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A NOTE ON A-CONTRACTIONS AND COMMON FIXED POINTS

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Abstract. This article represents an appreciable generalization of the recent results of M. Akram et al which extended several results of B. Ahmad and F. U. Rehman, N. Shioji et al and Z. Chuanyi to the A-contractions.

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1. Introduction and preliminaries

Recently, Akram et al[2] introduced the following new class of contraction maps called A-contractions.

Let $\mathbf{R}_+$ denote the set of all nonnegative real numbers and A the set of all functions $\alpha : \mathbf{R}_+^{\mathbf{3}} \longrightarrow \mathbf{R}_+$ satisfying the following conditions.
(i) α is continuous on the set $\mathbf{R}_+^{\mathbf{3}}$
(ii) $a \leq kb$ for some $k \in [0, 1)$ whenever $a \leq \alpha(a, b, b)$ or $a \leq \alpha(b, a, b)$ or $a \leq \alpha(b, b, a)$ for all $a, b \in \mathbf{R}_+$.

Definition 1. A selfmap T on a metric space X is said to be an A-contraction if it satisfies

$$d(Tx, Ty) \leq \alpha(d(x, y), d(x, Tx), d(y, Ty)) \tag{1}$$

for all $x, y \in X$ and some $\alpha \in A$.

It was shown in [2] that the classes of contractions studied by Bianchini[3], Kannan[5], Khan[6] and Reich[8] are all special cases of the A-contractions.

Definition 2. (See [3],[5],[6],[8]). Let X be a metric space. Then for all $x, y \in X$, $T : X \longrightarrow X$ is said to be

(i) B-contraction if there exists a number $b \in [0,1)$ such that

$$d(Tx,Ty) \leq bmax\{d(x,Tx), d(y,Ty)\}; \tag{2}$$

(ii) K-contraction if there exists a number $r \in [0, \frac{1}{2})$ such that

$$d(Tx,Ty) \leq r[d(x,Tx) + d(y,Ty)]; \tag{3}$$

(iii) M-contraction if there exists a number $h \in [0,1)$ such that

$$d(Tx,Ty) \leq h\sqrt{d(x,Tx)d(y,Ty)}; \tag{4}$$

(iv) R-contraction if there exist nonnegative numbers a, b, c satisfying $a + b + c \leq 1$ such that

$$d(Tx,Ty) \leq ad(x,Tx) + bd(y,Ty) + cd(x,y); \tag{5}$$

Theorems 1-4, and Example 1 of [2] can be summarized as follows.

Theorem 1. [2] *Let X be a metric space. Then*
(i) M-contractions and K-contractions are proper sub-classes of A-contractions,
(ii) Every B-contraction is an A-contraction and
(iii) Every R-contraction is an A-contraction.

The following fixed point theorems proved in [2] extended some of the results of Ahmad and Rehman[1] to the A-contractions.

Theorem 2. [2] *Let T be an A-contraction on a complete metric space X. Then T has a unique fixed point in X such that the sequence $\{T^n x_0\}$ converges to the fixed point, for any $x_0 \in X$.*

The purpose of this paper is to prove common fixed point theorems for two pairs of weakly compatible selfmaps of X, satisfying a generalized A-contractive condition, such that X need not be complete.

2.Main Results

Let F, G, S and T be selfmaps of a metric space X satisfying

$$SX \subseteq FX;\ TX \subseteq GX. \tag{7}$$

Then for any point $x_0 \in X$, we can find points x_1, x_2, x_3 ..., all in X, such that

$$Sx_0 = Fx_1,\ Tx_1 = Gx_2,\ Sx_2 = Fx_3 \ldots.$$

Therefore, by induction, we can define a sequence $\{y_n\}$ in X as

$$y_n = \begin{cases} Sx_n = Fx_{n+1}, \ \textit{when } n \textit{ is even} \\ Tx_n = Gx_{n+1}, \ \textit{when } n \textit{ is odd}, \end{cases}$$

where$n = 0, 1, 2, \ \ldots.$

The following theorem establishes existence of coincidence and unique common fixed point of F, G, S and T where the union of the ranges of F and G is required to be complete. The set of coincidence points of T and F is denoted by $C(T, F)$, and the set of natural numbers denoted by $\mathbf{N}$.

Theorem 3. *Let F, G, S and T be selfmaps of a metric space X satisfying (7) and, for all $x, y \in X$,*

$$d(Sx, Ty) \leq \alpha(d(Gx, Fy), d(Gx, Sx), d(Fy, Ty)), \tag{8}$$

where $\alpha \in A$. Suppose $FX \cup GX$ is a complete subspace of X, then the sets C(T,F) and C(S,G) are nonempty.
Suppose further that (T,F) and (S,G) are weakly compatible, then F, G, S and T have a unique common fixed point.

Proof. Assuming $n \in \mathbf{N}$ is even, then

$$\begin{aligned} d(y_n, y_{n+1}) &= d(Sx_n, Tx_{n+1}) \\ &\leq \alpha(d(Gx_n, Fx_{n+1}), d(Gx_n, Sx_n), d(Fx_{n+1}, Tx_{n+1})) \\ &= \alpha(d(y_{n-1}, y_n), d(y_{n-1}, y_n), d(y_n, y_{n+1})) \end{aligned}$$

which implies $d(y_n, y_{n+1}) \leq kd(y_{n-1}, y_n)$.

On the other hand, assuming $n \in \mathbf{N}$ is odd,

$$\begin{aligned} d(y_n, y_{n+1}) &= d(Tx_n, Sx_{n+1}) \\ &\leq \alpha(d(Gx_{n+1}, Fx_n), d(Gx_{n+1}, Sx_{n+1}), d(Fx_n, Tx_n)) \\ &= \alpha(d(y_n, y_{n-1}), d(y_n, y_{n+1}), d(y_{n-1}, y_n)). \end{aligned}$$

This means $d(y_n, y_{n+1}) \leq kd(y_{n-1}, y_n)$.
Thus whether n is odd or even, we have $d(y_n, y_{n+1}) \leq kd(y_{n-1}, y_n)$ for some $k \in [0, 1)$.
Inductively,

$$d(y_n, y_{n+1}) \leq kd(y_{n-1}, y_n) \leq k^2 d(y_{n-2}, y_{n-1}) \leq \ldots \leq k^n d(y_0, y_1).$$

That is, $d(y_n, y_{n+1}) \leq k^n d(y_0, y_1)$ for some $k \in [0, 1)$. Hence $\{y_n\}$ is Cauchy in X.
Observe that $\{y_n\}$ is contained in $FX \cup GX$. Now since $\{y_n\}$ is Cauchy and $FX \cup GX$ is complete, there exists a point $p \in FX \cup GX$ such that $\lim_{n\to\infty} y_n = p$.

Without loss of generality, let $p \in GX$. It means we can find a point $q \in X$ such that $p = Gq$. Putting $x = q,\ y = x_m$, m odd, into (8) yields

$$d(Sq, Ty) \leq \alpha(d(Gq, Fx_m), d(Gq, Sq), d(Fx_m, Tx_m)),$$

i.e.,

$$d(Sq, y_m) \leq \alpha(d(p, y_{m-1}), d(p, Sq), d(y_{m-1}, y_m)).$$

Letting $m \to \infty$, recalling that α is continuous on $\mathbf{R}_+^{\mathbf{3}}$, we obtain

$$d(Sq, p) \leq \alpha(d(p, p), d(p, Sq), d(p, p)).$$

That is, $d(Sq, p) \leq \alpha(0, d(p, Sq), 0)$, which implies that $d(Sq, p) \leq k0 = 0$.
Consequently, $Sq = p$.
From $SX \subseteq FX$ we know that there exists a point $u \in X$ such that $Fu = Sq = p = Gq$.
Choosing $x = q,\ y = u$, (8) gives $d(p, Tu) \leq \alpha(0, 0, d(p, Tu))$ so that $d(p, Tu) \leq k0 = 0$.
Hence, $Fu = Tu = p = Sq = Gq$. This proves the first part of the theorem.

Now suppose (F, T) and (S, G) are weakly compatible pairs, then F and T commute at u, and G and S commute at q so that

$$Fp = FFu = FTu = TFu = Tp \text{ and } Sp = SSq = SGq = GSq = Gp. \qquad (9)$$

Now with $x = p,\ y = u$, (8) and (9) yield $d(Sp, p) \leq \alpha(d(Sp, p), 0, 0)$, and this implies $d(Sq, p) \leq k0 = 0$. Therefore $p = Sp = Gp$.
In a similar way, letting $x = y = p$, (8) and (9) yield $p = Tp = Fp$.
Thus, $Sp = Gp = p = Tp = Fp$.

Finally, we show that p is unique in X.
Suppose p^* is another common fixed point of the four maps. Then from (8),

$$\begin{aligned} x = p^*,\ y = p \quad &\Rightarrow d(Sp^*, Tp) \leq \alpha(d(Gp^*, Fp), d(Gp^*, Sp^*), d(Fp, Tp)) \\ &\Rightarrow d(p^*, p) \leq \alpha(d(p^*, p), 0, 0) \\ &\Rightarrow d(p^*, p) \leq k0 = 0 \end{aligned}$$

Hence, $p^* = p$ and this completes the proof.

The following corollary is obtained by letting $F = G$ in the preceding theorem.

Corrolary 1. *Let F, S and T be selfmaps of a metric space X satisfying $SX \cup TX \subseteq FX$ and, for all $x, y \in X$,*

$$d(Sx, Ty) \leq \alpha(d(Fx, Fy), d(Fx, Sx), d(Fy, Ty)),$$

where $\alpha \in A$. Suppose FX is a complete subspace of X, then F, S and T have a coincidence point.
Suppose further that F commutes with both S and T at this coincidence point, then F, S and T have a unique common fixed point.

Choosing F to be the identity map of X in Corollary 1, the following result follows immediately.

Corrolary 2. *Let S and T be selfmaps of a complete metric space X satisfying*

$$d(Sx, Ty) \leq \alpha(d(x, y), d(x, Sx), d(y, Ty)), \quad for\ all\ x, y \in X,$$

where $\alpha \in A$. Then S and T have a unique common fixed point.

Remarks.
1. It is clear that Theorem 2 can be obtained from Corollary 2 by letting $T = S$.
2. By virtue of Theorems 1 and 3, suppose $\alpha \in A$ and F, G, S, T are selfmaps of a metric space X satisfying (7) and any of the following inequalities for all $x, y \in X$,

$(i)\ d(Sx, Ty) \leq bmax\{d(Gx, Tx), d(Fy, Ty)\}\ for\ some\ b \in [0, 1)$
$(ii)\ d(Sx, Ty) \leq r[d(Gx, Sx) + d(Fy, Ty)]\ for\ some\ r \in [0, \frac{1}{2})$
$(iii)\ d(Sx, Ty) \leq h\sqrt{d(Gx, Tx)d(Fy, Ty)}\ for\ some\ h \in [0, 1)$
$(iv)\ d(Sx, Ty) \leq ad(Gx, Tx) + bd(Fy, Ty) + cd(Gx, Fy)\ for\ some\ nonnegative\ numbers\ a, b, c\ satisfying\ a + b + c \leq 1.$

Then, F, G, S and T always have a unique common fixed point provided the pairs (T, F) and (S, G) are weakly compatible.

The following Example illustrates Theorem 3.

Example. Let $X = [\frac{1}{10}, 1]$, $d(x, y) = |x - y|$, $\alpha(a, b, c) = \frac{1}{4}(a + b + c)\ \ for\ all\ \ a, b, c \in \mathbf{R}_+$. It is clear that α is well-defined, for (i) α is continuous on $\mathbf{R}^{\mathbf{3}}_+$, and (ii) $a \leq \frac{2}{3}b$ whenever $a \leq \alpha(a, b, b)$ or $a \leq \alpha(b, a, b)$ or $a \leq \alpha(b, b, a)$ for all $a, b \in \mathbf{R}_+$.

Define the selfmaps S, G, T, F of $X = [\frac{1}{10}, 1]$ as follows

$$Sx = \begin{cases} \frac{1}{10} & if \quad x = \frac{1}{10} \\ \frac{3}{20} & if \ x > \frac{1}{10} \end{cases}, \quad Gx = \begin{cases} \frac{1}{10} & if \ x = \frac{1}{10} \\ \frac{3}{10} & if \ x > \frac{1}{10} \end{cases},$$

$$Ty = \begin{cases} \frac{1}{10} \; if \; y = \frac{1}{10} \; or \; y > \frac{1}{4} \\ \frac{3}{10} \; if \; \frac{1}{10} < y \le \frac{1}{4}, \end{cases} \quad Fy = \begin{cases} \frac{1}{10} \; if \; y = \frac{1}{10} \\ \frac{3}{5} \; if \; \frac{1}{10} < y \le \frac{1}{4} \\ y - \frac{3}{20} \; if \; y > \frac{1}{4}. \end{cases}$$

Clearly, $SX = \{\frac{1}{10}, \frac{3}{20}\}, GX = \{\frac{1}{10}, \frac{3}{10}\}, TX = \{\frac{1}{10}, \frac{3}{10}\}, SX = \{\frac{1}{10}\} \cup (\frac{1}{10}, \frac{17}{20}] = [\frac{1}{10}, \frac{17}{20}]$. We observe that condition (7) is satisfied, and $FX \cup GX = [\frac{1}{10}, \frac{17}{20}]$ is complete. To verify condition (8), we consider the six exhaustive cases below.

Case 1: When $x = y = \frac{1}{10}$, using (8), we have
$|\frac{1}{10} - \frac{1}{10}| \le \alpha(|\frac{1}{10} - \frac{1}{10}|, |\frac{1}{10} - \frac{1}{10}|, |\frac{1}{10} - \frac{1}{10}|) = \alpha(0,0,0) = 0$. True.

Case 2: $x > \frac{1}{10}, y = \frac{1}{10}$, using (8),
$\frac{1}{20} = |\frac{3}{20} - \frac{1}{10}| \le \alpha(|\frac{3}{10} - \frac{1}{10}|, |\frac{3}{10} - \frac{3}{20}|, |\frac{1}{10} - \frac{1}{10}|) = \alpha(\frac{1}{5}, \frac{3}{10}, 0) = \frac{1}{8}$. True.

Case 3: $x = \frac{1}{10}, \frac{1}{10} < y \le \frac{1}{4}$, using (8), $\frac{1}{5} = |\frac{1}{10} - \frac{3}{10}| \le \alpha(\frac{1}{2}, 0, \frac{3}{10}) = \frac{1}{5}$. True.

Case 4: $x > \frac{1}{10}, \frac{1}{10} < y \le \frac{1}{4}$ yields $\frac{3}{20} = |\frac{3}{20} - \frac{3}{10}| = \alpha(\frac{3}{10}, \frac{3}{20}, \frac{3}{10}) = \frac{3}{16}$. True.

Case 5: $x = \frac{1}{10}, y > \frac{1}{4}$ gives $0 \le \alpha(y - \frac{1}{4}, 0, y - \frac{1}{4}) = \frac{1}{4}(2y - \frac{1}{2})$, that is, $0 \le y - \frac{1}{4}$. True.

Case 6: When $x > \frac{1}{10}, y > \frac{1}{4}$, we have $\frac{1}{20} \le \alpha(|y - \frac{9}{20}|, \frac{3}{20}, |y - \frac{1}{4}|) = \frac{1}{4}(|y - \frac{9}{20}| + \frac{3}{20} + y - \frac{1}{4})$. That is, $\frac{3}{10} \le y + |y - \frac{9}{20}|$, which is true for $y > \frac{1}{4}$.

Moreover, it is obvious that $C(T,F) = C(S,G) = \{\frac{1}{10}\}$. Also, $TF(\frac{1}{10}) = FT(\frac{1}{10}) = \frac{1}{10}$ and $SG(\frac{1}{10}) = GS(\frac{1}{10}) = \frac{1}{10}$. Thus, (T,F) and (S,G) are weakly compatible pairs. The four maps have a unique common fixed point $\frac{1}{10} \in X$.

Finally, we present the following generalization of Theorem 3.

Theorem 4. *Let F, G, S and T be selfmaps of a metric space X, and let $\{S_n\}_{n=1}^{\infty}$ and $\{T_n\}_{n=1}^{\infty}$ be sequences on S and T satisfying*

$$S_nX \subseteq FX; \; T_n \subseteq GX, \; n = 1, 2, ... \tag{7'}$$

and, for all $x, y \in X$,

$$d(S_ix, T_jy) \le \alpha(d(Gx, Fy), d(Gx, S_ix), d(Fy, T_jy)), \tag{8'}$$

where $\alpha \in A$. Suppose $FX \cup GX$ is a complete subspace of X, then for each $n \in \mathbf{N}$,
(i) the sets $C(F, T_n)$ and $C(G, S_n)$ are nonempty.
Further, if T_n commutes with F and S_n commutes with G at their coincidence points, then
(ii) F, G, S_n and T_n have a unique common fixed point.

Proof. For any arbitrary $x_0 \in X$ and $n = 0, 1, 2, ...$, following a similar argument as in the beginning of this section, we can define a sequence $\{y'_n\}$ in X as

$$y'_n = \begin{cases} S_n x_n = F x_{n+1}, \ \textit{when } n \textit{ is even} \\ T_n x_n = G x_{n+1}, \ \textit{when } n \textit{ is odd.} \end{cases}$$

Now for each $i = 1, 3, 5, ...$ and $j = 2, 4, 6, ...$, from $(8')$ we have $d(y'_i, y'_{i+1}) \leq kd(y'_{i-1}, y'_i)$ and $d(y'_j, y'_{j+1}) \leq kd(y'_{j-1}, y'_j)$. That is,

$$d(y'_n, y'_{n+1}) \leq kd(y'_{n-1}, y'_n) \quad n = 1, 2, 3, \ \ldots.$$

By induction (as in the proof of Theorem 4), we have $d(y'_n, y'_{n+1}) \leq k^n d(y'_0, y'_1)$ for some $k \in [0, 1)$. Consequently, $\{y'_n\}$ is Cauchy in $FX \cup GX$, a complete subspace of X.
The rest of the proof is similar to the corresponding part of the proof of Theorem 4.

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