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DIFFERENTIAL SUBORDINATION DEFINED BY USING EXTENDED MULTIPLIER TRANSFORMATIONS OPERATOR AT THE CLASS OF MEROMORPHIC FUNCTIONS

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Abstract.By using the generalized multiplier transformations, $I^m(\lambda, \ell)f(z)$ $(z \in U)$, we obtain interesting properties of certain subclass of p-valent meromorphic functions.

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1. Introduction

Let $H(U)$ be the class of analytic functions in the unit disk $U = \{z : |z| < 1\}$ and denote by $U^* = U\backslash\{0\}$. We can let
$A(n) = \left\{f \in H\left(U\right), f\left(z\right) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + ..., z \in U\right\}$
with $A(1) = A$. Let $\sum_{p,k}$ denote the class of functions in U^* of the form:

$$f\left(z\right) = \frac{1}{z^p} + a_k z^k + a_{k+1}z^{z+1} + ..., \left(p \in N = \{1, 2, 3, ...\}\right),$$

where k is integer , $k \geq -p + 1, p \in N$ which are regular in the punctured disk U^*. If $f(z)$ and $g(z)$ are analytic in U, we say that $f(z)$ is subordinate to $g(z)$ written symbolically as follows:

$$f \prec g \ (z \in U) \ or \ f(z) \prec g(z) \ (z \in U),$$

if there exists a Schwarz function $w(z)$, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$, such that $f(z) = g(w(z))$ $(z \in U)$. Indeed it is known that $f(z) \prec g(z)$ $(z \in U) \Rightarrow f(0) = g(0)$ and $f(U) \subset g(U)$. Further, if the function $g(z)$ is univalent in U, then we have the following equivalent (see [7, p.4])

$$f(z) \prec g(z) \ (z \in U) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \prec g(U).$$

A function $f \in H(U)$ is said to be convex if it is univalent and $f(U)$ is a convex domain. It is well known that the function f is convex if and only if $f'(0) \neq 0$ and

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>0 \quad (z\in U).$$

We denote by that class of functions by K.

Definition 1. [2] *Let the function $f(z) \in A(n)$. For $m \in N_0 = N \cup \{0\}, \lambda \geq 0, \ell \geq 0$. The extended muliplier transformation $I^m(\lambda,\ell)$ on $A(n)$ is defined by the following infinite series:*

$$I^m(\lambda,\ell)f(z) = z + \sum_{k=n+1}^{\infty}\left[\frac{1+\lambda(k-1)+\ell}{1+\ell}\right]^m a_k z^k. \tag{1.1}$$

It follows from (1.1) that

$$I^0(\lambda,\ell)f(z) = f(z),$$

$$\lambda z(I^m(\lambda,\ell)f(z))' = (\ell+1-\lambda)I^m(\lambda,\ell)f(z) - (\ell+1)I^{m+1}(\lambda,\ell)f(z) \ \ (\lambda>0) \tag{1.2}$$

and

$$I^{m_1}(\lambda,\ell)(I^{m_2}(\lambda,\ell))f(z) = I^{m_1+m_2}(\lambda,\ell)f(z) = I^{m2}(\lambda,\ell)(I^{m_1}(\lambda,\ell))f(z). \tag{1.3}$$

for all integers m_1 and m_2.

We note that:

$$I^0(1,0)f(z) = f(z) \quad \text{and} \quad I^1(1,0)f(z) = zf'(z).$$

By specializing the parameters m, λ and ℓ, we obtain the following operators studied by various authors:

(i) $I^m(1,\ell)f(z) = I_\ell^m f(z)$ (see [3] and [4]);

(ii) $I^m(\lambda,0)f(z) = D_\lambda^m f(z)$ (see [1]);

(iii) $I^m(1,0)f(z) = D^m f(z)$ (see [9]);

(iv) $I^m(1,1)f(z) = I_m f(z)$ (see [10]).

Also if $f \in A(n)$, then we can write

$$I^m(\lambda,\ell)f(z) = (f * \varphi^m_{\lambda,\ell})(z),$$

where

$$\varphi^m_{\lambda,\ell}(z) = z + \sum_{k=n+1}^{\infty}\left[\frac{1+\lambda(k-1)+\ell}{1+\ell}\right]^m z^k. \tag{1.4}$$

To establish our main results, we shall need the following lemmas.

Lemma 1 [5]. *Let the function $h(z)$ be analytic and convex (univalent) in U with $h(0)=1$. Suppose also the function $\varphi(z)$ given by*

$$\varphi(z)=1+c_n z^n+c_{n+1}z^{n+1}+... \tag{1.5}$$

be analytic in U. If

$$\varphi(z)+\frac{z\varphi'(z)}{\delta}\prec h(z) \quad (\mathrm{Re}(\delta)\geqslant 0;\delta\neq 0; z\in U),$$

then

$$\varphi(z)\prec\psi(z)=\frac{\delta}{n}z^{-(\frac{\delta}{n})}\int_0^z t^{(\frac{\delta}{n})-1}h(t)dt\prec h(z) \quad (z\in U), \tag{1.6}$$

and ψ is the best dominant.

Lemma 2 [7, p.66, Corollary 2.6.g.2]. Let $f\in A$ and F is given by

$$F(z)=\frac{2}{z}\int_0^z f(t)dt$$

If

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>-\frac{1}{2} \ \ (z\in U),$$

then $F\in K$.

For the case when $F(z)$ has a more elaborate form, Lemma 2, can be rewritten in the following form.

Lemma 3 [8]. *Let $f\in A, \delta>1$ and F is given by*

$$F(z)=\frac{1+\delta}{\delta z^{\frac{1}{\delta}}}\int_0^z f(t)t^{\frac{1}{\delta}-1}dt.$$

If

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>-\frac{1}{2} \ \ (z\in U),$$

then $F\in K$.

2. Main results

Unless otherwise mentioned, we assume throughout this paper that $m \in N_0,\ p, n \in N, \ell \geq 0$ and $\lambda > 0$.

Theorem 1. *Let $h \in H(U)$, with $h(0) = 1$, which verifies the inequality:*

$$\mathrm{Re}\left[1 + \frac{zh''(z)}{h'(z)}\right] > -\frac{\left(\frac{\ell+1}{\lambda}\right)}{2(p+k)} \quad (z \in U). \qquad (2.1)$$

If $f \in \sum_{p,k}$ and verifies the differentional subordination

$$\left[I^{m+1}(\lambda, \ell)(z^{p+1}f(z))\right]' \prec h(z) \quad (z \in U), \qquad (2.2)$$

then

$$\left[I^{m}(\lambda, \ell)\left(z^{p+1}f(z)\right)\right]' \prec g(z) \quad (z \in U),$$

where

$$g(z) = \frac{\left(\frac{\ell+1}{\lambda}\right)}{(p+k)\, z^{\frac{\left(\frac{\ell+1}{\lambda}\right)}{p+k}}} \int_0^z h(t) t^{\frac{\left(\frac{\ell+1}{\lambda}\right)}{p+k} - 1} dt. \qquad (2.3)$$

The function g is convex and is the best (1,p+k)-dominant.

Proof. From the identity (1.2), we have

$$I^{m+1}(\lambda, \ell)(z^{p+1}f(z)) = \left(\tfrac{\lambda}{\ell+1}\right)\left[z\left(I^m(\lambda, \ell)\left[z^{p+1}f(z)\right]\right)' + \left(\tfrac{\ell+1}{\lambda} - 1\right) I^m(\lambda, \ell)\left(z^{p+1}f(z)\right)\right] (z \in U). \qquad (2.4)$$

Differentiating (2.4) with respect to z, we obtain

$$\left[I^{m+1}(\lambda, \ell)(z^{p+1}f(z))\right]' = \left(\tfrac{\lambda}{\ell+1}\right)\left\{z\left[I^m(\lambda, \ell)(z^{p+1}f(z))\right]'' + \left(\tfrac{\ell+1}{\lambda}\right)\left[I^m(\lambda, \ell)\left(z^{p+1}f(z)\right)\right]'\right\} \quad (\lambda > 0; z \in U). \qquad (2.5)$$

If we put

$$q(z) = \left[I^m(\lambda, \ell)\left(z^{p+1}f(z)\right)\right]' \quad (z \in U), \qquad (2.6)$$

then (2.5) becomes

$$\left[I^{m+1}(\lambda + \ell)\left(z^{p+1}f(z)\right)\right]' = q(z) + \left(\tfrac{\lambda}{\ell+1}\right) zq'(z) \quad (z \in U). \qquad (2.7)$$

Using (2.7), subordination (2.2) is equivalent to

$$q(z) + \left(\tfrac{\lambda}{\ell+1}\right) zq'(z) \prec h(z) \quad (z \in U), \qquad (2.8)$$

where

$$q(z) = 1 + c_{p+k+1}z^{p+k} + \dots .$$

By using Lemma 1, for $\delta = \frac{\ell+1}{\lambda}, n = p+k$, we have $q(z) \prec g(z) \prec h(z)$,
we have

$$g(z) = \frac{\left(\frac{\ell+1}{\lambda}\right)}{(p+k)\, z^{\frac{\left(\frac{\ell+1}{\lambda}\right)}{p+k}}} \int_0^z h(t) t^{\frac{\left(\frac{\ell+1}{\lambda}\right)}{p+k}-1} dt \quad (z \in U),$$

is the best (1,p+k)-dominant.

By applying Lemma 3 for the function given by (2.3) and function h with the property in (2.1) for $\delta = \frac{\ell+1}{\lambda}$, we obtain that the function g is convex.

Putting $p = \lambda = 1$ and $m = k = \ell = 0$ in Theorem 1, we obtain the result due to Libera [6] (see also [7 p. 64,Theorem 2.6.g]).

Corollary 1. *Let $h \in H(U)$, with $h(0) = 1$ which verifies the inequality*

$$\operatorname{Re}\left\{1 + \frac{zh''(z)}{h'(z)}\right\} > -\frac{1}{2} \quad (z \in U).$$

If $f \in \sum_{1,0}$ and verifies the differential subordination:

$$z\left[z^2 f(z)\right]'' + \left[z^2 f(z)\right]' \prec h(z) \quad (z \in U),$$

then

$$(z^2 f(z))' \prec g(z) \quad (z \in U),$$

where

$$g(z) = \frac{1}{z}\int_0^z h(t)dt \quad (z \in U),$$

the function $g(z)$ is convex and is the best (1,1)-dominat.

Theorm 2. *Let $h \in H(U)$, with $h(0) = 1$, which verifies the inequality:*

$$\operatorname{Re}\left\{1 + \frac{zh''(z)}{h'(z)}\right\} > -\frac{1}{2(p+k)} \quad (z \in U). \tag{2.9}$$

If $f \in \sum_{p,k}$ and verifies the differential subordination

$$\left[I^m(\lambda,\ell)(z^{p+1}f(z))\right]' \prec h(z) \quad (z \in U), \tag{2.10}$$

then

$$\frac{I^m(\lambda,\ell)(z^{p+1}f(z))}{z} \prec g(z) \quad (z \in U),$$

where

$$g(z) = \frac{1}{(p+k)\, z^{\left(\frac{1}{p+k}\right)}} \int_0^z h(t) t^{\left(\frac{1}{p+k}\right)-1} dt \quad (z \in U). \tag{2.11}$$

The function g is convex and is the best (1,p+k)-dominant.

Proof. We let

$$q(z) = \frac{I^m(\lambda, \ell)(z^{p+1} f(z))}{z} \quad (z \in U), \tag{2.12}$$

and we obtain

$$\left[I^m(\lambda, \ell)(z^{p+1} f(z))\right]' = q(z) + zq'(z) \quad (z \in U).$$

Then (2.9) gives

$$q(z) + zq'(z) \prec h(z)$$

where

$$q(z) = 1 + q_{p+k+1} z^{p+k} + ... \quad (z \in U).$$

By using Lemma 1 for $\delta = 1, n = p + k$, we have

$$q(z) \prec g(z) \prec h(z),$$

where

$$g(z) = \frac{1}{(p+k) z^{\left(\frac{1}{p+k}\right)}} \int_0^z h(t) t^{\left(\frac{1}{p+k}\right)-1} dt \quad (z \in U),$$

and g is the best (1,p+k)-dominent.

By applying Lemma 3 for the function $g(z)$ given by (2.11) and the function $h(z)$ with the property in (2.9) for $n = p + k$, we obtain that the function $g(z)$ is convex.

Putting $p = \lambda = 1$ and $m = k = \ell = 0$ in Theorem 2, we obtain the following corollary.

Corollary 2. *Let* $h \in H(U)$, *with* $h(0) = 1$, *which verifies the inequality:*

$$\text{Re}\left\{1 + \frac{zh''(z)}{h'(z)}\right\} > -\frac{1}{2} \quad (z \in U).$$

If $f \in \sum_{1,0}$ *and verifies the differential subordination:*

$$\left(z^2 f(z)\right)' \prec h(z) \quad (z \in U),$$

then

$$(zf(z)) \prec g(z) \quad (z \in U),$$

where

$$g(z) = \frac{1}{z}\int_0^z h(t)dt, \quad (z \in U).$$

The function $g(z)$ is convex and is the best (1,1)-dominant.

Remark. *Putting $\ell = 0$ and $\lambda = 1$ in the above results we obtain the results obtained by Oros [8].*

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