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SUBORDINATION RESULTS FOR A NEW CLASS OF ANALYTIC FUNCTIONS DEFINED BY HURWITZ–LERCH ZETA FUNCTION

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Abstract. In this paper, we drive several interesting subordination results for a new class of analytic function defined by the integral operator $J_{s,b}$ defined in terms of the Hurwitz–Lerch Zeta function.

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1. Introduction

Let A denote the class of functions f of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disc $U = \{z : |z| < 1\}$. A function $f \in A$ is said to be in the class $S^*(\alpha)$ of starlike functions of order α, if satisfies the following inequality

$$\text{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (0 \leq \alpha < 1; z \in U). \tag{1.2}$$

Also denote by K the class of functions $f \in A$ which are convex in U. Given two functions f and g in the class A, where f is given by (1.1) and g is given by $g(z) = z + \sum\limits_{k=2}^{\infty} b_k z^k$. The Hadamard product (or convolution) $(f * g)(z)$ is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z) \quad (z \in U). \tag{1.3}$$

If f and g are analytic functions in U, we say that f is subordinate to g, written $f \prec g$ if there exists a Schwarz function w, which (by definition) is analytic in U

with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = g(w(z))$, $z \in U$. Furthermore, if the function g is univalent in U, then we have the following equivalence (cf., e.g., [3] and [14]):

$$f(z) \prec g(z) \;\; (z \in U) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

We begin our investigation by recalling that the general Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ defined by (see [4])

$$\Phi(z, s, b) = \sum_{k=0}^{\infty} \frac{z^k}{(k+b)^s} \;, \tag{1.4}$$

$$(b \in C \backslash Z_0^- = \{0, -1, -2, ...\}; \mathbb{Z}_o^- = \mathbb{Z} \backslash \mathbb{N}, (\mathbb{Z} = \left\{0, \pm 1, \pm 2, ...\right\};$$
$$\mathbb{N} = \{1, 2, 3, ...\}); s \in C \text{ when } |z| < 1; R\{s\} > 1 \text{ when } |z| = 1).$$

Some interesting properties and characteristics of the Hurwitz-Lerch Zeta function $\Phi(z, s, b)$ can be found in [5], [10], [11], [13] and [19].

Recently, Srivastava and Attiya [18] introduced the linear operator $J_{s,b} : A \to A$, defined in terms of the Hadamard product, by

$$J_{s,b}(f)(z) = G_{s,b}(z) * f(z) \;\; (z \in U; b \in C \backslash Z_0^-; s \in C), = z + \sum_{k=2}^{\infty} \left(\frac{1+b}{k+b} \right)^s a_k z^k, \tag{1.5}$$

where, for convenience,

$$G_{s,b}(z) = (1+b)^s [\Phi(z, s, b) - b^{-s}] \; (z \in U). \tag{1.6}$$

We note that:
(i) $J_{1,0}(f)(z) = J[f](z)$ (see Alexander [1]);
(ii) $J_{1,v}(f)(z) = J_v f(z) \;\; (v > -1; z \in U)$ (see [2], [9], [12]);
(iii) $J_{\gamma,\beta}(f)(z) = P_\beta^\gamma f(z) \; (\gamma \geq 0; \beta > 1; z \in U)$ (see Patel and Sahoo [15]);
(iv) $J_{\gamma,1}(f)(z) = I^\gamma f(z) \; (\gamma > 0; z \in U)$ (see Jung et al. [8]);
(v) $J_{n,0}(f)(z) = I^n f(z) \; (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$ (see Salagean [16]).

For some α $(0 \leq \alpha < 1), b$ $(b \in \mathbb{C} \backslash \mathbb{Z}_o^-), s \in \mathbb{C}$ and for all $z \in U$, let $S_{s,b}^*(\alpha)$ denote the subclass of A consisting of functions $f(z)$ of the form (1.1) and satisfying the condition:

$$\text{Re} \left(\frac{z(J_{s,b} f(z))'}{J_{s,b} f(z)} \right) > \alpha. \tag{1.7}$$

The class $S^{*}_{s,b}(\alpha)$ was intreduce and studied by Răducanu and Srivastava [7].

Definition 1 *(Subordinating Factor Sequence) [20]. A sequence $\{b_k\}_{k=1}^{\infty}$ of complex numbers is said to be a subordinating factor sequence if, whenever f of the form (1.1) is analytic, univalent and convex in U, we have the subordination given by*

$$\sum_{k=1}^{\infty} b_k a_k z^k \prec f(z) \quad (z \in U;\ a_1 = 1). \tag{1.8}$$

2. Main result

Unless otherwise mentioned, we shall assume in the reminder of this paper that, $0 \leq \alpha < 1,\ b \in \mathbb{C}\backslash\mathbb{Z}_o^-,\ s \in \mathbb{C}$ and $z \in U$.

To prove our main results we need the following lemmas.

Lemma 1 [20]. *The sequence $\{b_k\}_{k=1}^{\infty}$ is a subordinating factor sequence if and only if*

$$\operatorname{Re}\left\{1 + 2\sum_{k=1}^{\infty} b_k z^k\right\} > 0. \tag{2.1}$$

Lemma 2 [7]. If $f(z) \in A$ *satisfy the following inequality:*

$$\sum_{k=2}^{\infty} (k-\alpha)\left|\left(\frac{1+b}{k+b}\right)^s\right| |a_k| \leq 1-\alpha, \tag{2.2}$$

then $f(z) \in S^{*}_{s,b}(\alpha)$.

Let $S^{**}_{s,b}(\alpha)$ denote the class of functions $f(z) \in A$ whose coefficients satisfy the condition (2.2). We note that $S^{**}_{s,b}(\alpha) \subseteq S^{*}_{s,b}(\alpha)$.

Theorem 1. *Let $f \in S^{**}_{s,b}(\alpha)$. Then*

$$\frac{(2-\alpha)\left|1+b\right|^s}{2[\left|2+b\right|^s(1-\alpha) + (2-\alpha)\left|1+b\right|^s]}(f * g)(z) \prec g(z) \tag{2.3}$$

for every function $g \in K$, and

$$\operatorname{Re}\{f(z)\} > -\frac{[\left|2+b\right|^s(1-\alpha) + (2-\alpha)\left|1+b\right|^s]}{(2-\alpha)\left|1+b\right|^s}. \tag{2.4}$$

The constant $\dfrac{(2-\alpha)\left|1+b\right|^s}{2[\left|2+b\right|^s(1-\alpha) + (2-\alpha)\left|1+b\right|^s]}$ *is the best estimate.*

Proof. Let $f \in S^{**}_{s,b}(\alpha)$ and let $g(z) = z + \sum\limits_{k=2}^{\infty} c_k z^k \in K$. Then we have

$$\frac{(2-\alpha)|1+b|^s}{2[|2+b|^s(1-\alpha)+(2-\alpha)|1+b|^s]}(f*g)(z) = \frac{(2-\alpha)|1+b|^s}{2[|2+b|^s(1-\alpha)+(2-\alpha)|1+b|^s]}\left(z+\sum_{k=2}^{\infty} a_k c_k z^k\right). \tag{2.5}$$

Thus, by Definition 1, the subordination result (2.3) will hold true if the sequence

$$\left\{\frac{(2-\alpha)\left|1+b\right|^s}{2[\left|2+b\right|^s(1-\alpha)+(2-\alpha)\left|1+b\right|^s]}a_k\right\}_{k=1}^{\infty}, \tag{2.6}$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 1, this is equivalent to the following inequality:

$$\operatorname{Re}\left\{1+\sum_{k=1}^{\infty}\frac{(2-\alpha)\left|1+b\right|^s}{[\left|2+b\right|^s(1-\alpha)+(2-\alpha)\left|1+b\right|^s]}a_k z^k\right\} > 0. \tag{2.7}$$

Now, since

$$(k-\alpha)\left|\left(\frac{1+b}{k+b}\right)^s\right|,$$

is an increasing function of k $(k \geq 2)$, we have

$$\operatorname{Re}\left\{1+\sum_{k=1}^{\infty}\frac{(2-\alpha)\left|1+b\right|^s}{2[\left|2+b\right|^s(1-\alpha)+(2-\alpha)\left|1+b\right|^s]}a_k z^k\right\}$$

$$= \operatorname{Re}\left\{1+\frac{(2-\alpha)|1+b|^s}{[|2+b|^s(1-\alpha)+(2-\alpha)|1+b|^s]}z+\frac{1}{[|2+b|^s(1-\alpha)+(2-\alpha)|1+b|^s]}\sum_{k=2}^{\infty}(2-\alpha)\left|1+b\right|^s a_k z^k\right\}$$

$$\geq 1-\frac{(2-\alpha)|1+b|^s}{[|2+b|^s(1-\alpha)+(2-\alpha)|1+b|^s]}r-\frac{1}{|2+b|^s(1-\alpha)+(2-\alpha)|1+b|^s}\sum_{k=2}^{\infty}(k-\alpha)\left|1+b\right|^s\left|a_k\right|r^k$$

$$> 1-\frac{(2-\alpha)|1+b|^s}{[|2+b|^s(1-\alpha)+(2-\alpha)|1+b|^s]}r-\frac{(1-\alpha)|2+b|^s}{[|2+b|^s(1-\alpha)+(2-\alpha)|1+b|^s]}r = 1-r > 0 \ (|z| = r < 1),$$

where we have also made use of assertion (2.2) of Lemma 2. Thus (2.7) holds true in U. This proves the inequality (2.3). The inequality (2.4) follows from (2.3) by taking the convex function $g(z) = \frac{z}{1-z} = z + \sum\limits_{k=2}^{\infty} z^k \in K$.

To prove the sharpness of the constant $\dfrac{(2-\alpha)\left|1+b\right|^s}{2[\left|2+b\right|^s(1-\alpha)+(2-\alpha)\left|1+b\right|^s]}$, we consider the function $f_0(z) \in S^{**}_{s,b}(\alpha)$ given by

$$f_0(z) = z - \frac{(1-\alpha)\left|(2+b)^s\right|}{(2-\alpha)\left|(1+b)^s\right|}z^2. \tag{2.8}$$

Thus from (2.3), we have

$$\frac{(2-\alpha)\left|1+b\right|^s}{2[\left|2+b\right|^s(1-\alpha)+(2-\alpha)\left|1+b\right|^s]}f_0(z) \prec \frac{z}{1-z}. \tag{2.9}$$

Moreover, it can easily be verified for the function $f_0(z)$ given by (2.8) that

$$\min_{|z|\le r}\left\{\operatorname{Re}\frac{(2-\alpha)\left|1+b\right|^s}{2[\left|2+b\right|^s(1-\alpha)+(2-\alpha)\left|1+b\right|^s]}f_0(z)\right\} = -\frac{1}{2}. \tag{2.10}$$

.

This show that the constant $\dfrac{(2-\alpha)\left|1+b\right|^s}{2[\left|2+b\right|^s(1-\alpha)+(2-\alpha)\left|1+b\right|^s]}$ is the best possible. This completes the proof of Theorem 1.

Putting $s=1$ and $b=0$ in Theorem 1, we obtain the following corollary:

Corollary 1. *Let f defined by (1.1) be in the class $S^{**}_{1,0}(\alpha)$, $g \in K$, and satisfy the condition*

$$\sum_{k=2}^{\infty} k^{-1}(k-\alpha)\left|a_k\right| \le 1-\alpha. \tag{2.11}$$

Then

$$\frac{2-\alpha}{8-6\alpha}(f*g)(z) \prec g(z), \tag{2.12}$$

and

$$\operatorname{Re}\{f(z)\} > -\frac{4-3\alpha}{2-\alpha}. \tag{2.13}$$

The constant $\dfrac{2-\alpha}{8-6\alpha}$ is the best estimate.

Putting $s=1$ and $b=v$ $(v>-1)$ in Theorem 1, we obtain the following corollary:

Corollary 2. *Let f defined by (1.1) be in the class $S^{**}_{1,v}(\alpha)$, $g \in K$, and satisfy the condition*

$$\sum_{k=2}^{\infty}(k-\alpha)\left(\frac{1+v}{k+v}\right)\left|a_k\right| \le 1-\alpha,$$

then

$$\frac{(2-\alpha)(1+v)}{2[(2+v)(1-\alpha)+(2-\alpha)(1+v)]}(f*g)(z) \prec g(z) \tag{2.14}$$

and

$$\mathrm{Re}\{f(z)\} > -\frac{[(2+v)(1-\alpha)+(2-\alpha)(1+v)]}{(2-\alpha)(1+v)}. \tag{2.15}$$

The constant $\frac{(2-\alpha)(1+v)}{2[(2+v)(1-\alpha)+(2-\alpha)(1+v)]}$ *is the best estimate.*

Putting $s=\gamma$ and $b=\beta$ $(\gamma \geq 0, \beta > 1)$ in Theorem 1, we obtain the following corollary:

Corollary 3. *Let* f *defined by (1.1) be in the class* $S^{**}_{\gamma,\beta}(\alpha)$, $g \in K$, *and satisfy the condition*

$$\sum_{k=2}^{\infty}(k-\alpha)\left(\frac{1+\beta}{k+\beta}\right)^{\gamma}|a_k| \leq 1-\alpha, \tag{2.16}$$

then

$$\frac{(2-\alpha)(1+\beta)^{\gamma}}{2[(2+\beta)^{\gamma}(1-\alpha)+(2-\alpha)(1+\beta)^{\gamma}]}(f*g)(z) \prec g(z), \tag{2.17}$$

and

$$\mathrm{Re}\{f(z)\} > -\frac{(2+\beta)^{\gamma}(1-\alpha)+(2-\alpha)(1+\beta)^{\gamma}}{(2-\alpha)(1+\beta)^{\gamma}}. \tag{2.18}$$

The constant $\frac{(2-\alpha)(1+\beta)^{\gamma}}{2[(2+\beta)^{\gamma}(1-\alpha)+(2-\alpha)(1+\beta)^{\gamma}]}$ *is the best estimate.*

Putting $s=\gamma$ $(\gamma > 0)$ and $b=1$ in Theorem 1, we obtain the following corollary:

Corollary 4. *Let* f *defined by (1.1) be in the class* $S^{**}_{\gamma,1}(\alpha)$, $g \in K$, *and satisfy the condition*

$$\sum_{k=2}^{\infty}(k-\alpha)\left(\frac{2}{k+1}\right)^{\gamma}|a_k| \leq 1-\alpha, \tag{2.19}$$

then

$$\frac{(2-\alpha)2^{\gamma}}{2[3^{\gamma}(1-\alpha)+(2-\alpha)2^{\gamma}]}(f*g)(z) \prec g(z) \tag{2.20}$$

and

$$\operatorname{Re}\{f(z)\} > -\frac{3^{\gamma}(1-\alpha)+(2-\alpha)2^{\gamma}}{(2-\alpha)2^{\gamma}}. \tag{2.21}$$

The constant $\dfrac{(2-\alpha)2^{\gamma}}{2[3^{\gamma}(1-\alpha)+(2-\alpha)2^{\gamma}]}$ *is the best estimate.*

Putting $s = n$ $(n \in \mathbb{N}_0)$ and $b = 0$ in Theorem 1, we obtain the following corollary:
Corollary 5. *Let* f *defined by (1.1) be in the class* $S^{**}_{n,0}(\alpha)$, $g \in K$, *and satisfy the condition*

$$\sum_{k=2}^{\infty} k^{-n}(k-\alpha)\,|a_k| \le 1-\alpha, \tag{2.22}$$

then

$$\frac{(2-\alpha)}{2[2^{n}(1-\alpha)+(2-\alpha)]}\;(f*g)(z) \prec g(z) \tag{2.23}$$

and

$$\operatorname{Re}\{f(z)\} > -\frac{[2^{n}(1-\alpha)+(2-\alpha)]}{(2-\alpha)}. \tag{2.24}$$

The constant $\dfrac{(2-\alpha)}{2[2^{n}(1-\alpha)+(2-\alpha)]}$ *is the best estimate.*

Remarks.

(i) Putting $s = 0$ *in Theorem 1, we obtain the result obtained by Frasin [6, Corollary 2.3];*

(ii) Putting $s = \alpha = 0$ *in Theorem 1, we obtain the result obtained by Singh [17, Corollary 2.2].*

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