Acta Universitatis Apulensis No. 23/2010
ISSN: 1582-5329 pp. 39-47

INTEGRAL MEANS OF UNIVALENT SOLUTION FOR FRACTIONAL EQUATION IN COMPLEX PLANE

Rabha W. Ibrahim and Maslina Darus

Abstract. Integral means inequalities are established for the univalent solution of fractional differential equation involving fractional integral operator of order $0 < \alpha + \beta < 1,\ 0 < \alpha,\, \beta < 1$.

2000 *Mathematics Subject Classification*: 34G10, 26A33, 30C45.

Key words: Fractional calculus; Univalent solution;.

1. Introduction and preliminaries

Let $\mathcal{A}$ be the class of all normalized analytic functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

in the open unit disk $U := \{z \in \mathbb{C} : |z| < 1\}$ satisfying $f(0) = 0$ and $f'(0) = 1$. Let $\mathcal{H}$ be the class of analytic functions in U and for any $a \in \mathbb{C}$ and $n \in \mathbb{N}$, $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_n z^n + \dots$.

For given two functions F and G, which are analytic in U, the function F is said to be subordinate to G in U if there exists a function h analytic in U with

$$h(0) = 0 \quad and \quad |h(z)| < 1 \ \ for\ all\ z \in U$$

such that

$$F(z) = G(h(z)) \ \ for\ all\ z \in U.$$

We denote this subordination by $F \prec G$. If G is univalent in U, then the subordination $F \prec G$ is equivalent to $F(0) = G(0)$ and $F(U) \subset G(U)$.

Definition 1.1. **[1]** *Denote by Q the set of all functions f that are analytic and injective on $\overline{U} - E(f)$ where $E(f) := \{\zeta \in \partial U : lim_{z\to\zeta} f(z) = \infty\}$ and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U - E(f)$. The subclass of Q for which $f(0) = a$ is denoted by $Q(a)$.*

We need the following results in the sequel.

Lemma 1.1. **[2]***Let $\Omega \subset \mathbb{C}$, $q \in \mathcal{H}[q(0), 1]$, $\varphi : \mathbb{C}^2 \times \overline{U} \to \mathbb{C}$ and $\varphi(q, tzq'; \zeta) \in \Omega$ for $z \in U$ and $\zeta \in \partial U$ and $0 < t \leq \frac{1}{n} < 1$. If $p \in Q(a)$ and $\varphi(p, zp'; z)$ is univalent in U, then*

$$\Omega \subset \{\varphi(p, zp'; z)|z \in U\} \Rightarrow q(z) \prec p(z).$$

Lemma 1.2. **[3]***If the functions f and g are analytic in U then*

$$g(z) \prec f(z) \Rightarrow \int_0^{2\pi} |g(re^{i\theta})|^{\mu} d\theta \leq \int_0^{2\pi} |f(re^{i\theta})|^{\mu} d\theta, \ \ \mu > 0, \ 0 < r < 1.$$

In [4], Srivastava and Owa, gave definitions for fractional operators (derivative and integral) in the complex z-plane $\mathbb{C}$ as follows:

Definition 1.2. *The fractional integral of order α is defined, for a function f, by*

$$I_z^{\alpha} f(z) := \frac{1}{\Gamma(\alpha)} \int_0^z f(\zeta)(z-\zeta)^{\alpha-1} d\zeta; \ \ \alpha > 0,$$

where the function $f(z)$ is analytic in simply-connected region of the complex z-plane $\mathbb{C}$ containing the origin and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $log(z-\zeta)$ to be real when$(z-\zeta) > 0$.

Definition 1.3. *The fractional derivative of order α is defined, for a function f by*

$$D_z^{\alpha} f(z) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\alpha}} d\zeta; \ \ 0 < \alpha < 1,$$

where the function f is analytic in the simply-connected region of the complex z-plane $\mathbb{C}$ containing the origin and the multiplicity of $(z-\zeta)^{\alpha}$ is removed by requiring $log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Remark 1.1. **[4,5]***From Definitions 1.2 and 1.3, we have $D_z^0 f(z) = f(z)$, $lim_{\alpha\to 0} I_z^{\alpha} f(z) = f(z)$ and $lim_{\alpha\to 0} D_z^{1-\alpha} f(z) = f'(z)$. Moreover,*

$$D_z^{\alpha}\{z^{\mu}\} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}\{z^{\mu-\alpha}\}, \ \ \mu > -1; \ 0 < \alpha < 1$$

and

$$I_z^{\alpha}\{z^{\mu}\} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}\{z^{\mu+\alpha}\}, \ \ \mu > -1; \ \alpha > 0, \ z \neq 0.$$

Our work is organized as follows: In Section 2, we will derive the integral means for normalized analytic functions involving fractional integral in the open unit disk U

$$p(z) := I_z^{\alpha} f(z) \prec q(z) := I_z^{\beta} g(z), \ \ \alpha, \ \beta \in (0,1).$$

In Section 3, we study the existence of locally univalent solution for the fractional diffeo-integral equation

$$D_z^{\alpha} u(z) = h(z, u(z), I_z^{\beta} k(z, u(z))), \ \ \alpha, \ \beta \in (0,1), \tag{1}$$

subject to the initial condition $u(0) = 0$, where $u : U \to \mathbb{C}$ is an analytic function for all $z \in U$ and $k : U \times \mathbb{C} \to \mathbb{C}$, $h : U \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ are analytic functions in $z \in U$. The existence is shown by using Schauder fixed point theorem while the uniqueness is verified by using Banach fixed point theorem. The univalent solutions for various type of fractional differential equations have been rigorously studied by the authors (see [6-8]).

For that purpose we need the following definitions and results:

Let M be a subset of Banach space X and $A : M \to M$ an operator. The operator A is called *compact* on the set M if it carries every bounded subset of M into a compact set. If A is continuous on M (that is, it maps bounded sets into bounded sets) then it is said to be *completely continuous* on M. A mapping $A : X \to X$ is said to be a contraction if there exists a real number ρ, $0 \leq \rho < 1$ such that

$$\|Ax - Ay\| \leq \rho \|x - y\|, \quad \text{for all}\, x, y \in X.$$

Theorem 1.1. Arzela-Ascoli (see [9]) *Let E be a compact metric space and $\mathcal{C}(E)$ be the Banach space of real or complex valued continuous functions normed by*

$$\|f\| := sup_{t\in E}|f(t)|.$$

If $A = \{f_n\}$ is a sequence in $\mathcal{C}(E)$ such that f_n is uniformly bounded and equi-continuous, then $\overline{A}$ is compact.

Theorem 1.2. (Schauder) (see [9]) *Let X be a Banach space, $M \subset X$ a nonempty closed bounded convex subset and $P : M \to M$ is compact. Then P has a fixed point.*

Theorem 1.3. (Banach) (see [10]) *If X is a Banach space and $P : X \to X$ is a contraction mapping then P has a unique fixed point.*

2. Subordination Result

In this section, we discuss the relation between two fractional integral operators of different order. For this purpose we need the following result which can be found in [8].

Theorem 2.1. *For $\alpha \in (0,1)$ and f is a continuous function, then*

$$1)\ \ DI_z^\alpha f(z) = \frac{z^{\alpha-1}}{\Gamma(\alpha)} f(0) + I_z^\alpha Df(z); \ \ D = \frac{d}{dz}.$$

$$2)\ \ I_z^\alpha D_z^\alpha f(z) = D_z^\alpha I_z^\alpha f(z) = f(z).$$

The next result shows the subordination between two fractional integral operators of different order $p(z) := I_z^\alpha f(z)$ and $q(z) := I_z^\beta g(z)$.

Theorem 2.2. *Let f, g be analytic function in U. Assume that $I_z^\beta g(z) \in \mathcal{H}[0,1]$, $I_z^\beta f(z) \in Q(a)$ and $I_z^\beta f(z)[1 + \frac{zf'(z)}{f(z)}]$ is univalent in U. If*

$$\frac{|g(z)| + |g'(z)|}{\Gamma(\beta)} < \frac{|f(z)| + |f'(z)|}{\Gamma(\alpha)},$$

then $q(z) \prec p(z)$.

Proof. Our aim is to apply Lemma 1.1. Define a set

$$\Omega := \{z \in \mathbb{C} : |z| \leq \frac{|g(z)| + |g'(z)|}{\Gamma(\beta)}\}$$

and a function $\varphi(q, tzq'; \zeta) := [q(z) + tzq'(z)]\zeta$. For $z \in U$ we have

$$\begin{aligned}
|\varphi(q, tzq'; \zeta)| &= |[q(z) + tzq'(z)]\zeta| \\
&\leq |q(z) + tzq'(z)||\zeta| \\
&\leq |q(z) + tzq'(z)| \\
&\leq |q(z)| + |tzq'(z)| \\
&\leq |q(z)| + |t||z||q'(z)| \\
&< |q(z)| + |q'(z)| \\
&= |\frac{z^{\beta-1}}{\Gamma(\beta)} g(z)| + |\frac{z^{\beta-1}}{\Gamma(\beta)} g'(z)| \\
&\leq \frac{|z|^{\beta-1}}{\Gamma(\beta)} |g(z)| + \frac{|z|^{\beta-1}}{\Gamma(\beta)} |g'(z)| \\
&< \frac{|g(z)| + |g'(z)|}{\Gamma(\beta)}.
\end{aligned}$$

Hence $\varphi(q, tzq'; \zeta) \in \Omega$. Let $z \in \Omega$, then we obtain that

$$|z| < \frac{|g(z)| + |g'(z)|}{\Gamma(\beta)} < \frac{|f(z)| + |f'(z)|}{\Gamma(\alpha)}.$$

This last inequality implies that $\Omega \subset \{\varphi(p, tzp'; z) | z \in U\}$. Thus in view of Lemma 1.1, we obtained $q(z) \prec p(z)$.

3. Univalent solution for fractional diffeo-integral equation

In this section we established the existence and uniqueness solution for the diffeo-integral equation (1). Let $\mathcal{B} := \mathcal{C}[U, \mathbb{C}]$ be a Banach space of all continuous functions on U endowed with the sup. norm

$$\|u\| := sup_{z \in U} |u(z)|.$$

By using the properties in Theorem 2.1, we easily obtain the following result.

Lemma 3.1. *If the function h is analytic, then the initial value problem (1) is equivalent to the nonlinear Volterra integral equation*

$$u(z) = \int_0^z \frac{(z-\zeta)^{\alpha-1}}{\Gamma(\alpha)} h(\zeta, u(\zeta), v(\zeta)) d\zeta; \;\; \alpha \in (0,1). \tag{2}$$

In other words, every solution of the Volterra equation (2) is also a solution of the initial value problem (1) and vice versa.

The following assumptions are needed in the next theorem:

(H1) There exists a continuous function $\rho(z)$ on U and increasing positive function $\Psi \in C[\mathbb{R}_+, \mathbb{R}_+]$ such that

$$0 < |h(z, u, v)| \leq |\rho(z)| \Psi(\|u\| + \|v\|)$$

with the property that $\Psi(a(z)\|u\| + b(z)\|v\|) \leq a(z)\Psi(\|u\|) + b(z)\Psi(\|v\|)$.

Note that $C[\mathbb{R}_+, \mathbb{R}_+]$ is the Banach space of all continuous positive functions.

(H2) There exists a continuous function p in U, such that $|k(z, u)| \leq |p(z)| \|u\|$ implies that

$$I^\beta |k(z, u)| \leq \frac{\|p\|}{\Gamma(\beta + 1)} \|u\|.$$

Remark 3.1. By using fractional calculus we easily discover that equation (2) is equivalent to the integral equation of the form

$$u(z) = I_z^\alpha h(z, u(z), I_z^\beta k(z, u(z))), \tag{3}$$

that is, the existence of equation (2) is the existence of the equation (3).

Theorem 3.1. *Let the assumptions (H1) and (H2) hold. Then equation (1) has a univalent solution $u(z)$ on U.*

Proof. We need only to show that $P : \mathcal{B} \to \mathcal{B}$ has a fixed point by using Theorem 1.2 where

$$(Pu)(z) := I_z^\alpha h(z, u(z), I_z^\beta k(z, u(z))), \;\; \textit{then} \tag{4}$$

$$\begin{aligned} |(Pu)(z)| &= |I_z^\alpha h(z, u(z), I_z^\beta k(z, u(z)))| \\ &\leq I_z^\alpha |h(z, u(z), I_z^\beta k(z, u(z)))| \\ &\leq \frac{\|\rho\|}{\Gamma(\alpha+1)} \Psi(\|u\| + |I_z^\beta k(z, u(z))|) \\ &\leq \frac{\overline{\Psi}\|\rho\|}{\Gamma(\alpha+1)} [1 + \frac{\|p\|}{\Gamma(\beta+1)}] \\ &= \frac{\overline{\Psi}\|\rho\|[\Gamma(\beta+1) + \|p\|]}{\Gamma(\beta+1)\Gamma(\alpha+1)} \end{aligned}$$

where $\overline{\Psi} := sup_{u \in \mathcal{B}} \Psi(\|u\|)$. Thus we obtain that

$$\|P\| \leq \frac{\overline{\Psi}\|\rho\|[\Gamma(\beta+1) + \|p\|]}{\Gamma(\beta+1)\Gamma(\alpha+1)} =: r$$

that is $P : B_r \to B_r$. Then P mapped B_r into itself. Now we proceed to prove that P is equicontinuous. For $z_1, z_2 \in U$ such that $z_1 \neq z_2$, $|z_2 - z_1| < \delta, \;\; \delta > 0$. Then for all $u \in S$, where

$$S := \{u \in \mathbb{C}, : |u| \leq \frac{\overline{\Psi}\|\rho\|[\Gamma(\beta+1) + \|p\|]}{\Gamma(\beta+1)\Gamma(\alpha+1)}\},$$

we obtained

$$\begin{aligned}
|(Pu)(z_1)-(Pu)(z_2)| &= |\int_0^{z_1}\frac{(z_1-\zeta)^{\alpha-1}}{\Gamma(\alpha)}h(\zeta,u(\zeta),v(\zeta))d\zeta- \\
&-\int_0^{z_2}\frac{(z_2-\zeta)^{\alpha-1}}{\Gamma(\alpha)}h(\zeta,u(\zeta),v(\zeta))d\zeta| \\
&\le \frac{\|h\|}{\Gamma(\alpha)}|\int_0^{z_1}[(z_1-\zeta)^{\alpha-1}-(z_2-\zeta)^{\alpha-1}]d\zeta+\int_{z_1}^{z_2}(z_2-\zeta)^{\alpha-1}d\zeta| \\
&= \frac{\|h\|}{\Gamma(\alpha+1)}|[2(z_2-z_1)^{\alpha}+z_2^{\alpha}-z_1^{\alpha}]| \\
&\le 2\frac{\|h\|}{\Gamma(\alpha+1)}|z_2-z_1|^{\alpha} \\
&\le 2\frac{\|h\|}{\Gamma(\alpha+1)}\delta^{\alpha},
\end{aligned}$$

which is independent of u.

Hence P is an equicontinuous mapping on S. Moreover, for $z_1\neq 0, z_2\neq 0\in U$ such that $z_1\neq z_2$ and under assumption (H1), we show that P is a univalent function. The Arzela-Ascoli theorem yields that every sequence of functions $\{u_n\}$ from $P(S)$ has a uniformly convergent subsequence, and therefore $P(S)$ is relatively compact. Schauder's fixed point theorem asserts that P has a fixed point. By construction, a fixed point of P is a univalent solution of the initial value problem (1).

Now we discuss the uniqueness solution for the problem (1). For this purpose let us state the following assumptions:

(H3) Assume that there exists a positive number L such that for each u_1, v_1 and $u_2, v_2\in\mathcal{B}$,

$$|h(z,u_1(z),v_1(z))-h(z,u_2(z),v_2(z))|\le L[\|u_1-u_2\|+\|v_1-v_2\|].$$

(H4) Assume that there exists a positive number ℓ such that for each $u_1,u_2\in\mathcal{B}$ we have

$$|k(z,u_1(z))-k(z,u_2(z))|\le \ell\|u_1-u_2\|.$$

Theorem 3.2. *Let the hypotheses (H1-H4) be satisfied. If* $\frac{L[\Gamma(\beta+1)+\ell]}{\Gamma(\beta+1)\Gamma(\alpha+1)}<1$, *then (1) admits a unique univalent solution* $u(z)$.

Proof. Assume the operator P defined in equation (4), we only need to show that P is a contraction mapping that is P has a unique fixed point which is corresponding to the unique solution of the equation (1). Let $u_1, u_2 \in \mathcal{B}$, then for all $z \in U$, we obtain that

$$\begin{aligned}
|(Pu_1)(z) - (Pu_2)(z)| &\leq I_z^\alpha |h(z, u_1(z), I_z^\beta k(z, u_1(z))) - h(z, u_2(z), I_z^\beta k(z, u_2(z)))| \\
&\leq \frac{L}{\Gamma(\alpha+1)}[\|u_1 - u_2\| + \|I_z^\beta k(z, u_1(z)) - I_z^\beta k(z, u_2(z))\|] \\
&\leq \frac{L}{\Gamma(\alpha+1)}[\|u_1 - u_2\| + I_z^\beta \|k(z, u_1(z)) - k(z, u_2(z))\|] \\
&\leq \frac{L}{\Gamma(\alpha+1)}[\|u_1 - u_2\| + \frac{\ell}{\Gamma(\beta+1)}\|u_1 - u_2\|] \\
&= \frac{L[\Gamma(\beta+1) + \ell]}{\Gamma(\beta+1)\Gamma(\alpha+1)}\|u_1 - u_2\|.
\end{aligned}$$

Thus by the assumption of the theorem we have that P is a contraction mapping. Then in view of Banach fixed point theorem, P has a unique fixed point which corresponds to the univalent solution (Theorem 3.1) of equation (1). Hence the proof.

The next result shows the integral means of univalent solutions of problem (1).

Theorem 3.3. *Let $u_1(z), u_2(z)$ be two analytic univalent solutions for the equation (1) satisfy the assumptions of Theorem 2.2 with $p(z) := u_1(z)$ and $q(z) := u_2(z)$ then*

$$\int_0^{2\pi} |q(re^{i\theta})|^\mu d\theta \leq \int_0^{2\pi} |p(re^{i\theta})|^\mu d\theta, \;\; \mu > 0, \;\; 0 < r < 1.$$

Proof. Setting $\alpha = \beta, f(z) := h(z, u_1, v_1), g(z) := h(z, u_2, v_2)$, Theorem 2.2 implies that $q(z) \prec p(z)$. Hence in view of Lemma 1.2, we obtain the result.

Recently, the authors established the existence solutions for different types of fractional differential equations and also involving the integral operator(see [11],[12]).

Acknowledgement. The work presented here was supported by Fundamental Research Grant: UKM-ST-06-FRGS0107-2009.

References

[1] S. S. Miller and P. T. Mocanu, *Subordinants of differential superordinations*, Complex Variables, 48(10)(2003), 815-826.

[2] S. S. Miller and P. T. Mocanu, *Differential Subordinants: Theory and Applications*, Monographs and Texbooks in Pure and Applied Mathematics, vol 225, Marcel Dekker, New York, 2000.

[3] J. E. Littlwood., *On inequalities in the theory of functions*, Proc. London Math. Soc.(2), 23(1925), 481-519.

[4] H. M. Srivastava and S. Owa, *Univalent Functions, Fractional Calculus, and Their Applications*, Halsted Press, John Wiley and Sons, New York, Chichester, Brisbane, and Toronto, 1989.

[5] K. S. Miller and B. Ross, *An Introduction to The Fractional Calculus and Fractional Differential Equations*, John-Wiley and Sons, Inc., 1993.

[6] R. W. Ibrahim and M. Darus, *Subordination and superordination for analytic functions involving fractional integral operator*, Complex Variables and Elliptic Equations, 53(11) (2008), 1021-1031.

[7] R. W. Ibrahim and M. Darus, *Sandwich Theorem for Analytic Functions Involving a Fractional Operator*, ASM Sci. J., 2(1)(2008), 93-100.

[8] R. W. Ibrahim and M. Darus, *Subordination and superordination for univalent solutions for fractional differential equations*, J. Math. Anal. Appl. 345 (2008), 871-879.

[9] R. F. Curtain and A. J. Pritchard, *Functional Analysis in Modern Applied Mathematics*, Academic Press. 1977.

[10] K. Balachandar and J. P. Dauer, *Elements of Control Theory*, Narosa Publishing House, 1999.

[11] R. W. Ibrahim and M. Darus, *Subordination and superordination for functions based on Dziok-Srivastava linear operator*, Bulletin of Mathematical Analysis and Applications. 2(3) (2010), 15-26.

[12] R. W. Ibrahim, M. Darus and S. Momani, *Subordination and superordination for certain analytic functions containing fractional integral*, Survey in mathematics and its applications, 4 (2009), 111 117.

Rabha W. Ibrahim and Maslina Darus
[1,2]School of Mathematical Sciences
Faculty of Science and Technology
Universiti Kebangsaan Malaysia
Bangi 43600, Selangor D. Ehsan, Malaysia
email:[1]*rabhaibrahim@yahoo.com*, [2]*maslina@ukm.my*