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TRIGONOMETRIC PROOF OF STEINER-LEHMUS THEOREM IN HYPERBOLIC GEOMETRY

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Abstract In this note, we present a short trigonometric proof to the Steiner - Lehmus Theorem in hyperbolic geometry.

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1. Introduction

Hyperbolic Geometry appeared in the first half of the 19^{th} century as an attempt to understand Euclid's axiomatic basis of Geometry. It is also known as a type of non-Euclidean Geometry, being in many respects similar to Euclidean Geometry. Hyperbolic Geometry includes similar concepts as distance and angle. Both these geometries have many results in common but many are different.

There are known many models for Hyperbolic Geometry, such as: Poincaré disc model, Poincaré half-plane, Klein model, Einstein relativistic velocity model, etc. In this note we choose the Poincaré disc model in order to present the hyperbolic version of the Steiner-Lehmus theorem. We mention that N.Sonmez[9] has presented a trigonometric proof for the Poincaré half plane model but his approach is different than ours. The Euclidean version of this well-known theorem states that if the internal angle bisectors of two angles of a triangle are equal, then the triangle is isosceles (see the book of H.S.M.Coxeter and S.L.Greitzer [2,pp.14-16]). This result has a simple statement but it is of great interest. We just mention here few different proofs given by O.A.AbuArqob, H.E.Rabadi, J.S.Khitan[1], G.Gilbert, D.MacDonnell[3], H.Hajja[4], M.Levin[5], J.V.Malesevic[6] and A.P.Pargeter[8].

We begin with the recall of some basic geometric notions and properties in the Poincaré disc. Let D denote the unit disc in the complex z - plane, i.e.

$$D = \{z \in \mathbb{C} : |z| < 1\}$$

The most general Möbius transformation of D is

$$z \to e^{i\theta} \frac{z_0 + z}{1 + \overline{z_0} z} = e^{i\theta}(z_0 \oplus z),$$

which induces the Möbius addition $\oplus$ in D, allowing the Möbius transformation of the disc to be viewed as a Möbius left gyro-translation

$$z \to z_0 \oplus z = \frac{z_0 + z}{1 + \overline{z_0} z}$$

followed by a rotation. Here $\theta \in \mathbb{R}$ is a real number, $z, z_0 \in D$, and $\overline{z_0}$ is the complex conjugate of z_0. Let $Aut(D, \oplus)$ be the automorphism group of the grupoid $(D, \oplus)$. If we define

$$gyr : D \times D \to Aut(D, \oplus), gyr[a, b] = \frac{a \oplus b}{b \oplus a} = \frac{1 + a\overline{b}}{1 + \overline{a}b},$$

then is true gyro-commutative law

$$a \oplus b = gyr[a, b](b \oplus a).$$

A gyro-vector space $(G, \oplus, \otimes)$ is a gyro-commutative gyro-group $(G, \oplus)$ that obeys the following axioms:

(1) $gyr[\mathbf{u}, \mathbf{v}]\mathbf{a} \cdot gyr[\mathbf{u}, \mathbf{v}]\mathbf{b} = \mathbf{a} \cdot \mathbf{b}$ for all points $\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v} \in G$.

(2) G admits a scalar multiplication, $\otimes$, possessing the following properties. For all real numbers $r, r_1, r_2 \in \mathbb{R}$ and all points $\mathbf{a} \in G$:

(G1) $1 \otimes \mathbf{a} = \mathbf{a}$

(G2) $(r_1 + r_2) \otimes \mathbf{a} = r_1 \otimes \mathbf{a} \oplus r_2 \otimes \mathbf{a}$

(G3) $(r_1 r_2) \otimes \mathbf{a} = r_1 \otimes (r_2 \otimes \mathbf{a})$

(G4) $\frac{|r| \otimes \mathbf{a}}{\|r \otimes \mathbf{a}\|} = \frac{\mathbf{a}}{\|\mathbf{a}\|}$

(G5) $gyr[\mathbf{u}, \mathbf{v}](r \otimes \mathbf{a}) = r \otimes gyr[\mathbf{u}, \mathbf{v}]\mathbf{a}$

(G6) $gyr[r_1 \otimes \mathbf{v}, r_1 \otimes \mathbf{v}] = 1$

(3) Real vector space structure $(\|G\|, \oplus, \otimes)$ for the set $\|G\|$ of one-dimensional "vectors"

$$\|G\| = \{\pm \|\mathbf{a}\| : \mathbf{a} \in G\} \subset \mathbb{R}$$

with vector addition $\oplus$ and scalar multiplication $\otimes$, such that for all $r \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b} \in G$,

(G7) $\|r \otimes \mathbf{a}\| = |r| \otimes \|\mathbf{a}\|$
(G8) $\|\mathbf{a} \oplus \mathbf{b}\| \leq \|\mathbf{a}\| \oplus \|\mathbf{b}\|$

Lemma. *Let ABC be a gyro-triangle in a Möbius gyro-vector space $(V_s, \oplus, \otimes)$, with vertices A, B, C, corresponding gyro-angles $\alpha, \beta, \gamma, 0 < \alpha + \beta + \gamma < \pi$, and side gyro-lengths (or, simply, sides) a, b, c. The gyro-angles of the gyro-triangle ABC are determined by its sides :*

$$\begin{aligned}
\cos\alpha &= \frac{-a_s^2 + b_s^2 + c_s^2 - a_s^2 b_s^2 c_s^2}{2b_s c_s} \cdot \frac{1}{1 - a_s^2}, \\
\cos\beta &= \frac{a_s^2 - b_s^2 + c_s^2 - a_s^2 b_s^2 c_s^2}{2a_s c_s} \cdot \frac{1}{1 - b_s^2}, \\
\cos\gamma &= \frac{a_s^2 + b_s^2 - c_s^2 - a_s^2 b_s^2 c_s^2}{2b_s a_s} \cdot \frac{1}{1 - c_s^2},
\end{aligned}$$

with $a_s = \frac{a}{s}$ (see [10, pp.259]).

For further details we refer to the recent book of A.Ungar [10].

2.Main result

The hyperbolic version of the classical Steiner-Lehmus Theorem is the following.

Theorem. *If the internal angle bisectors of two angles of a triangle are equal, then the triangle is not isosceles.*

Proof. Let ΔABC be a hyperbolic triangle in the Poincaré disc, whose vertices are the points A, B and C of the disc whose sides (directed counterclockwise) are $\mathbf{a} = -B \oplus C, \mathbf{b} = -C \oplus A$ and $\mathbf{c} = -A \oplus B$. Let BB' and CC' be the respective internal angle bisectors of angles B and C in triangle ABC (See Figure 1).

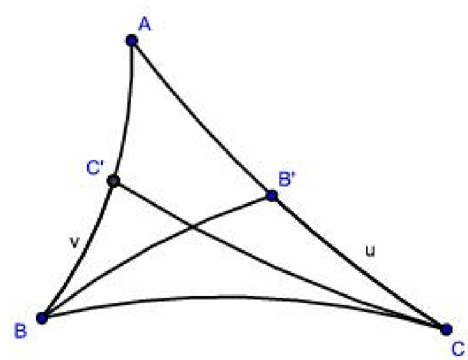

Figure 1

We take $a := |-B \oplus C|, b := |-c \oplus A|, c := |-A \oplus B|, x := d(B, B') = d(C, C'), u := d(C, B'), v := d(B, C'), B = 2\beta, C = 2\gamma$. Let $B > C$ (î.e. $\beta > \gamma$). Then, $\cos\beta < \cos\gamma$ $(\beta, \gamma \in (0, \frac{\pi}{2}))$. If we use the result contained in the previous Lemma in triangles $BB'C$ and $CC'B$ then we get:

$$\cos\beta = \frac{-u^2 + x^2 + a^2 - u^2x^2a^2}{2xa} \cdot \frac{1}{1-u^2},$$

$$\cos\gamma = \frac{-v^2 + x^2 + a^2 - v^2x^2a^2}{2xa} \cdot \frac{1}{1-v^2}.$$

This implies

$$\cos\beta - \cos\gamma = \frac{1}{2xa}\left[\frac{-u^2 + x^2 + a^2 - u^2x^2a^2}{1-u^2} - \frac{-v^2 + x^2 + a^2 - v^2x^2a^2}{1-v^2}\right] =$$

$$\frac{(v^2 - u^2)(1 + a^2x^2 - x^2 - a^2)}{2xa(1-u^2)(1-v^2)} = \frac{(v-u)(v+u)(1-x^2)(1-a^2)}{2xa(1-u^2)(1-v^2)} < 0$$

Now we use the following theorem: *If triangles ABC and $A'B'C'$ have $AB = A'B'$ and $AC = A'C'$,then $BC < B'C'$ if and only if $\sphericalangle A < \sphericalangle A'$* . E.Moise [7, p.121] calls this the "Hinge Theorem" and the result is valid in Absolute Geometry. Applying this result in triangles BCB' and BCC' it follows $v < u$, hence the relation $\frac{(v-u)(v+u)(1-x^2)(1-a^2)}{2xa(1-u^2)(1-v^2)} < 0$ is true. Consequently, the case $B > C$ is satisfied while $d(B, B') = d(C, C')$, therefore the triangle ABC cannot be isosceles. □

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