Acta Universitatis Apulensis No. 23/2010
ISSN: 1582-5329 pp. 153-157

ON THE UNIVALENCE OF CERTAIN INTEGRAL OPERATORS

Horiana Tudor

Abstract. In this note we shall study the analyticity and the univalence of some integral operators if the functions involved belongs to some special subclasses of univalent functions.

2000 *Mathematics Subject Classification*: 30C45.

1. Introduction

Let A be the class of functions f which are analytic in the unit disk $U = \{ z \in C \ : |z| < 1 \}$ such that $f(0) = 0,\ f'(0) = 1$.

Let S denote the class of functions $f \in A$, f univalent in U.

We recall here the well known integral operators due to Kim and Merkes [1], Pfaltzgraff [4], Moldoveanu and N. N. Pascu [3] and the recently generalization of these results obtained by the author in [6].

Theorem 1.[1]. *Let $f \in S$, $\beta \in C$. If $|\beta| \le 1/4$, then the function*

$$F(z) = \int_0^z \left(\frac{f(u)}{u} \right)^\beta du \tag{1}$$

is univalent in U.

Theorem 2.[4]. *Let $f \in S$, $\gamma \in C$. If $|\gamma| \le 1/4$, then the function*

$$F(z) = \int_0^z \left(f'(u) \right)^\gamma du \tag{2}$$

is univalent in U.

Theorem 3.[3]. *Let $f \in S$, $\alpha \in C$. If $|\alpha - 1| \le 1/4$, then the function*

$$F(z) = \left(\alpha \int_0^z f^{\alpha - 1}(u) du \right)^{1/\alpha} \tag{3}$$

is analytic and univalent in U, where the principal branch is intended.

Theorem 4.[6]. *Let f, $g \in S$ and α, β, γ be complex numbers. If*

$$4|\alpha - 1| + 4|\beta| + 4|\gamma| \leq 1 \tag{4}$$

then the function

$$H(z) = \left(\alpha \int_0^z g^{\alpha-1}(u) \left(\frac{f(u)}{u}\right)^{\beta} \left(f'(u)\right)^{\gamma} du\right)^{1/\alpha} \tag{5}$$

is analytic and univalent in U, where the principal branch is intended.

The usual subclasses of the class S consisting of starlike and convex functions will be denoted by S^* , respectively CV. Also we consider the subclasses of φ-spiral and convex φ-spiral functions of order ρ defined as follows

$$S^*(\varphi, \rho) = \left\{ f \in S : \ \ Re\left(e^{i\varphi} \frac{zf'(z)}{f(z)} \right) > \rho \cos\varphi, \quad z \in U \right\}$$

and

$$C(\varphi, \rho) = \left\{ f \in S : \ \ Re\left[e^{i\varphi} \left(1 + \frac{zf''(z)}{f'(z)}\right)\right] > \rho \cos\varphi, \ \ z \in U \right\},$$

where $\varphi \in (-\pi/2,\ \pi/2)$, $\rho \in [0, 1)$.

We observe that $S^* = S^*(0, 0)$ and $CV = C(0, 0)$.

2. Preliminaries

We first recall here some results which will be used in the sequel.

Theorem 5.[2]. *If $f \in S^*(\varphi, \rho)$ and a is a fixed point from the unit disk U, then the function h,*

$$h(z) = \frac{a \cdot z}{f(a)(z + a)(1 + \bar{a}z)^{\psi}} \cdot f\left(\frac{z + a}{1 + \bar{a}z}\right) \tag{6}$$

where

$$\psi = e^{-2i\varphi} - 2\rho \cos\varphi e^{-i\varphi} \tag{7}$$

is a function of the class $S^*(\varphi, \rho)$.

The results obtained are proved by using the following univalence criteria:

Theorem 6.[5]. *Let α, c be complex numbers, $|\alpha - 1| < 1$, $|c| < 1$, $g \in A$ and h an analytic function in U, $h(z) = 1 + c_1 z + \dots$. If the inequality*

$$\left| c|z|^2 + (1-|z|^2)\left[(\alpha - 1)\frac{zg'(z)}{g(z)} + \frac{zh'(z)}{h(z)} \right] \right| \leq 1 \tag{8}$$

is true for all $z \in U$, then the function

$$H(z) = \left(\alpha \int_0^z g^{\alpha-1}(u)h(u)du \right)^{1/\alpha} \tag{9}$$

is analytic and univalent in U, where the principal branch is intended.

In the next section we will consider the case when the functions f and g belongs to some subsets of S and we expect that the hypothesis (4) of the Theorem 4 becomes larger.

3. Main results

Theorem 7. *Let f, $g \in S^*(\varphi, \rho)$, α, β be complex numbers. If*

$$(1 + 2(1-\rho)\cos\varphi) \cdot |\alpha - 1| + 2(1-\rho)\cos\varphi \cdot |\beta| < 1 \tag{10}$$

then the function

$$H(z) = \left(\alpha \int_0^z g^{\alpha-1}(u) \left(\frac{f(u)}{u} \right)^{\beta} du \right)^{1/\alpha} \tag{11}$$

is analytic and univalent in U, where the principal branch is intended.

Proof. Let $f \in S^*(\varphi, \rho)$, $f(z) = z + b_2 z^2 + \dots$ and h be the function defined by Theorem 5, $h(z) = z + c_2 z^2 + \dots$, $h \in S^*(\varphi, \rho)$. From (6) we obtain

$$c_2 = \frac{h''(0)}{2} = (1-|a|^2)\frac{f'(a)}{f(a)} - \frac{1 + \psi|a|^2}{a} \ ,$$

where ψ is given by (7). It follows that

$$\frac{af'(a)}{f(a)} = \frac{1 + a \cdot c_2 + \psi|a|^2}{1-|a|^2} \tag{12}$$

It is known that if $g \in S^*(\varphi, \rho)$, $g(z) = z + a_2 z^2 + \dots$, we have

$$|\ a_2\ | \leq 2(1-\rho)\cos\varphi \tag{13}$$

Since the function f is univalent in U we can choose the uniform branch of $\left(\frac{f(u)}{u}\right)^{\beta}$ equal to 1 at the origin. Then the function h,

$$h(z) = \left(\frac{f(u)}{u}\right)^{\beta} \tag{14}$$

is analytic in U, $h(z) = 1 + c_1 z + \dots$
From (12) and (14)we deduce

$$c|z|^2 + (1-|z|^2)\left[\ (\alpha-1)\frac{zg'(z)}{g(z)} + \frac{zh'(z)}{h(z)}\ \right] \tag{15}$$

$$= c|z|^2 + (\alpha-1)(1 + c_2 z + \psi|z|^2) + \beta(b_2 z + \psi|z|^2 + |z|^2)$$

$$= [\ c + (\alpha-1)\psi + \beta(\psi+1)\ ]|z|^2 + (\alpha-1)(1+c_2 z) + \beta b_2 z$$

If we take $c = -[\ (\alpha-1)\psi + \beta(\psi+1)\ ]$, then

$$|\ c\ | \le |\alpha - 1\ | \cdot |\ \psi + 1\ | + |\beta| \cdot |\ \psi + 1\ | + |\alpha - 1\ |$$

and since $|\psi+1| = 2(1-\rho)\cos\varphi$, in view of (10), it is clear that $|c| < 1$. The relation (15) becomes:

$$\left|\ c|z|^2 + (1-|z|^2)\left[\ (\alpha-1)\frac{zg'(z)}{g(z)} + \frac{zh'(z)}{h(z)}\ \right]\ \right|$$

$$\le [\ 1 + |\ c_2\ |\ ] \cdot |\alpha - 1| + |\ b_2\ | \cdot |\beta|$$

Taking into account (13), in view of assertion (10), the conditions of Theorem 6 are satisfied. It follows that the function H defined by (11) is analytic and univalent in U.

Corollary 1. *Let $f,\ g \in S^*$ and $\alpha,\ \beta$ be complex numbers. If*

$$3|\alpha - 1| + 2|\beta| < 1 \tag{16}$$

then the function

$$H(z) = \left(\alpha \int_0^z g^{\alpha-1}(u)\left(\frac{f(u)}{u}\right)^{\beta} du\right)^{1/\alpha}$$

is analytic and univalent in U, where the principal branch is intended.

Theorem 8. *Let $g \in S^*(\varphi, \rho)$, $f \in C(\varphi, \rho)$, α, γ be complex numbers. If*

$$(1 + 2(1-\rho)\cos\varphi) \cdot |\alpha - 1| + 2(1-\rho)\cos\varphi \cdot |\gamma| < 1$$

then the function

$$H(z) = \left(\alpha \int_0^z g^{\alpha-1}(u)\left(f'(u)\right)^{\gamma} du\right)^{1/\alpha}$$

is analytic and univalent in U, where the principal branch is intended.

Proof. The proof of Theorem 8 is analogous to that of Theorem 7 and it uses the relationship between the classes $S^*(\varphi, \rho)$ and $C(\varphi, \rho)$: if $f \in C(\varphi, \rho)$ then $h \in S^*(\varphi, \rho)$, where $h(z) = zf'(z)$.

Corollary 2. *Let $g \in S^*$, $f \in CV$ and α, γ be complex numbers. If*

$$3|\alpha - 1| + 2|\gamma| < 1 \tag{17}$$

then the function

$$H(z) = \left(\alpha \int_0^z g^{\alpha-1}(u)\left(f'(u)\right)^{\gamma} du\right)^{1/\alpha}$$

is analytic and univalent in U, where the principal branch is intended.

References

[1] Y. J. Kim, E. P. Merkes, *On an integral of power of a spirallike functions* , Kyungpook Math. Journal, 12(2),(1972), 249-253.

[2] R. J. Libera, M. R. Ziegler,*Regular function f(z) for which zf'(z) is α-spiral* , Transactions of the American Math. Soc., 166,(1972), 361-368.

[3] S. Moldoveanu, N. N. Pascu, *Integral operators which preserves the univalence* , Mathematica, Cluj-Napoca, 32(55)(1990), 159-166.

[4] J. Pfaltzgraff, *Univalence of the integral* $(f'(z))^{\lambda}$, Bulletin of the London Math. Soc., 7(3)(1975), 254-256.

[5] H. Tudor, *An univalence condition* , Bulletin of the Transilvania University of Braşov, 3(52), (2010)(to appear).

[6] H. Tudor, *An extension of Kim and Merkes' and of Pfaltzgraff's integral operators* , General Mathematics, 2010 (to appear).

Horiana Tudor
Department of Mathematics
"Transilvania" University of Braşov
500091 Braşov, Romania
email: *horianatudor@yahoo.com*