \tag{3.23}$$

for all $z \in U$ *and* $\theta \in [0, 2\pi]$, *then the integral operator* M_γ *define by* (1.1) *is in the class* S.

Proof. In Theorem 3.6. we take $n = 1$.

Corollary 3.8. *Let* $n \in N - \{0\}$ *and* $f \in A$, $f(z) = z + a_2 z^2 + ...$ *If*

$$\operatorname{Re}\left\{e^{i\theta}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} \leq \frac{1}{4(1 + |n-1|)} \tag{3.24}$$

for all $z \in U$ *and* $\theta \in [0, 2\pi]$, *then the integral operator* K_n *given by* (1.3) *is in the class* S.

Proof. For $\gamma = 1$, from Theorem 3.6. we obtain Corollary 3.8.

Corollary 3.9. *Let* γ *be a complex number* $\operatorname{Re}\frac{1}{\gamma} > 0$ *and* $f \in A$, $f(z) = z + a_2 z^2 + ...$

If

$$\operatorname{Re}\left\{e^{i\theta}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} \leq \frac{|\gamma| \operatorname{Re}\frac{1}{\gamma}}{4(1 + |\gamma|)}, \quad \operatorname{Re}\frac{1}{\gamma} \in (0, 1) \tag{3.25}$$

or

$$\operatorname{Re}\left\{e^{i\theta}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} \leq \frac{|\gamma|}{4(1 + |\gamma|)}, \quad \operatorname{Re}\frac{1}{\gamma} \in [1, \infty) \tag{3.26}$$

then the integral operator G_γ *given by* (1.5) *is in the class* S.

Proof. In Theorem 3.6. we take $n = 0$.

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