Acta Universitatis Apulensis
ISSN: 1582-5329

No. 23/2010
pp. 233-249

SOLUTION AND BEHAVIOR OF A RATIONAL DIFFERENCE EQUATIONS

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Abstract. We obtain in this paper the form of the solutions of the following difference equations

$$x_{n+1} = \frac{x_{n-7}}{\pm 1 \pm \alpha x_{n-1}x_{n-3}x_{n-5}x_{n-7}}, \quad n = 0, 1, ...,$$

where the initial conditions are arbitrary nonzero real numbers and α is constant. Also we study the behavior of the solutions.

2000 *Mathematics Subject Classification*: 39A10.

1. Introduction

In this paper we get the form of the solutions of the following difference equations

$$x_{n+1} = \frac{x_{n-7}}{\pm 1 \pm \alpha x_{n-1}x_{n-3}x_{n-5}x_{n-7}}, \quad n = 0, 1, ..., \tag{1}$$

where the initial conditions are arbitrary nonzero real numbers and α is constant. Also we study the behavior of the solutions.

Recently there has been a great interest in studying the qualitative properties of rational difference equations. For the systematical studies of rational and nonrational difference equations, one can refer to the papers [1–43] and references therein.

The study of difference equations has been growing continuously for the last decade. This is largely due to the fact that difference equations manifest themselves as mathematical models describing real life situations in probability theory, queuing theory, statistical problems, stochastic time series, combinatorial analysis, number theory, geometry, electrical network, quanta in radiation, genetics in biology, economics, psychology, sociology, etc. In fact, now it occupies a central position in applicable analysis and will no doubt continue to play an important role in mathematics as a whole.

The study of rational difference equations of order greater than one is quite challenging and rewarding because some prototypes for the development of the basic theory of the global behavior of nonlinear difference equations of order greater than one come from the results for rational difference equations. However, there have not been any effective general methods to deal with the global behavior of rational difference equations of order greater than one so far. Therefore, the study of rational difference equations of order greater than one is worth further consideration.

Recently there has been a lot of interest in studying the global attractivity, boundedness character, periodicity and the solution form of nonlinear difference equations. For some results in this area, for example: Aloqeili [5] has obtained the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-1}}{a - x_n x_{n-1}}.$$

Cinar [6]-[8] investigated the solutions of the following difference equations

$$x_{n+1} = \frac{x_{n-1}}{1 + a x_n x_{n-1}}, \quad x_{n+1} = \frac{x_{n-1}}{-1 + a x_n x_{n-1}}, \quad x_{n+1} = \frac{a x_{n-1}}{1 + b x_n x_{n-1}}.$$

Cinar et al. [9] studied the solutions and attractivity of the difference equation

$$x_{n+1} = \frac{x_{n-3}}{-1 + x_n x_{n-1} x_{n-2} x_{n-3}}.$$

Elabbasy et al. [11]-[13] investigated the global stability, periodicity character and gave the solution of special case of the following recursive sequences

$$x_{n+1} = a x_n - \frac{b x_n}{c x_n - d x_{n-1}}, \quad x_{n+1} = \frac{d x_{n-l} x_{n-k}}{c x_{n-s} - b} + a, \quad x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^{k} x_{n-i}}.$$

Elabbasy et al. [15] gave the solution of the following difference equations

$$x_{n+1} = \frac{x_{n-7}}{\pm 1 \pm x_{n-3} x_{n-7}}.$$

Ibrahim [20] obtained the solution of the third order rational difference equation

$$x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1}(a + b x_n x_{n-2})}$$

Karatas et al. [21-22] get the form of the solution of the difference equations

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2} x_{n-5}}, \quad x_{n+1} = \frac{a x_{n-(2k+2)}}{-a + \prod_{i=0}^{2k+2} x_{n-i}}.$$

Simsek et al. [29] obtained the solution of the difference equation

$$x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}}.$$

In [30] Stevic solved the following problem

$$x_{n+1} = \frac{x_{n-1}}{1 + x_n}.$$

In [36] Yalcinkaya get the solution of the difference equation

$$x_{n+1} = \frac{x_{n-(2k+1)}}{1 + x_{n-k}x_{n-(2k+1)}}.$$

Here, we recall some notations and results which will be useful in our investigation.

Let I be some interval of real numbers and let

$$f : I^{k+1} \to I,$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, ..., x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, ..., \tag{2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$ [25].

Definition 1. (Equilibrium Point)
A point $\overline{x} \in I$ is called an equilibrium point of Eq.(2) if

$$\overline{x} = f(\overline{x}, \overline{x}, ..., \overline{x}).$$

That is, $x_n = \overline{x}$ for $n \geq 0$, is a solution of Eq.(2), or equivalently, $\overline{x}$ is a fixed point of f.

Definition 2. (Stability)
(i) The equilibrium point $\overline{x}$ of Eq.(2) is locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \delta,$$

we have

$$|x_n - \overline{x}| < \epsilon \quad \text{for all} \quad n \geq -k.$$

(ii) The equilibrium point $\overline{x}$ of Eq.(2) is locally asymptotically stable if $\overline{x}$ is locally stable solution of Eq.(2) and there exists $\gamma > 0$, such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \gamma,$$

we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iii) The equilibrium point $\overline{x}$ of Eq.(2) is global attractor if for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$, we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iv) The equilibrium point $\overline{x}$ of Eq.(2) is globally asymptotically stable if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of Eq.(2).
(v) The equilibrium point $\overline{x}$ of Eq.(2) is unstable if $\overline{x}$ is not locally stable.

The linearized equation of Eq.(2) about the equilibrium $\overline{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f(\overline{x}, \overline{x}, ..., \overline{x})}{\partial x_{n-i}} y_{n-i}.$$

Definition 3. (Periodicity)

A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.

Theorem A [25]: Assume that $p_i \in R,\ i = 1, 2, ..., k$ and $k \in \{0, 1, 2, ...\}$. Then

$$\sum_{i=1}^{k} |p_i| < 1,$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+k} + p_1 x_{n+k-1} + ... + p_k x_n = 0, \quad n = 0, 1, ... \ . \tag{3}$$

2. The Main Results

2.1. The First Difference Equation $x_{n+1} = \dfrac{x_{n-7}}{1 + \alpha x_{n-1} x_{n-3} x_{n-5} x_{n-7}}$

In this section we give a specific form of the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-7}}{1 + \alpha x_{n-1} x_{n-3} x_{n-5} x_{n-7}}, \quad n = 0, 1, ..., \tag{4}$$

where the initial conditions are arbitrary nonzero positive real numbers.

Theorem 1. Let $\{x_n\}_{n=-7}^{\infty}$ be a solution of Eq.(4). Then for $n = 0, 1, ...$

$$x_{8n-7} = h\prod_{i=0}^{n-1}\left(\frac{(1+4i\alpha bdfh)}{(1+(4i+1)\alpha bdfh)}\right), \quad x_{8n-3} = d\prod_{i=0}^{n-1}\left(\frac{(1+(4i+2)\alpha bdfh)}{(1+(4i+3)\alpha bdfh)}\right),$$

$$x_{8n-6} = g\prod_{i=0}^{n-1}\left(\frac{(1+4i\alpha aceg)}{(1+(4i+1)\alpha aceg)}\right), \quad x_{8n-2} = c\prod_{i=0}^{n-1}\left(\frac{(1+(4i+2)\alpha aceg)}{(1+(4i+3)\alpha aceg)}\right),$$

$$x_{8n-5} = f\prod_{i=0}^{n-1}\left(\frac{(1+(4i+1)\alpha bdfh)}{(1+(4i+2)\alpha bdfh)}\right), \quad x_{8n-1} = b\prod_{i=0}^{n-1}\left(\frac{(1+(4i+3)\alpha bdfh)}{(1+(4i+4)\alpha bdfh)}\right),$$

$$x_{8n-4} = e\prod_{i=0}^{n-1}\left(\frac{(1+(4i+1)\alpha aceg)}{(1+(4i+2)\alpha aceg)}\right), \quad x_{8n} = a\prod_{i=0}^{n-1}\left(\frac{(1+(4i+3)\alpha aceg)}{(1+(4i+4)\alpha aceg)}\right),$$

where $x_{-7} = h,\ x_{-6} = g,\ x_{-5} = f,\ x_{-4} = e,\ x_{-3} = d,\ x_{-2} = c,\ x_{-1} = b,\ x_{-0} = a,\ \prod_{i=0}^{-1} A_i = 1.$

Proof. For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n-1$. That is;

$$x_{8n-15} = h\prod_{i=0}^{n-2}\left(\frac{(1+4i\alpha bdfh)}{(1+(4i+1)\alpha bdfh)}\right), \quad x_{8n-11} = d\prod_{i=0}^{n-2}\left(\frac{(1+(4i+2)\alpha bdfh)}{(1+(4i+3)\alpha bdfh)}\right),$$

$$x_{8n-14} = g\prod_{i=0}^{n-2}\left(\frac{(1+4i\alpha aceg)}{(1+(4i+1)\alpha aceg)}\right), \quad x_{8n-10} = c\prod_{i=0}^{n-2}\left(\frac{(1+(4i+2)\alpha aceg)}{(1+(4i+3)\alpha aceg)}\right),$$

$$x_{8n-13} = f\prod_{i=0}^{n-2}\left(\frac{(1+(4i+1)\alpha bdfh)}{(1+(4i+2)\alpha bdfh)}\right), \quad x_{8n-9} = b\prod_{i=0}^{n-2}\left(\frac{(1+(4i+3)\alpha bdfh)}{(1+(4i+4)\alpha bdfh)}\right),$$

$$x_{8n-12} = e\prod_{i=0}^{n-2}\left(\frac{(1+(4i+1)\alpha aceg)}{(1+(4i+2)\alpha aceg)}\right), \quad x_{8n-8} = a\prod_{i=0}^{n-2}\left(\frac{(1+(4i+3)\alpha aceg)}{(1+(4i+4)\alpha aceg)}\right).$$

Now, it follows from Eq.(4) that

$$\begin{aligned} x_{8n-7} &= \frac{x_{8n-15}}{1+\alpha x_{8n-9}x_{8n-11}x_{8n-13}x_{8n-15}} \\ &= \frac{h\prod_{i=0}^{n-2}\left(\frac{(1+4i\alpha bdfh)}{(1+(4i+1)\alpha bdfh)}\right)}{\begin{array}{c}1+\alpha b\prod_{i=0}^{n-2}\left(\frac{(1+(4i+3)\alpha bdfh)}{(1+(4i+4)\alpha bdfh)}\right)d\prod_{i=0}^{n-2}\left(\frac{(1+(4i+2)\alpha bdfh)}{(1+(4i+3)\alpha bdfh)}\right)\\ f\prod_{i=0}^{n-2}\left(\frac{(1+(4i+1)\alpha bdfh)}{(1+(4i+2)\alpha bdfh)}\right)h\prod_{i=0}^{n-2}\left(\frac{(1+4i\alpha bdfh)}{(1+(4i+1)\alpha bdfh)}\right)\end{array}} \end{aligned}$$

$$= \frac{h\prod_{i=0}^{n-2}(1+4ibdfh)}{\prod_{i=0}^{n-2}(1+(4i+1)\alpha bdfh)}\left(\frac{1}{1+\frac{\alpha bdfh}{\prod_{i=0}^{n-2}(1+(4i+4)\alpha bdfh)}\prod_{i=0}^{n-2}(1+4i\alpha bdfh)}\right)$$

$$= h\prod_{i=0}^{n-2}\left(\frac{(1+4ibdfh)}{(1+(4i+1)\alpha bdfh)}\right)\left(\frac{1}{1+\frac{\alpha bdfh}{(1+(4n-4)\alpha bdfh)}}\left\{\frac{(1+(4n-4)\alpha bdfh)}{(1+(4n-4)\alpha bdfh)}\right\}\right)$$

$$\begin{aligned} &= h\prod_{i=0}^{n-2}\left(\frac{(1+4ibdfh)}{(1+(4i+1)\alpha bdfh)}\right)\left(\frac{1+(4n-4)\alpha bdfh}{1+(4n-4)\alpha bdfh+\alpha bdfh}\right) \\ &= h\prod_{i=0}^{n-2}\left(\frac{(1+4ibdfh)}{(1+(4i+1)\alpha bdfh)}\right)\left(\frac{1+(4n-4)\alpha bdfh}{1+(4n-3)\alpha bdfh}\right). \end{aligned}$$

Hence, we have

$$x_{8n-7} = h\prod_{i=0}^{n-1}\left(\frac{(1+4ibdfh)}{(1+(4i+1)\alpha bdfh)}\right).$$

Similarly, one can easily obtain the other relations. Thus, the proof is completed.

Theorem 2. Eq.(4) has a unique equilibrium point which is the number zero and this equilibrium point is not locally asymptotically stable.

Proof. For the equilibrium points of Eq.(4), we can write

$$\overline{x} = \frac{\overline{x}}{1+\alpha\overline{x}^4}.$$

Then

$$\overline{x} + \alpha\overline{x}^5 = \overline{x},$$

or

$$\alpha\overline{x}^5 = 0.$$

Thus the equilibrium point of Eq.(4) is $\overline{x} = 0$. Let $f : (0,\infty)^4 \longrightarrow (0,\infty)$ be a function defined by

$$f(u,v,w,t) = \frac{u}{1+\alpha uvwt}.$$

Therefore it follows that

$$\begin{aligned} f_u(u,v,w,t) &= \frac{1}{(1+\alpha uvwt)^2}, \quad f_v(u,v,w,t)=\frac{-\alpha u^2 wt}{(1+\alpha uvwt)^2}, \\ f_w(u,v,w,t) &= \frac{-\alpha u^2 vt}{(1+\alpha uvwt)^2}, \quad f_t(u,v,w,t)=\frac{-\alpha u^2 vw}{(1+\alpha uvwt)^2}, \end{aligned}$$

we see that

$$f_u(\overline{x},\overline{x},\overline{x},\overline{x})=1, \quad f_v(\overline{x},\overline{x},\overline{x},\overline{x}x)=0, \quad f_w(\overline{x},\overline{x},\overline{x},\overline{x})=0, \quad f_t(\overline{x},\overline{x},\overline{x},\overline{x})=0.$$

The proof follows by using Theorem A.

Theorem 3. Every positive solution of Eq.(4) is bounded and $\lim_{n\to\infty} x_n = 0$.

Proof. It follows from Eq.(4) that

$$x_{n+1} = \frac{x_{n-7}}{1+\alpha x_{n-1}x_{n-3}x_{n-5}x_{n-7}} \leq x_{n-7}.$$

Then the subsequences $\{x_{8n-7}\}_{n=0}^{\infty}$, $\{x_{8n-6}\}_{n=0}^{\infty}$, $\{x_{8n-5}\}_{n=0}^{\infty}$, $\{x_{8n-4}\}_{n=0}^{\infty}$, $\{x_{8n-3}\}_{n=0}^{\infty}$, $\{x_{8n-2}\}_{n=0}^{\infty}$, $\{x_{8n-1}\}_{n=0}^{\infty}$, $\{x_{8n}\}_{n=0}^{\infty}$ are decreasing and so are bounded from above by $M = \max\{x_{-7}, x_{-6}, x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0\}$.

Numerical examples

For confirming the results of this section, we consider numerical examples which represent different types of solutions to Eq. (4).

Example 1. Consider $\alpha = 2,\ x_{-7} = 2,\ x_{-6} = 4,\ x_{-5} = 11,\ x_{-4} = 2,\ x_{-3} = 6,\ x_{-2} = 5,\ x_{-1} = 9,\ x_0 = 1$. See Fig. 1.

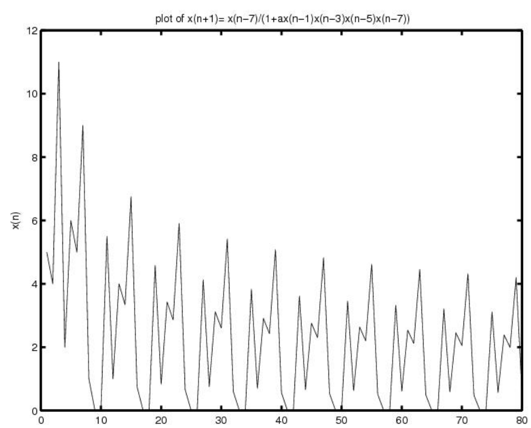

Figure 1.

Example 2. See Fig. 2, since $\alpha = 2,\ x_{-7} = 5,\ x_{-6} = 2,\ x_{-5} = 11,\ x_{-4} = 0.7,\ x_{-3} = 6,\ x_{-2} = 4,\ x_{-1} = 0.3,\ x_0 = 7$.

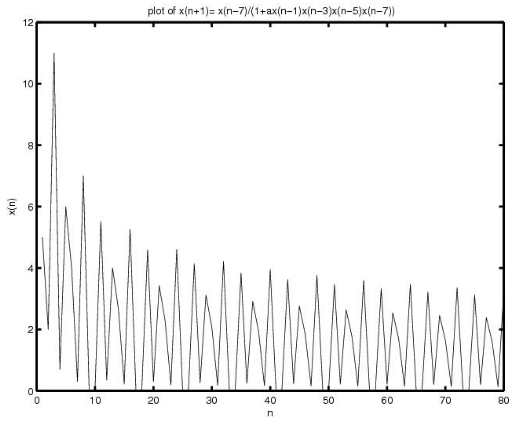

Figure 2.

2.2. The Second Difference Equation $x_{n+1} = \dfrac{x_{n-7}}{1-\alpha x_{n-1}x_{n-3}x_{n-5}x_{n-7}}$

In this section we give a specific form of the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-7}}{1-\alpha x_{n-1}x_{n-3}x_{n-5}x_{n-7}}, \quad n = 0, 1, ..., \tag{5}$$

where the initial conditions are arbitrary nonzero real numbers.

Theorem 4. Let $\{x_n\}_{n=-7}^{\infty}$ be a solution of Eq.(5). Then for $n = 0, 1, ...$

$$\begin{aligned}
x_{8n-7} &= h\prod_{i=0}^{n-1}\left(\frac{(1-4i\alpha bdfh)}{(1-(4i+1)\alpha bdfh)}\right), \qquad x_{8n-3} = d\prod_{i=0}^{n-1}\left(\frac{(1-(4i+2)\alpha bdfh)}{(1-(4i+3)\alpha bdfh)}\right),\\
x_{8n-6} &= g\prod_{i=0}^{n-1}\left(\frac{(1-4i\alpha aceg)}{(1-(4i+1)\alpha aceg)}\right), \qquad x_{8n-2} = c\prod_{i=0}^{n-1}\left(\frac{(1-(4i+2)\alpha aceg)}{(1-(4i+3)\alpha aceg)}\right),\\
x_{8n-5} &= f\prod_{i=0}^{n-1}\left(\frac{(1-(4i+1)\alpha bdfh)}{(1-(4i+2)\alpha bdfh)}\right), \qquad x_{8n-1} = b\prod_{i=0}^{n-1}\left(\frac{(1-(4i+3)\alpha bdfh)}{(1-(4i+4)\alpha bdfh)}\right),\\
x_{8n-4} &= e\prod_{i=0}^{n-1}\left(\frac{(1-(4i+1)\alpha aceg)}{(1-(4i+2)\alpha aceg)}\right), \qquad x_{8n} = a\prod_{i=0}^{n-1}\left(\frac{(1-(4i+3)\alpha aceg)}{(1-(4i+4)\alpha aceg)}\right),
\end{aligned}$$

where $j\alpha bdfh \neq 1, j\alpha aceg \neq 1$ for $j = 1, 2, 3, ...$.

Proof. As the proof of Theorem 1 and will be omitted.

Theorem 5. Eq.(5) has a unique equilibrium point which is the number zero and this equilibrium point is not locally asymptotically stable.

Numerical examples:-

Example 3. Consider $\alpha = 4,\ x_{-7} = 0.5,\ x_{-6} = 2, x_{-5} = 11, x_{-4} = 7, x_{-3} = 16, x_{-2} = 8, x_{-1} = 3, x_0 = 0.7$. See Fig. 3.

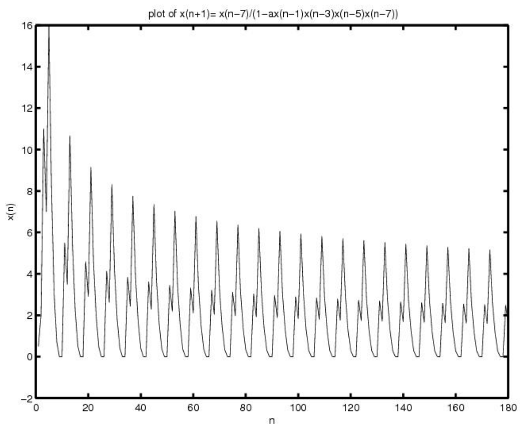

Figure 3.

Example 4. See Fig. 4, since $\alpha = 1,\ x_{-7} = 0.5,\ x_{-6} = 0.1,\ x_{-5} = 0.8,\ x_{-4} = 0.7,\ x_{-3} = 0.4,\ x_{-2} = 0.9,\ x_{-1} = 0.2,\ x_0 = 1.3$.

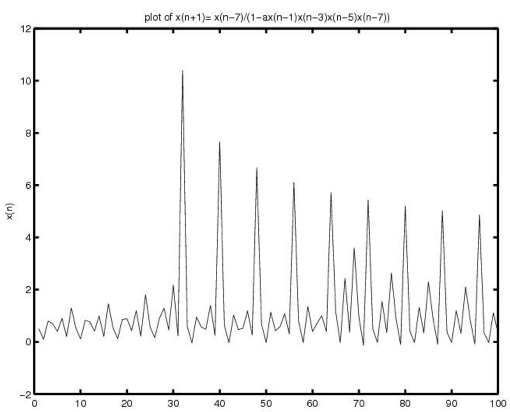

Figure 4.

2.3. The Third Difference Equation $x_{n+1} = \dfrac{x_{n-7}}{-1 + \alpha x_{n-1}x_{n-3}x_{n-5}x_{n-7}}$

In this section we investigate the solutions of the following difference equation

$$x_{n+1} = \frac{x_{n-7}}{-1 + \alpha x_{n-1}x_{n-3}x_{n-5}x_{n-7}}, \quad n = 0, 1, ..., \tag{6}$$

where the initial conditions are arbitrary nonzero real numbers with $\alpha x_{-7}x_{-5}x_{-3}x_{-1} \neq 1$, $\alpha x_{-6}x_{-4}x_{-2}x_0 \neq 1$.

Theorem 6. Let $\{x_n\}_{n=-7}^{\infty}$ be a solution of Eq.(6). Then Eq.(6) has unbounded solutions and for $n = 0, 1, ...$

$$\begin{aligned} x_{8n-7} &= \frac{h}{(-1+\alpha bdfh)^n}, \quad x_{8n-3} = \frac{d}{(-1+\alpha bdfh)^n}, \\ x_{8n-6} &= \frac{g}{(-1+\alpha aceg)^n}, \quad x_{8n-2} = \frac{c}{(-1+\alpha aceg)^n}, \end{aligned}$$

$$\begin{aligned} x_{8n-5} &= f\left(-1+\alpha bdfh\right)^n, \quad x_{8n-1} = b\left(-1+\alpha bdfh\right)^n, \\ x_{8n-4} &= e\left(-1+\alpha aceg\right)^n, \quad x_{8n} = a\left(-1+\alpha aceg\right)^n. \end{aligned}$$

Proof. For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n-1$. That is;

$$\begin{aligned} x_{8n-15} &= \frac{h}{(-1+\alpha bdfh)^{n-1}}, \quad x_{8n-11} = \frac{d}{(-1+\alpha bdfh)^{n-1}}, \\ x_{8n-14} &= \frac{g}{(-1+\alpha aceg)^{n-1}}, \quad x_{8n-10} = \frac{c}{(-1+\alpha aceg)^{n-1}}, \end{aligned}$$

$$\begin{aligned} x_{8n-13} &= f\left(-1+\alpha bdfh\right)^{n-1}, \quad x_{8n-9} = b\left(-1+\alpha bdfh\right)^{n-1}, \\ x_{8n-12} &= e\left(-1+\alpha aceg\right)^{n-1}, \quad x_{8n-8} = a\left(-1+\alpha aceg\right)^{n-1}. \end{aligned}$$

Now, it follows from Eq.(6) that

$$\begin{aligned} x_{8n-7} &= \frac{x_{8n-15}}{-1+\alpha x_{8n-9}x_{8n-11}x_{8n-13}x_{8n-15}} \\ &= \frac{\dfrac{h}{(-1+\alpha bdfh)^{n-1}}}{-1+\alpha b\left(-1+\alpha bdfh\right)^{n-1}\dfrac{d}{(-1+\alpha bdfh)^{n-1}} f\left(-1+\alpha bdfh\right)^{n-1}\dfrac{h}{(-1+\alpha bdfh)^{n-1}}} \\ &= \frac{\dfrac{h}{(-1+\alpha bdfh)^{n-1}}}{-1+\alpha bdfh}. \end{aligned}$$

Hence, we have

$$x_{8n-7} = \frac{h}{(-1+\alpha bdfh)^{n-1}}.$$

Similarly

$$\begin{aligned}
x_{8n-2} &= \frac{x_{8n-10}}{-1+\alpha x_{8n-4}x_{8n-6}x_{8n-8}x_{8n-10}} \\
&= \frac{\frac{c}{(-1+\alpha aceg)^{n-1}}}{-1+\alpha e\left(-1+\alpha aceg\right)^{n}\frac{g}{(-1+\alpha aceg)^{n}}a\left(-1+\alpha aceg\right)^{n-1}\frac{c}{(-1+\alpha aceg)^{n-1}}} \\
&= \frac{\frac{c}{(-1+\alpha aceg)^{n-1}}}{-1+\alpha aceg}.
\end{aligned}$$

Hence, we have

$$x_{8n-2} = \frac{c}{(-1+\alpha aceg)^{n}}.$$

Similarly, one can easily obtain the other relations. Thus, the proof is completed.

Theorem 7. Eq.(6) has three equilibrium points which are $0, \pm\sqrt[4]{\frac{2}{\alpha}}$ and these equilibrium points are not locally asymptotically stable.

Proof. The proof as in Theorem 2.

Theorem 8. Eq.(6) has a periodic solutions of period eight iff $\alpha aceg = \alpha bdfh = 2$ and will be take the form $\{h, g, f, e, d, c, b, a, h, g, f, e, d, c, b, a, ...\}$.

Proof. First suppose that there exists a prime period eight solution

$$h, g, f, e, d, c, b, a, h, g, f, e, d, c, b, a, ...,$$

of Eq.(6), we see from Eq.(6) that

$$\begin{aligned}
h &= \frac{h}{(-1+\alpha bdfh)^{n}}, \quad d = \frac{d}{(-1+\alpha bdfh)^{n}}, \\
g &= \frac{g}{(-1+\alpha aceg)^{n}}, \quad c = \frac{c}{(-1+\alpha aceg)^{n}},
\end{aligned}$$

$$\begin{aligned}
f &= f\left(-1+\alpha bdfh\right)^{n}, \quad b = b\left(-1+\alpha bdfh\right)^{n}, \\
e &= e\left(-1+\alpha aceg\right)^{n}, \quad a = a\left(-1+\alpha aceg\right)^{n}.
\end{aligned}$$

or

$$(-1+\alpha bdfh)^{n} = 1, \quad (-1+\alpha aceg)^{n} = 1.$$

Then

$$\alpha bdfh = 2, \quad \alpha aceg = 2.$$

Second suppose $\alpha aceg = 2,\ \alpha bdfh = 2$. Then we see from Eq.(6) that

$$\begin{aligned} x_{8n-7} &= h, \quad x_{8n-6} = g, \quad x_{8n-5} = f, \quad x_{8n-4} = e, \\ x_{8n-3} &= d, \quad x_{8n-2} = c, \quad x_{8n-1} = b, \quad x_{8n} = a. \end{aligned}$$

Thus we have a periodic solution with period eight solution and the proof is complete.

Numerical examples:-

Example 5. We consider $\alpha = 2,\ x_{-7} = 0.5,\ x_{-6} = 0.1,\ x_{-5} = 0.7,\ x_{-4} = 0.8,\ x_{-3} = 0.4,\ x_{-2} = 0.6,\ x_{-1} = 0.2,\ x_0 = 1.3$. See Fig. 5.

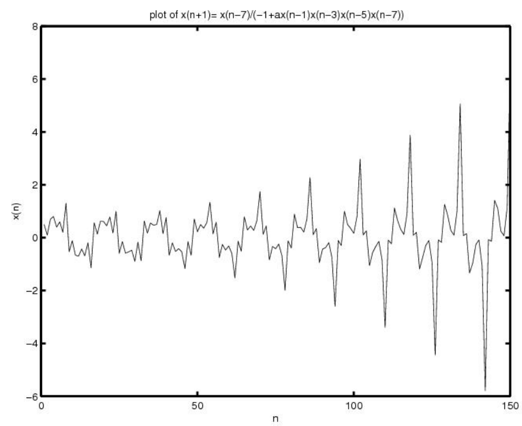

Figure 5.

Example 6. See Fig. 6, since $\alpha = 2,\ x_{-7} = 5,\ x_{-6} = 0.1,\ x_{-5} = 7,\ x_{-4} = 10,\ x_{-3} = 1/35,\ x_{-2} = 6,\ x_{-1} = 1,\ x_0 = 1/6$.

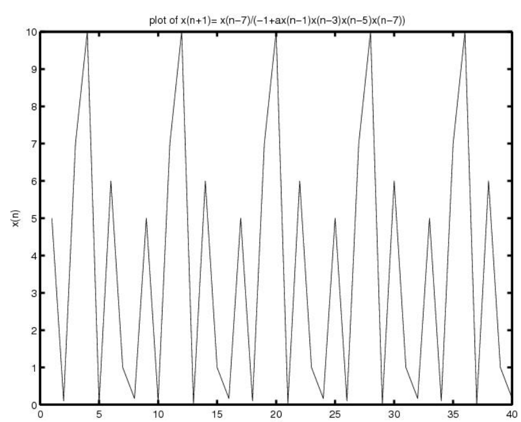

Figure 6.

2.4. The Fourth Difference Equation $x_{n+1}=\dfrac{x_{n-7}}{-1-\alpha x_{n-1}x_{n-3}x_{n-5}x_{n-7}}$

In this section we investigate the solutions of the following difference equation

$$x_{n+1}=\frac{x_{n-7}}{-1-\alpha x_{n-1}x_{n-3}x_{n-5}x_{n-7}},\quad n=0,1,..., \tag{7}$$

where the initial conditions are arbitrary nonzero real numbers with $\alpha x_{-5}x_{-3}x_{-1}\neq -1,\ \alpha x_{-4}x_{-2}x_{0}\neq -1.$

Theorem 9. Let $\{x_n\}_{n=-7}^{\infty}$ be a solution of Eq.(7). Then Eq.(7) has unbounded solutions and for $n=0,1,...$

$$\begin{aligned} x_{8n-7} &= \frac{(-1)^n h}{(1+\alpha bdfh)^n}, \quad x_{8n-3}=\frac{(-1)^n d}{(1+\alpha bdfh)^n},\\ x_{8n-6} &= \frac{(-1)^n g}{(1+\alpha aceg)^n}, \quad x_{8n-2}=\frac{(-1)^n c}{(1+\alpha aceg)^n},\end{aligned}$$

$$\begin{aligned} x_{8n-5} &= f(-1)^n(1+\alpha bdfh)^n, \quad x_{8n-1}=b(-1)^n(1+\alpha bdfh)^n,\\ x_{8n-4} &= e(-1)^n(1+\alpha aceg)^n, \quad x_{8n}=a(-1)^n(1+\alpha aceg)^n.\end{aligned}$$

Theorem 10. Eq.(7) has one equilibrium point which is number zero and this equilibrium point is not locally asymptotically stable.

Theorem 11. Eq.(7) has a periodic solutions of period eight iff $\alpha aceg=\alpha bdfh=-2$ and will be take the form $\{h,g,f,e,d,c,b,a,h,g,f,e,d,c,b,a,...\}$.

Numerical examples:-

Example 7. Fig. 7 shows the solution when $\alpha=0.1,\ x_{-7}=0.5,\ x_{-6}=1,\ x_{-5}=-0.4,\ x_{-4}=0.3,\ x_{-3}=1.3,\ x_{-2}=0.6,\ x_{-1}=-1.9,\ x_0=0.8.$

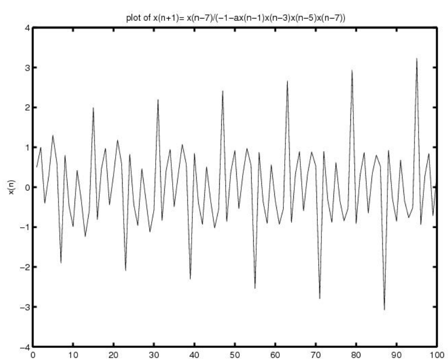

Figure 7.

Example 8. See Fig. 8, since $\alpha=-2,\ x_{-7}=5,\ x_{-6}=10,\ x_{-5}=1/40,\ x_{-4}=0.1,\ x_{-3}=8,\ x_{-2}=1/6,\ x_{-1}=1,\ x_0=6.$

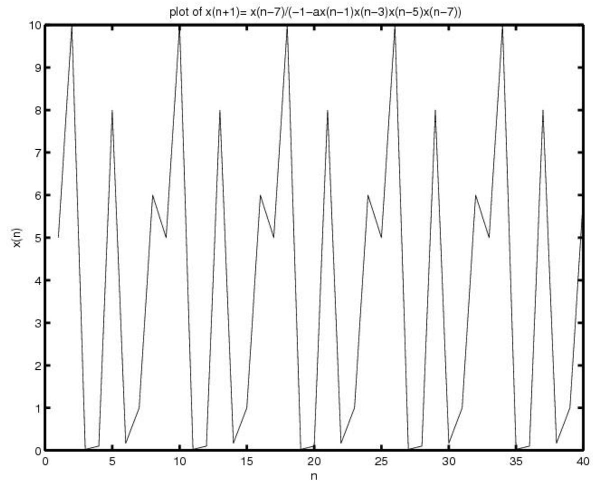

Figure 8.

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