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SUBORDINATION AND SUPERORDINATION PROPERTIES OF MULTIVALENT FUNCTIONS DEFINED BY EXTENDED MULTIPLIER TRANSFORMATION

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Abstract. In this paper, we study different applications of the theory of differential subordination and superordination results for certain normalized analytic functions in the open unit disc, which are acted upon by a class of extended multiplier transformation.

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1. Introduction

Let $H(U)$ denotes the class of analytic functions in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ and let $H[a,p]$ denotes the subclass of the functions $f \in H(U)$ of the form

$$f(z) = a + a_p z^p + a_{p+1} z^{p+1} + \dots \quad (a \in \mathbb{C},\ p \in \mathbb{N} = \{1, 2, \dots\}). \tag{1.1}$$

Also, let $A(p)$ be the subclass of the functions $f \in H(U)$ of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in \mathbb{N}), \tag{1.2}$$

and set $A \equiv A(1)$.

For $f, g \in H(U)$, we say that the function $f\ (z)$is subordinate to $g(z)$, written symbolically as follows:

$$f \prec g \quad or \quad f(z) \prec g(z)\ ,$$

if there exists a Schwarz function $w(z)$, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$, $(z \in U)$, such that $f(z) = g(w(z))$ for all $z \in U$. In particular, if the function $g(z)$ is univalent in U, then we have the following equivalence (cf., e.g., [11]; see also [12, p.4]):

$$f(z) \prec g\ (z)\ \Leftrightarrow f(0) \prec g(0)\ \ and\ \ \ f(U) \subset g(U).$$

Supposing that p and h are two analytic functions in U, let

$$\varphi(r, s, t; z) : \mathbb{C}^3 \times U \to \mathbb{C}.$$

If p and $\varphi(p(z), zp^{'}(z), z^2p^{''}(z); z)$ are univalent functions in U and if p satisfies the second-order superordination

$$h(z) \prec \varphi(p(z), zp^{'}(z), z^2p^{''}(z); z), \tag{1.3}$$

then p is called to be a solution of the differential superordination (1.3). (If f is subordinate to F, then F is superordination to f). An analytic function q is called a subordinant of (1.3), if $q(z) \prec p(z)$ for all the functions p satisfying (1.3). A univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all of the subordinants q of (1.3), is called the best subordinant (cf., e.g.,[11], see also [12]).

Recently, Miller and Mocanu [13] obtained sufficient conditions on the functions h, q and φ for which the following implication holds:

$$k(z) \prec \varphi(p(z), zp^{'}(z), z^2p^{''}(z); z) \Rightarrow q(z) \prec p(z). \tag{1.4}$$

Using the results Miller and Mocanu [13], Bulboaca [5] considered certain classes of first-order differential superordinations as well as superordination preserving integral operators [4]. Ali et al. [1], have used the results of Bulboaca [5] and obtained sufficient conditions for certain normalized analytic functions $f(z)$to satisfy

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z), \tag{1.5}$$

where q_1 and q_2 are given univalent functions in U with $q_1(0) = 1$. Shanmugam et al. [17] obtained sufficient conditions for normalized analytic functions $f(z)$ to satisfy

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z),$$

and

$$q_1(z) \prec \frac{z^2f'(z)}{\{f(z)\}^2} \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in U with $q_1(0) = 1$ and $q_2(0) = 1$,

Many essentially equivalent definitions of multiplier transformation have been given in literature (see [7], [8], and [19]). In [6] Catas defined the operator $I_p^m(\lambda, \ell)$ as follows:

Definition 1[6]. *Let the function* $f(z) \in A(p)$. *For* $m \in N_0 = N \cup \{0\}$, $\lambda \geq 0, \ell \geq 0$.The extended multiplier transformation $I_p^m(\lambda,\ell)$ *on* $A(p)$ is defined by the following infinite series:

$$I_p^m(\lambda,\ell)f(z) = z^p + \sum_{k=p+1}^{\infty} \left[\frac{p+\lambda(k-p)+\ell}{p+\ell}\right]^m a_k z^k. \tag{1.6}$$

$$(\lambda \geq 0; \ell \geq 0; p \in N; m \in N_0; z \in U).$$

We can write (1.6) as follows:

$$I_p^m(\lambda,\ell)f(z) = (\Phi_{\lambda,\ell}^{p,m} * f)(z),$$

where

$$\Phi_{\lambda,\ell}^{p,m}(z) = z^p + \sum_{k=p+1}^{\infty} \left[\frac{p+\lambda(k-p)+\ell}{p+\ell}\right]^m z^k. \tag{1.7}$$

It is easily verified from (1.6), that

$$\lambda z(I_p^m(\lambda,\ell)f(z))' = (p+\ell)I_p^{m+1}(\lambda,\ell)f(z) - [p(1-\lambda)+\ell]I_p^m(\lambda,\ell)f(z) \ (\lambda > 0). \tag{1.8}$$

We note that:

$$I_p^0(\lambda,\ell)f(z) = f(z)\ ,\ I_p^1(1,0)f(z) = \frac{zf'(z)}{p} \ \textit{and}\ \ I_p^2(1,0)f(z) = \frac{z(zf'(z))'}{p^2}.$$

Also by specilizing the parameters λ, ℓ, m and p, we obtain the following operators studied by various authors:

(i) $I_p^m(1,\ell) = I_p(m,\ell)f(z)$ (see Kumar et al. [10] and Srivastava et al. [18]);
(ii) $I_p^m(1,0)f(z) = D_p^m f(z)$ (see [3], [9] and [15]);
(iii) $I_1^m(1,\ell)f(z) = I_\ell^m f(z)$ (see Cho and Kim [7] and Cho and Srivastava [8]);
(iv) $I_1^m(1,0) = D^m f(z)$ $(m \in N_0)$ (see Salagean [16]);
(v) $I_1^m(\lambda,0) = D_\lambda^m$ (see Al-Aboudi [2]);
(vi) $I_1^m(1,1) = I^m f(z)$ (see Uralegaddi and Somanatha [19]);
(vii) $I_p^m(\lambda,0) = D_{\lambda,p}^m f(z)$, where $D_{\lambda,p}^m f(z)$ is defined by

$$D_{\lambda,p}^m f(z) = z^p + \sum_{k=p+1}^{\infty} \left[\frac{p+\lambda(k-p)}{p}\right]^m a_k z^k.$$

2. Preliminaries

In order to prove our subordination and superordination results, we make use of the following known definition and lemmas.

Definition 2[13]. *Denote by Q the set of all functions $f\ (z)$ that are analytic and injective on $\overline{U} \setminus E(f)$, where*

$$E(f) = \{\zeta : \zeta \in \partial U \ \ and \ \ \lim_{z\to\zeta} f(z) = \infty\}, \tag{2.1}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$.

Lemma 1 [12]. *Let the function $q(z)$ be univalent in the unit disc U, and let θ and φ be analytic in a domain D containing $q(U)$, with $\varphi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = zq'(z)\varphi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$ and suppose that*

(i) Q is a starlike function in U,

(ii) $\text{Re}\left(\dfrac{zh'(z)}{Q(z)}\right) > 0\ for\ z \in U$.

If p is analytic in U with $p(0) = q(0)$, $p(U) \subseteq D$ and

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z)), \tag{2.2}$$

then $p(z) \prec q(z)$, and q is the best dominant .

Lemma 2 [17]. *Let q be a convex function in U and let $\psi \in C$ with $\delta \in C^* = C\backslash\{0\}$ with*

$$\text{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left\{0; -\text{Re}\,\frac{\psi}{\delta}\right\},\ z \in U.$$

If $p(z)$ is analytic in U, and

$$\psi p(z) + \delta zp'(z) \prec \psi q(z) + \delta zq'(z), \tag{2.3}$$

then $p(z) \prec q(z)$, and q is the best dominant .

Lemma 3 [4]. *Let $q(z)$ be a convex univalent function in the unit disc U and let θ and φ be analytic in a domain D containing $q(U)$. Suppose that*

(i) $\text{Re}\left\{\dfrac{\theta'(q(z))}{\varphi(q(z))}\right\} > 0$ for $z \in U$;

(ii) $zq'(z)\varphi(q(z))$ is starlike in U.

If $p \in H[q(0), 1] \cap Q$ with $p(U) \subseteq D$, and $\theta(p(z)) + zp'(z)\varphi(p(z))$ is univalent in U, and

$$\theta(q(z)) + zq'(z)\varphi(q(z)) \prec \theta(p(z)) + zp'(z)\varphi(p(z)),$$

then $q(z) \prec p(z)$, and q is the best subordinant.

Lemma 4[13]. *Let q be convex univalent in U and let $\delta \in C$, with $\operatorname{Re}(\delta) > 0$. If $p \in H[q(0), 1] \cap Q$ and $p(z) + \delta z p'(z)$ is univalent in U, then*

$$q(z) + \delta z q'(z) \prec p(z) + \delta z p'(z), \tag{2.4}$$

implies

$$q(z) \prec p(z) \qquad (z \in U)$$

and q is the best subordinant .

3.Subordination results for analytic functions

Unless otherwise mentioned we shall assume throught this paper that $\lambda > o$, $\ell \geq 0$, $p \in N$ and $m \in N_0$.

Theorem 1. *Let q be univalent in U, with $q(0) = 1$, $\beta \in C^*$. Suppose q satisfies*

$$\operatorname{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left\{0; -\operatorname{Re}\frac{1}{\beta}\right\}. \tag{3.1}$$

If $f \in A(p)$, $I_p^{m+1}(\lambda, \ell)f(z) \neq 0$ $(z \in U^* = U - \{0\})$ and satisfies the subordination

$$\Phi(f, \beta, p, m, \lambda, \ell) \prec q(z) + \beta z q'(z), \tag{3.2}$$

where

$$\Phi(f, \beta, p, m, \lambda, \ell) = \frac{I_p^m(\lambda, \ell)f(z)}{I_p^{m+1}(\lambda, \ell)f(z)} + \beta\left(\frac{p+\ell}{\lambda}\right)\left\{1 - \frac{I_p^{m+2}(\lambda, \ell)f(z) I_p^m(\lambda, \ell)f(z)}{\left[I_p^{m+1}(\lambda, \ell)f(z)\right]^2}\right\}, \tag{3.3}$$

then

$$\frac{I_p^m(\lambda, \ell)f(z)}{I_p^{m+1}(\lambda, \ell)f(z)} \prec q(z), \tag{3.4}$$

and q is the best dominant of (3.2).

Proof Define the function $p(z)$ by

$$p(z) = \frac{I_p^m(\lambda, \ell)f(z)}{I_p^{m+1}(\lambda, \ell)f(z)} \ (z \in U). \tag{3.5}$$

Then the function p is analytic in U and $p(0) = 1$. Therefore, differentiating (3.5) logarithmically with respect to z, and using the identity (1.8) in the resulting equation, we have

$$\frac{zp'(z)}{p(z)} = \beta\left(\frac{p+\ell}{\lambda}\right)\left\{\frac{1}{p(z)} - \frac{I_p^{m+2}(\lambda, \ell)f(z)}{I_p^{m+1}(\lambda, \ell)f(z)}\right\} \tag{3.6}$$

and

$$p(z)+\beta zp'(z)=$$
$$\frac{I_p^m(\lambda,\ell)f(z)}{I_p^{m+1}(\lambda,\ell)f(z)}+\beta\left(\frac{p+\ell}{\lambda}\right)\left\{1-\frac{I_p^{m+2}(\lambda,\ell)f(z)I_p^m(\lambda,\ell)f(z)}{\left[I_p^{m+1}(\lambda,\ell)f(z)\right]^2}\right\}. \tag{3.7}$$

The subordination (3.1) from hypothesis becomes

$$p(z)+\beta zp'(z)\prec q(z)+\beta zq'(z). \tag{3.8}$$

The assertion(3.4) of Theorem 1 now follows by an application of Lemma 2.

Putting $m=\ell=0$ in Theorem 1, we obtain the following corollary.

Corollary 1. *Assume that (3.1) holds. If* $f\in A(p),\beta\in C^*$, *and*

$$\Psi(f,\beta,p,\lambda)\prec q(z)+\beta zq'(z), \tag{3.9}$$

where

$$\Psi(f,\beta,p,\lambda)=\left[1-\frac{\beta}{\lambda}(2-\lambda)p\right]\frac{f(z)}{\left[(1-\lambda)f(z)+\frac{\lambda}{p}zf'(z)\right]}+$$
$$\beta\frac{p}{\lambda}\left\{1-\frac{\left(\frac{\lambda}{p}\right)^2z^2f''(z)f(z)-(1-\frac{\lambda}{p})f^2(z)}{\left[(1-\lambda)f(z)+\frac{\lambda}{p}zf'(z)\right]^2}\right\} \tag{3.10}$$

then

$$\frac{f(z)}{\left[(1-\lambda)f(z)+\frac{\lambda}{p}zf'(z)\right]}\prec q(z), \tag{3.11}$$

and q is the best dominant .

Remark 1. *Putting* $p=1$ *in Corollary 1, we obtain the result obtained by Nechita [14, Corollary 6].*

Putting $\lambda=1$ and $\ell=0$ in Theorem 1 we obtain the following corollary.

Corollary 2. *Assume that (3.1) holds. If* $f\in A(p),\beta\in C^*$, *and*

$$\frac{D_p^mf(z)}{D_p^{m+1}f(z)}+\beta p\left\{1-\frac{D_p^{m+2}f(z).D_p^mf(z)}{\left[D_p^{m+1}f(z)\right]^2}\right\}\prec q(z)+\beta zq'(z), \tag{3.14}$$

then

$$\frac{D_p^m f(z)}{D_p^{m+1} f(z)} \prec q(z), \tag{3.15}$$

and q is the best dominant .

Remark 2. *Putting $p = 1$ in Corollary 2, we obtain the result obtained by Nechita [14, Corollary 7]* and correct the result obtained by Shanmugam et al. [17, Theorem 5.1].

Putting $m = \ell = 0$ and $\lambda = 1$ in Theorem 1, we obtain the following corollary.

Corollary 3. *Assume that (3.1) holds. If $f \in A(p), \beta \in C^*$, and*

$$(1-\beta p)\frac{pf(z)}{zf'(z)} + \beta p\left\{1 - \frac{z^2 f''(z)f(z) - p(p-1)f^2(z)}{[zf'(z)]^2}\right\} \prec q(z) + \beta z q'(z), \tag{3.16}$$

then

$$\frac{pf(z)}{zf'(z)} \prec q(z) \tag{3.17}$$

and q is the best dominant .

Remark 3. *Putting $p = 1$ in Corollary 3, we obtain the result obtained by Shanmugam et al. [17, Theorem 3.2].*

Putting $q(z) = \dfrac{1+Az}{1+Bz}$ $(-1 \le B < A \le 1)$ in Theorem 1, we obtain the following corollary.

Corollary 4. *Let $-1 \le B < A \le 1,\ \beta \in C^*$, and suppose that*

$$\operatorname{Re}\left\{\frac{1-Bz}{1+Bz}\right\} > \max\left\{0, -\operatorname{Re}\frac{1}{\beta}\right\} . \tag{3.18}$$

If $f \in A(p)$, and

$$\Phi(f,\beta,p,m,\lambda,\ell) \prec \frac{1+Az}{1+Bz} + \beta\frac{(A-B)z}{(1+Bz)^2},$$

where $\Phi(f,\beta,p,m,\lambda,\ell)$ is given by (3.3), then

$$\frac{I_p^m(\lambda,\ell)f(z)}{I_p^{m+1}(\lambda,\ell)f(z)} \prec \frac{1+Az}{1+Bz} \tag{3.19}$$

and $q(z) = \dfrac{1+Az}{1+Bz}$ is the best dominant.

Putting $A = 1$ and $B = -1$ in Corollary 4, we obtain the following corollary.

Corollary 5. *If* $f \in A(p)$ *and* $\beta \in C^*$ satisfy

$$\Phi(f, \beta, p, m, \lambda, \ell) \prec \frac{1+z}{1-z} + \frac{2\beta z}{(1-z)^2},$$

where $\Phi(f, \beta, p, m, \lambda, \ell)$ is given by (3.3), then

$$\frac{I_p^m(\lambda, \ell)f(z)}{I_p^{m+1}(\lambda, \ell)f(z)} \prec \frac{1+z}{1-z}$$

and $q(z) = \dfrac{1+z}{1-z}$ is the best dominant.

4.Superordination and sandwich results

Theorem 2. *Let* q *be convex univalent in* U, $\beta \in C$. *Suppose*

$$\operatorname{Re} \beta > 0. \tag{4.1}$$

If $f \in A(p)$, $\frac{I_p^m(\lambda,\ell)f(z)}{I_p^{m+1}(\lambda,\ell)f(z)} \in H[q(0), 1] \cap Q$, $\Phi(f, \beta, p, m, \lambda, \ell)$ is univalent in the unit disc U, where $\Phi(f, \beta, p, m, \lambda, \ell)$ is defined by (3.3). and

$$q(z) + \beta z q'(z) \prec \Phi(f, \beta, p, m, \lambda, \ell), \tag{4.2}$$

then

$$q(z) \prec \frac{I_p^m(\lambda, \ell)f(z)}{I_p^{m+1}(\lambda, \ell)f(z)}$$

and q is the best subordinant of (4.1).

Proof. Define the function $p(z)$ by (3.5). Differentiating (3.5) logarithmically with respect to z, and using the identity (1.8) in the resulting equation, we have

$$p(z) + \beta z p'(z) \prec \Phi(f, \beta, p, m, \lambda, \ell). \tag{4.3}$$

Theorem 2 follows by an applying of Lemma 4.

Putting $m = \ell = 0$ in Theorem 2, we obtain the following corollary.

Corollary 6. *Let* q *be convex in* U *with* $q(0) = 1$, *and* $\beta \in C$, $\operatorname{Re} \beta > 0$. *If* $f \in A(p)$, $\frac{f(z)}{\left[(1-\lambda)f(z)+\frac{\lambda}{p}zf'(z)\right]} \in H[q(0), 1] \cap Q$, $\Psi(f, \beta, p, \lambda)$ is univalent in the unit disc U, where $\Psi(f, \beta, p, \lambda)$ is defined by (3.10), and

$$q(z) + \beta z q'(z) \prec \Psi(f, \beta, p, \lambda), \tag{4.4}$$

then

$$q(z) \prec \frac{f(z)}{\left[(1-\lambda)f(z)+\frac{\lambda}{p}zf'(z)\right]} \tag{4.5}$$

and q is the best subordinant.

Putting $\lambda = 1$ and $\ell = 0$ in Theorem 2, we obtain the following corollary.

Corollary 7. *Let q be convex in U with $q(0) = 1$, and $\beta \in C$, $\operatorname{Re}\beta > 0$.If $f \in A(p)$* $\frac{D_p^m f(z)}{D_p^{m+1} f(z)} \in H[q(0), 1] \cap Q, \frac{D_p^m f(z)}{D_p^{m+1} f(z)} + \beta p \left\{1 - \frac{D_p^{m+2} f(z).D_p^m f(z)}{\left[D_p^{m+1} f(z)\right]^2}\right\}$ is univalent in the unit disc U, and

$$q(z) + \beta z q'(z) \prec \frac{D_p^m f(z)}{D_p^{m+1} f(z)} + \beta p \left\{1 - \frac{D_p^{m+2} f(z).D_p^m f(z)}{\left[D_p^{m+1} f(z)\right]^2}\right\}, \tag{4.6}$$

then

$$q(z) \prec \frac{D_p^m f(z)}{D_p^{m+1} f(z)},$$

and q is the best subordinant .

Remark 4. *Putting $p = 1$ in Corollary 7, we obtain the result obtained by Nechita [14, Corollary 12]* and correct the result obtained by Shanmugam et al. [17, Theorem 5.3].

Combining Theorem 1 and Theorem 2, we get the following sandawich theorem.

Theorem 3. *Let q_1, q_2 be convex in U with $q_1(0) = q_2(0) = 1$, $\beta \in C$, $\operatorname{Re}\beta > 0$ and $q_2(z)$ satisfies (3.1). If $f \in A(p)$, $\frac{I_p^m(\lambda,\ell)f(z)}{I_p^{m+1}(\lambda,\ell)f(z)} \in H[q(0), 1] \cap Q$, $\Phi(f, \beta, p, m, \lambda, \ell)$ is univalent in the unit disc U, where $\Phi(f, \beta, p, m, \lambda, \ell)$ is defined by (3.3) and*

$$q_1(z) + \beta z q_1'(z) \prec \Phi(f, \beta, m, \lambda, \ell) \prec q_2(z) + \beta z q_2'(z), \tag{4.7}$$

then

$$q_1(z) \prec \frac{I_p^m(\lambda,\ell)f(z)}{I_p^{m+1}(\lambda,\ell)f(z)} \prec q_2(z)$$

and the functions q_1, q_2 are respectively the best subordinant and the best dominant.

Theorem 4. Let $q(z)$ be univalent in U with $q(0) = 1, \beta \in C^*$. *Assume that (3.1) holds. If $f \in A(p)$,*

$$\zeta(f, \beta, m, p, \lambda, \ell) \prec q(z) + \beta z q'(z) \tag{4.8}$$

where

$$\zeta(f, \beta, m, p, \lambda, \ell) =$$

$$\left[1+\beta\left(\frac{p+\ell}{\lambda}\right)\right] z^{p}\frac{I_p^{m+1}(\lambda,\ell)f(z)}{\left[I_p^{m}(\lambda,\ell)f(z)\right]^2}+\beta\left(\frac{p+\ell}{\lambda}\right) \tag{4.9}$$

$$\frac{z^{p}I_p^{m+2}(\lambda,\ell)f(z)}{\left[I_p^{m}(\lambda,\ell)f(z)\right]^2}-2\beta\left(\frac{p+\ell}{\lambda}\right)z^{p}\frac{\left[I_p^{m+1}(\lambda,\ell)f(z)\right]^2}{\left[I_p^{m}(\lambda,\ell)f(z)\right]^3},$$

then

$$z^{p}\frac{I_p^{m+1}(\lambda,\ell)f(z)}{\left[I_p^{m}(\lambda,\ell)f(z)\right]^2}\prec q(z) \tag{4.10}$$

and q is the best dominant .

Proof. Define the function $p(z)$ by

$$p(z)=z^{p}\frac{I_p^{m+1}(\lambda,\ell)f(z)}{\left[I_p^{m}(\lambda,\ell)f(z)\right]^2}\ (z\in U). \tag{4.11}$$

Then, simple computations show that

$$\frac{zp'(z)}{p(z)}=p+\frac{z\left[I_p^{m+1}(\lambda,\ell)f(z)\right]'}{I_p^{m+1}(\lambda,\ell)f(z)}-2\frac{z\left[I_p^{m}(\lambda,\ell)f(z)\right]'}{I_p^{m}(\lambda,\ell)f(z)}. \tag{4.12}$$

We use the identity (1.8) in (4.12) we obtain

$$\frac{zp'(z)}{p(z)}=\frac{p+\ell}{\lambda}\left\{1+\frac{I_p^{m+2}(\lambda,\ell)f(z)}{I_p^{m}(\lambda,\ell)f(z)}-2\frac{I_p^{m+1}(\lambda,\ell)f(z)}{I_p^{m}(\lambda,\ell)f(z)}\right\}$$

and

$$\begin{aligned} p(z)+\beta zp'(z) &= \left[1+\beta\left(\frac{p+\ell}{\lambda}\right)\right] z^{p}\frac{I_p^{m+1}(\lambda,\ell)f(z)}{\left[I_p^{m}(\lambda,\ell)f(z)\right]^2}+\beta\left(\frac{p+\ell}{\lambda}\right). \\ &\cdot\frac{z^{p}I_p^{m+2}(\lambda,\ell)f(z)}{\left[I_p^{m}(\lambda,\ell)f(z)\right]^2}-2\beta\left(\frac{p+\ell}{\lambda}\right)z^{p}\frac{\left[I_p^{m+1}(\lambda,\ell)f(z)\right]^2}{\left[I_p^{m}(\lambda,\ell)f(z)\right]^3}. \end{aligned}$$

The subordination (4.9) becomes

$$p(z)+\beta zp'(z)\prec q(z)+\beta zq'(z).$$

Theorem 4 follows by an applying of Lemma 2.

Putting $m=\ell=0$ in Theorem 4, we obtain the following corollary.

Corollary 8. Let $q(z)$ be univalent in U with $q(0)=1, \beta \in C^{*}$. *Assume that (3.1) holds. If* $f \in A(p)$,

$$(1+\beta p)\frac{(1-\lambda)\,z^{p}}{f(z)}+\left[\frac{\lambda}{p}+(2\lambda-1)\,\beta\right]\frac{z^{p+1}f'(z)}{[f(z)]^{2}}+$$

$$\beta\frac{\lambda}{p}\frac{z^{p+2}f''(z)}{[f(z)]^{2}}-2\beta\frac{\lambda}{p}\frac{z^{p+2}\left[f'(z)\right]^{2}}{[f(z)]^{3}}\prec q(z)+\beta zq'(z) \tag{4.13}$$

then

$$(1-\lambda)\frac{z^{p}}{f(z)}+\frac{\lambda}{p}\frac{z^{p+1}f'(z)}{[f(z)]^{2}}\prec q(z), \tag{4.14}$$

and q is the best dominant .

Remark 5. *Putting* $p=1$ *in Corollary 8, we obtain the result obtained by Nechita [14, Corollary 15].*

Putting $\lambda=1$ and $\ell=0$ in Theorem 4, we obtain the following corollary.

Corollary 9. Let $q(z)$ be univalent in U with $q(0)=1, \beta \in C^{*}$. *Assume that (3.1) holds. If* $f \in A(p)$,

$$(1+\beta p)\frac{z^{p}D_{p}^{m+1}f(z)}{\left[D_{p}^{m}f(z)\right]^{2}}+\beta p\frac{z^{p}D_{p}^{m+2}f(z)}{\left[D_{p}^{m}f(z)\right]^{2}}-2\beta p\frac{z^{p}\left[D_{p}^{m+1}f(z)\right]^{2}}{\left[D_{p}^{m}f(z)\right]^{3}}\prec q(z)+\beta zq'(z) \tag{4.15}$$

then

$$\frac{z^{p}D_{p}^{m+1}f(z)}{\left[D_{p}^{m}f(z)\right]^{2}}\prec q(z),$$

and q is the best dominant .

Remark 6. *Putting* $p=1$ *in Corollary 9, we obtain the result obtained by* Shanmugam et al. [17, Theorem 5.4].

Putting $m=\ell=0$ and $\lambda=1$ in Theorem 4, we obtain the following corollary.

Corollary 10. Let $q(z)$ be univalent in U with $q(0)=1, \beta \in C^{*}$. *Assume that (3.1) holds. If* $f \in A(p)$,

$$\frac{z^{p+1}f'(z)}{p\,[f(z)]^{2}}-\beta\frac{z^{p+1}}{p}\left(\frac{z^{p}}{f(z)}\right)''\prec q(z)+\beta zq'(z), \tag{4.16}$$

then

$$\frac{z^{p+1}f'(z)}{p\left[f(z)\right]^2} \prec q(z)$$

and q is the best dominant.

Remark 7. Putting $p = 1$ in Corollary 10, we obtain the result obtained by Shanmugam et al. [17, Theorem 3.4].

Putting $q(z) = \dfrac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$ in Theorem 4, we obtain the following corollary.

Corollary 11. Let $q(z)$ be univalent in U with $q(0) = 1, \beta \in C^*$. *Assume that (3.1) holds. If* $f \in A(p)$,

$$\zeta(f, \beta, p, m, \lambda, \ell) \prec \frac{1+Az}{1+Bz} + \beta\frac{(A-B)z}{(1+Bz)^2}, \tag{4.17}$$

where $\zeta(f, \beta, p, m, \lambda, \ell)$ is given by (4.9), then

$$\frac{z^p I_p^{m+1}(\lambda, \ell)f(z)}{\left[I_p^m(\lambda, \ell)f(z)\right]^2} \prec \frac{1+Az}{1+Bz} \tag{4.18}$$

and $q(z) = \dfrac{1+Az}{1+Bz}$ is the best dominant .

Next, applying Lemma 4, we have the following theorem.

Theorem 5. *Let q be convex in U with $q(0) = 1$, and $\beta \in C$,* $\mathrm{Re}\,\beta > 0$. *If* $f \in A(p)$, $\frac{z^p I_p^{m+1}(\lambda,\ell)f(z)}{\left[I_p^m(\lambda,\ell)f(z)\right]^2} \in H[q(0), 1] \cap Q$, $\zeta(f, \beta, p, m, \lambda, \ell)$ is univalent in U, where $\zeta(f, \beta, p, m, \lambda, \ell)$ is defined by (4.9), and

$$q(z) + \beta z q'(z) \prec \zeta(f, \beta, p, m, \lambda, \ell), \tag{4.19}$$

then

$$q(z) \prec \frac{z^p I_p^{m+1}(\lambda, \ell)f(z)}{\left[I_p^m(\lambda, \ell)f(z)\right]^2}$$

and q is the best subordinant .

Putting $m = \ell = 0$ in Theorem 5, we obtain the following corollary.

Corollary 12. *Let q be convex in U with $q(0) = 1$, and $\beta \in C$,* $\mathrm{Re}\,\beta > 0$. *If* $f \in A(p)$, $(1-\lambda)\frac{z^p}{f(z)} + \frac{\lambda}{p}\frac{z^{p+1}f'(z)}{[f(z)]^2} \in H[q(0), 1] \cap Q$. $\zeta(f, \beta, p, \lambda)$ is univalent in U, where $\zeta(f, \beta, p, \lambda)$ is defined by (4.9), and

$$q(z) + \beta z q'(z) \prec \zeta(f, \beta, p, \lambda), \tag{4.20}$$

then

$$q(z) \prec (1-\lambda)\frac{z^p}{f(z)} + \frac{\lambda}{p}\frac{z^{p+1}f'(z)}{[f(z)]^2}$$

and q is the best subordinant .

Remark 8. *Putting $p = 1$ in Corollary 12, we obtain the result obtained by Nechita [14, Corollary 20].*

Combining Theorem 4 and Theorem 5, we get the following sandawich theorem.

Theorem 6. *Let q_1, q_2 be convex in U with $q_1(0) = q_2(0) = 1$, $\beta \in C$, $\operatorname{Re}\beta > 0$ and $q_2(z)$ satisfies (3.1).* If $f \in A(p)$, $\frac{z^p I_p^{m+1}(\lambda,\ell)f(z)}{\left[I_p^m(\lambda,\ell)f(z)\right]^2} \in H[q(0),1] \cap Q$, $\zeta(f,\beta,p,m,\lambda,\ell)$ is univalent in disc U, where $\zeta(f,\beta,p,m,\lambda,\ell)$ is defined by (4.9) and

$$q_1(z) + \beta z q_1'(z) \prec \zeta(f,\beta,m,\lambda,\ell) \prec q_2(z) + \beta z q_2'(z), \tag{4.21}$$

then

$$q_1(z) \prec \frac{z^p I_p^{m+1}(\lambda,\ell)f(z)}{\left[I_p^m(\lambda,\ell)f(z)\right]^2} \prec q_2(z)$$

and the functions q_1, q_2 are, respectively, the best subordinant and the best dominant.

References

[1] R. M. Ali, V. Ravichandran, M. H. Khan and K. G. Subramaniam, *Differential sandwich theorems for certain analytic functions*, Far East J. Math. Sci., 15 (2004), no. 1, 87–94.

[2] F. M. Al-Oboudi, *On univalent function defined by a generalized Salagean operator*, Internat. J. Math. Math. Sci., 27(2004),1429-1436.

[3] M. K. Aouf and A.O. Mostaafa, *On a subclass of p- valent functions prestar-like functions*, Comput. Math. Appl., (2008), no. 55, 851-861.

[4] T. Bulboacă, *A class of superordination preserving integral operators*, Indag. Math. (New Series), 13 (2002), no. 3, 301–311.

[5] T. Bulboacă, *Classes of first-order differential subordinations*, Demonstratio Math., 35(2002), no. 2, 287–392.

[6] A. Catas, *On certain classes of p-valent functions defined by multiplier transformations*, in Proceedings of the International Symposium on Geometric Function Theory and Applications: GFTA 2007 Proceedings (İstanbul, Turkey; 20-24 August 2007) (S. Owa and Y. Polatoglu, Editors), pp. 241–250, TC İstanbul Kűltűr University Publications, Vol. 91, TC İstanbul Kűltűr University, İstanbul, Turkey, 2008.

[7] N. E. Cho and T. H. Kim, *Multiplier transformations and strongly close-to-convex functions*, Bull. Korean Math. Soc., 40(2003), no. 3, 399-410.

[8] N. E. Cho and H. M. Srivastava, *Argument estimates of certain anaytic functions defined by a class of multipier transformations*, Math. Comput. Modelling, 37(2003), no. 1-2, 39-49.

[9] M. Kamali and H. Orhan, *On a subclass of certian starlike functions with negative coefficients*, Bull. Korean Math. Soc. 41 (2004), no. 1, 53-71.

[10] S. S. Kumar, H. C. Taneja and V. Ravichandran, *Classes multivalent functions defined by Dziok-Srivastava linear operator and multiplier transformations*, Kyungpook Math. J. (2006), no. 46, 97-109.

[11] S. S. Miller and P. T. Mocanu, *Differential subordinations and univalent functions*, Michigan Math. J. 28(1981), 157-171.

[12] S. S. Miller and P. T. Mocanu, *Differential Subordinations Theory and Applications*, Series on Monographs and Textbooks in Pure and Applied Math., No. 225, Marcel Dekker Inc. New York, Basel, 2000.

[13] S. S. Miller and P. T. Mocanu, *Subordinatinations of differential superordinations*, Complex Variables 48(2003), no.10, 815–826.

[14] V. O. Nechita, *Differential subordinations and superordinations for analytic functions defined by the generalized Salagean derivative*, Acta Universitatis Apulensis, no.16(2008), 143-156.

[15] H. Orhan and H. Kiziltunc, *A generalization on subfamily of p-valent functions with negative coefficients*, Appl. Math. Comput., 155 (2004), 521-530.

[16] G. S. Salagean, *Subclasses of univalent functions*, Lecture Notes in Math. (Springer-Verlag) 1013(1983), 362-372.

[17] T. N. Shanmugam, V. Ravichandran and S. Sivasubramanian, *Differential sandwich theorems for some subclasses of analyitc functions*, Austral. J. Math. Anal. Appl., 3(2006), no.1, Art. 8, 1-11.

[18] H. M. Srivastava, K. Suchithra, B. Adolf Stephen and S. Sivasubramanian, *Inclusion and neighborhood properties of certian subclsaaes of multivalent functions of complex order*, J. Ineq. Pure Appl. Math. 7(2006), no. 5, Art. 191, 1-8.

[19] B. A. Uralegaddi and C. Somanatha, *Certain classes of univalent functions*, In Current Topics in Analytic Function Theory, (Edited by H. M. Srivastava and S. Owa), World Scientfic Publishing Company, Singapore, 1992, 371-374.

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