

**DIFFERENTIAL SUBORDINATION AND SUPERORDINATION OF
ANALYTIC FUNCTIONS DEFINED BY CERTAIN INTEGRAL
OPERATOR**

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ABSTRACT. Differential subordination and superordination results are obtained for analytic functions in the open unit disk which are associated with an integral operator. These results are obtained by investigating appropriate classes of admissible functions. Sandwich-type results are also obtained. Some of the results established in this paper would provide extensions of those given in earlier works.

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1. INTRODUCTION

Let $H(U)$ be the class of functions analytic in $U = \{z \in \mathbb{C} : |z| < 1\}$ and $H[a, n]$ be the subclass of $H(U)$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots$, with $H_0 = H[0, 1]$ and $H = H[1, 1]$. Let $A(p)$ denote the class of all analytic functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \in \mathbb{N} = \{1, 2, 3, \dots\}; z \in U) \quad (1.1)$$

and let $A(1) = A$. Let f and F be members of $H(U)$. The function $f(z)$ is said to be subordinate to $F(z)$, or $F(z)$ is said to be superordinate to $f(z)$, if there exists a function $\omega(z)$ analytic in U with $\omega(0) = 0$ and $|\omega(z)| < 1 (z \in U)$, such that $f(z) = F(\omega(z))$. In such a case we write $f(z) \prec F(z)$. If F is univalent, then $f(z) \prec F(z)$ if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$ (see [8] and [9]).

For two functions $f(z)$ given by (1.1) and

$$g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n,$$

the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n = (g * f)(z).$$

Motivated essentially by Jung et al. [6], Shams et al. [10] introduced the integral operator $I_p^\alpha : A(p) \rightarrow A(p)$ as follows (see also Aouf et al. [4]):

$$I_p^\alpha f(z) = \frac{(p+1)^\alpha}{z\Gamma(\alpha)} \int_0^z \left(\log \frac{z}{t}\right)^{\alpha-1} f(t) dt, \quad (\alpha > 0; p \in \mathbb{N}), \quad (1.2)$$

and

$$I_p^0 f(z) = f(z), \quad (\alpha = 0; p \in \mathbb{N}).$$

For $f \in A(p)$ given by (1.1), then from (1.2), we deduce that

$$I_p^\alpha f(z) = z^p + \sum_{n=p+1}^{\infty} \left(\frac{p+1}{n+1}\right)^\alpha a_n z^n, \quad (\alpha \geq 0; p \in \mathbb{N}).$$

Using the above relation, it is easy to verify the identity:

$$z (I_p^\alpha f(z))' = (p+1) I_p^{\alpha-1} f(z) - I_p^\alpha f(z). \quad (1.3)$$

We note that the one-parameter family of integral operator $I_1^\alpha = I^\alpha$ was defined by Jung et al. [6].

To prove our results, we need the following definitions and lemmas.

Denote by Q the set of all functions $q(z)$ that are analytic and injective on $\bar{U} \setminus E(q)$ where

$$E(q) = \left\{ \zeta \in \partial U : \lim_{z \rightarrow \zeta} q(z) = \infty \right\},$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(q)$. Further let the subclass of Q for which $q(0) = a$ be denoted by $Q(a)$, $Q(0) \equiv Q_0$ and $Q(1) \equiv Q_1$.

Definition 1 [8, Definition 2.3a, p. 27]. Let Ω be a set in \mathbb{C} , $q \in Q$ and n be a positive integer. The class of admissible functions $\Psi_n[\Omega, q]$, consists of those functions $\psi : \mathbb{C}^3 \times U \rightarrow \mathbb{C}$ that satisfy the admissibility condition:

$$\psi(r, s, t; z) \notin \Omega$$

whenever $r = q(\zeta)$, $s = k\zeta q'(\zeta)$,

$$\Re \left\{ \frac{t}{s} + 1 \right\} \geq k \Re \left\{ 1 + \frac{\zeta q''(\zeta)}{q'(\zeta)} \right\},$$

where $z \in U$, $\zeta \in \partial U \setminus E(q)$ and $k \geq n$. We write $\Psi_1[\Omega, q]$ as $\Psi[\Omega, q]$.

In particular when $q(z) = M \frac{Mz+a}{M+\bar{a}z}$, with $M > 0$ and $|a| < M$, then $q(U) = U_M = \{w : |w| < M\}$, $q(0) = a$, $E(q) = \emptyset$ and $q \in Q$. In this case, we set $\Psi_n[\Omega, M, a] = \Psi_n[\Omega, q]$, and in the special case when the set $\Omega = U_M$, the class is simply denoted by $\Psi_n[M, a]$.

Definition 2 [9, Definition 3, p. 817]. Let Ω be a set in \mathbb{C} , $q(z) \in H[a, n]$ with $q'(z) \neq 0$. The class of admissible functions $\Psi'_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^3 \times \bar{U} \rightarrow \mathbb{C}$ that satisfy the admissibility condition

$$\psi(r, s, t; \zeta) \in \Omega$$

whenever $r = q(z)$, $s = \frac{zq'(z)}{m}$,

$$\Re \left\{ \frac{t}{s} + 1 \right\} \geq \frac{1}{m} \Re \left\{ 1 + \frac{zq''(z)}{q'(z)} \right\},$$

where $z \in U, \zeta \in \partial U$ and $m \geq n \geq 1$. In particular, we write $\Psi'_1[\Omega, q]$ as $\Psi'[\Omega, q]$.

Lemma 1 [8, Theorem 2.3b, p. 28]. Let $\psi \in \Psi_n[\Omega, q]$ with $q(0) = a$. If the analytic function $g(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots$ satisfies

$$\psi(g(z), zg'(z), z^2 g''(z); z) \in \Omega,$$

then $g(z) \prec q(z)$.

Lemma 2 [9, Theorem 1, p. 818]. Let $\psi \in \Psi'_n[\Omega, q]$ with $q(0) = a$. If $g(z) \in Q(a)$ and

$$\psi(g(z), zg'(z), z^2 g''(z); z)$$

is univalent in U , then

$$\Omega \subset \{\psi(g(z), zg'(z), z^2 g''(z); z) : z \in U\},$$

implies $q(z) \prec g(z)$.

In the present investigation, the differential subordination result of Miller and Mocanu [8, Theorem 2.3b, p.28] is extended for functions associated with the integral operator I_p^α , and we obtain certain other related results. A similar problem for analytic functions was studied by Aghalary et al. [1], Ali et al. [2], Aouf [3], Aouf et al. [5], and Kim and Srivastava [7]. Additionally, the corresponding differential superordination problem is investigated, and several sandwich-type results are obtained.

2. SUBORDINATION RESULTS INVOLVING THE INTEGRAL OPERATOR

Definition 3. Let Ω be a set in \mathbb{C} and $q(z) \in Q_0 \cap H[0, p]$. The class of admissible functions $\Phi_I[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times U \rightarrow \mathbb{C}$ that satisfy the admissibility condition

$$\phi(u, v, w; z) \notin \Omega$$

whenever $u = q(\zeta), v = \frac{k\zeta q'(\zeta) + q(\zeta)}{p+1},$

$$\Re \left\{ \frac{(p+1)^2 w - 2(p+1)v + u}{(p+1)v - u} \right\} \geq k \Re \left\{ 1 + \frac{\zeta q''(\zeta)}{q'(\zeta)} \right\}$$

where $z \in U, \zeta \in \partial U \setminus E(q)$ and $k \geq p$.

Theorem 1. Let $\phi \in \Phi_I[\Omega, q]$. If $f(z) \in A(p)$ satisfies

$$\{\phi(I_p^\alpha f(z), I_p^{\alpha-1} f(z), I_p^{\alpha-2} f(z); z) : z \in U\} \subset \Omega \quad (\alpha > 2; p \in \mathbb{N}), \quad (2.1)$$

then

$$I_p^\alpha f(z) \prec q(z).$$

Proof. Define the analytic function $g(z)$ in U by

$$g(z) = I_p^\alpha f(z) \quad (\alpha > 2; p \in \mathbb{N}; z \in U). \quad (2.2)$$

Using the identity (1.3) in (2.2), we get

$$I_p^{\alpha-1} f(z) = \frac{z g'(z) + g(z)}{(p+1)}, \quad (2.3)$$

and

$$I_p^{\alpha-2} f(z) = \frac{z^2 g''(z) + 3z g'(z) + g(z)}{(p+1)^2}. \quad (2.4)$$

Define the transformations from \mathbb{C}^3 to \mathbb{C} by

$$u = r, \quad v = \frac{s+r}{p+1}, \quad w = \frac{t+3s+r}{(p+1)^2}. \quad (2.5)$$

Let

$$\psi(r, s, t; z) = \phi(u, v, w; z) = \phi\left(r, \frac{s+r}{p+1}, \frac{t+3s+r}{(p+1)^2}; z\right). \quad (2.6)$$

The proof shall make use of Lemma 1. Using equations (2.2) – (2.4), and from (2.6), we obtain

$$\psi(g(z), z g'(z), z^2 g''(z); z) = \phi(I_p^\alpha f(z), I_p^{\alpha-1} f(z), I_p^{\alpha-2} f(z); z) \quad (\alpha > 2; p \in \mathbb{N}; z \in U). \quad (2.7)$$

Hence (2.1) becomes

$$\psi(g(z), zg'(z), z^2g''(z); z) \in \Omega.$$

The proof is completed if it can be shown that the admissibility condition for $\phi \in \Phi_I[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1. Note that

$$\frac{t}{s} + 1 = \frac{(p+1)^2 w - 2(p+1)v + u}{(p+1)v - u},$$

and hence $\psi \in \Psi_p[\Omega, q]$. By Lemma 1,

$$g(z) \prec q(z) \quad \text{or} \quad I_p^\alpha f(z) \prec q(z).$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping $h(z)$ of U onto Ω . In this case the class $\Phi_I[h(U), q]$ is written as $\Phi_I[h, q]$. The following result is an immediate consequence of Theorem 1.

Theorem 2. *Let $\phi \in \Phi_I[h, q]$. If $f(z) \in A(p)$ satisfies*

$$\phi(I_p^\alpha f(z), I_p^{\alpha-1} f(z), I_p^{\alpha-2} f(z); z) \prec h(z) \quad (\alpha > 2; p \in \mathbb{N}), \quad (2.8)$$

then

$$I_p^\alpha f(z) \prec q(z).$$

Our next result is an extension of Theorem 1 to the case where the behavior of $q(z)$ on ∂U is not known.

Corollary 1. *Let $\Omega \subset \mathbb{C}$ and let $q(z)$ be univalent in U , $q(0) = 0$. Let $\phi \in \Phi_I[\Omega, q_\rho]$ for some $\rho \in (0, 1)$ where $q_\rho(z) = q(\rho z)$. If $f(z) \in A(p)$ and*

$$\phi(I_p^\alpha f(z), I_p^{\alpha-1} f(z), I_p^{\alpha-2} f(z); z) \in \Omega \quad (\alpha > 2; p \in \mathbb{N}; z \in U),$$

then

$$I_p^\alpha f(z) \prec q(z).$$

Proof. Theorem 1 yields $I_p^\alpha f(z) \prec q_\rho(z)$. The result is now deduced from $q_\rho(z) \prec q(z)$.

Theorem 3. *Let $h(z)$ and $q(z)$ be univalent in U , with $q(0) = 0$ and set $q_\rho(z) = q(\rho z)$ and $h_\rho(z) = h(\rho z)$. Let $\phi : \mathbb{C}^3 \times U \rightarrow \mathbb{C}$ satisfy one of the following conditions:*

- (1) $\phi \in \Phi_I[h, q_\rho]$, for some $\rho \in (0, 1)$, or
- (2) there exists $\rho_0 \in (0, 1)$ such that $\phi \in \Phi_I[h_\rho, q_\rho]$, for all $\rho \in (\rho_0, 1)$.

If $f(z) \in A(p)$ satisfies (2.8), then

$$I_p^\alpha f(z) \prec q(z).$$

Proof. The proof is similar to the proof of [8, Theorem 2.3d, p.30] and is therefore omitted.

The next theorem yields the best dominant of the differential subordination (2.8).

Theorem 4. Let $h(z)$ be univalent in U . Let $\phi : \mathbb{C}^3 \times U \rightarrow \mathbb{C}$. Suppose that the differential equation

$$\phi(q(z), zq'(z), z^2q''(z); z) = h(z) \tag{2.9}$$

has a solution $q(z)$ with $q(0) = 0$ and satisfy one of the following conditions:

- (1) $q(z) \in Q_0$ and $\phi \in \Phi_I[h, q]$,
- (2) $q(z)$ is univalent in U and $\phi \in \Phi_I[h, q_\rho]$, for some $\rho \in (0, 1)$, or
- (3) $q(z)$ is univalent in U and there exists $\rho_0 \in (0, 1)$ such that $\phi \in \Phi_I[h_\rho, q_\rho]$, for all $\rho \in (\rho_0, 1)$.

If $f(z) \in A(p)$ satisfies (2.8), then

$$I_p^\alpha f(z) \prec q(z),$$

and $q(z)$ is the best dominant.

Proof. Following the same arguments in [8, Theorem 2.3e, p. 31], we deduce that $q(z)$ is a dominant from Theorems 2 and 3. Since $q(z)$ satisfies (2.9) it is also a solution of (2.8) and therefore $q(z)$ will be dominated by all dominants. Hence $q(z)$ is the best dominant.

In the particular case $q(z) = Mz$, $M > 0$, and in view of the Definition 1, the class of admissible functions $\Phi_I[\Omega, q]$, denoted by $\Phi_I[\Omega, M]$, is described below.

Definition 4. Let Ω be a set in \mathbb{C} and $M > 0$. The class of admissible functions $\Phi_I[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^3 \times U \rightarrow \mathbb{C}$ such that

$$\phi \left(Me^{i\theta}, \frac{k+1}{p+1} Me^{i\theta}, \frac{L + (3k+1) Me^{i\theta}}{(p+1)^2}; z \right) \notin \Omega \tag{2.10}$$

whenever $z \in U$, $\theta \in \mathbb{R}$, $\Re(L e^{-i\theta}) \geq (k-1)kM$ for all real $\theta, p \in \mathbb{N}$ and $k \geq p$.

Corollary 2. Let $\phi \in \Phi_I[\Omega, M]$. If $f(z) \in A(p)$ satisfies

$$\phi(I_p^\alpha f(z), I_p^{\alpha-1} f(z), I_p^{\alpha-2} f(z); z) \in \Omega \quad (\alpha > 2; p \in \mathbb{N}; z \in U),$$

then

$$|I_p^\alpha f(z)| < M \quad (z \in U).$$

In the special case $\Omega = q(U) = \{\omega : |\omega| < M\}$, the class $\Phi_I[\Omega, M]$ is simply denoted by $\Phi_I[M]$.

Corollary 3. Let $\phi \in \Phi_I[M]$. If $f(z) \in A(p)$ satisfies

$$|\phi(I_p^\alpha f(z), I_p^{\alpha-1} f(z), I_p^{\alpha-2} f(z); z)| < M \quad (\alpha > 2; p \in \mathbb{N}; z \in U),$$

then

$$|I_p^\alpha f(z)| < M \quad (z \in U).$$

Remark 1. Putting $M = 1$ in Corollary 3 we obtain the result obtained by Aouf [3, Theorem 1].

Corollary 4. If $k \geq p$ and $f(z) \in A(p)$ satisfies

$$|I_p^{\alpha-1} f(z)| < M \quad (\alpha > 1; p \in \mathbb{N}; z \in U).$$

then

$$|I_p^\alpha f(z)| < M \quad (z \in U).$$

Proof. This follows from Corollary 3 by taking $\phi(u, v, w; z) = v = \frac{k+1}{p+1} M e^{i\theta}$.

Remark 2. For $M = 1$, Corollary 4 yields the result obtained by Aouf [3, Corollary 1].

Definition 5. Let Ω be a set in \mathbb{C} and $q(z) \in Q_0 \cap H_0$. The class of admissible functions $\Phi_{I,1}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times U \rightarrow \mathbb{C}$ that satisfy the admissibility condition:

$$\phi(u, v, w; z) \notin \Omega$$

whenever $u = q(\zeta)$, $v = \frac{k\zeta q'(\zeta) + pq(\zeta)}{p+1}$,

$$\Re \left\{ \frac{(p+1)^2 w - 2p(p+1)v + p^2 u}{(p+1)v - pu} \right\} \geq k \Re \left\{ 1 + \frac{\zeta q''(\zeta)}{q'(\zeta)} \right\},$$

where $z \in U$, $\zeta \in \partial U \setminus E(q)$ and $k \geq 1$.

Theorem 5. Let $\phi \in \Phi_{I,1}[\Omega, q]$. If $f(z) \in A(p)$ satisfies

$$\left\{ \phi \left(\frac{I_p^\alpha f(z)}{z^{p-1}}, \frac{I_p^{\alpha-1} f(z)}{z^{p-1}}, \frac{I_p^{\alpha-2} f(z)}{z^{p-1}}; z \right) : z \in U \right\} \subset \Omega \quad (\alpha > 2; p \in \mathbb{N}), \quad (2.11)$$

then

$$\frac{I_p^\alpha f(z)}{z^{p-1}} \prec q(z).$$

Proof. Define an analytic function $g(z)$ in U by

$$g(z) = \frac{I_p^\alpha f(z)}{z^{p-1}} \quad (\alpha > 2; p \in \mathbb{N}; z \in U). \quad (2.12)$$

By making use of (1.3) and (2.12), we get

$$\frac{I_p^{\alpha-1} f(z)}{z^{p-1}} = \frac{z g'(z) + p g(z)}{p+1}. \quad (2.13)$$

Further computations show that

$$\frac{I_p^{\alpha-2} f(z)}{z^{p-1}} = \frac{z^2 g''(z) + (2p+1) z g'(z) + p^2 g(z)}{(p+1)^2}. \quad (2.14)$$

Define the transformations from \mathbb{C}^3 to \mathbb{C} by

$$u = r, \quad v = \frac{s + pr}{p+1}, \quad w = \frac{t + (2p+1)s + p^2 r}{(p+1)^2}. \quad (2.15)$$

Let

$$\psi(r, s, t; z) = \phi(u, v, w; z) = \phi\left(r, \frac{s + pr}{p+1}, \frac{t + (2p+1)s + p^2 r}{(p+1)^2}; z\right). \quad (2.16)$$

The proof shall make use of Lemma 1. Using equations (2.12) – (2.14), and from (2.16), we obtain

$$\psi\left(g(z), z g'(z), z^2 g''(z); z\right) = \phi\left(\frac{I_p^\alpha f(z)}{z^{p-1}}, \frac{I_p^{\alpha-1} f(z)}{z^{p-1}}, \frac{I_p^{\alpha-2} f(z)}{z^{p-1}}; z\right). \quad (2.17)$$

Hence (2.11) becomes

$$\psi\left(g(z), z g'(z), z^2 g''(z); z\right) \in \Omega.$$

The proof is completed if it can be shown that the admissibility condition for $\phi \in \Phi_{I,1}[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.

Note that

$$\frac{t}{s} + 1 = \frac{(p+1)^2 w - 2p(p+1)v + p^2 u}{(p+1)v - pu},$$

and hence $\psi \in \Psi[\Omega, q]$. By Lemma 1,

$$g(z) \prec q(z) \quad \text{or} \quad \frac{I_p^\alpha f(z)}{z^{p-1}} \prec q(z).$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(U)$, for some conformal mapping $h(z)$ of U onto Ω . In this case the class $\Phi_{I,1}[h(U), q]$ is written as $\Phi_{I,1}[h, q]$. In the particular case $q(z) = Mz$, $M > 0$, the class of admissible functions $\Phi_{I,1}[\Omega, q]$, denoted by $\Phi_{I,1}[\Omega, M]$.

Proceeding similiary as in the previous section, the following result is an immediate consequence of Theorem 5.

Theorem 6. *Let $\phi \in \Phi_{I,1}[h, q]$. If $f(z) \in A(p)$ satisfies*

$$\phi \left(\frac{I_p^\alpha f(z)}{z^{p-1}}, \frac{I_p^{\alpha-1} f(z)}{z^{p-1}}, \frac{I_p^{\alpha-2} f(z)}{z^{p-1}}; z \right) \prec h(z) \quad (\alpha > 2; p \in \mathbb{N}), \quad (2.18)$$

then

$$\frac{I_p^\alpha f(z)}{z^{p-1}} \prec q(z).$$

Definition 6. *Let Ω be a set in \mathbb{C} and $M > 0$. The class of admissible functions $\Phi_{I,1}[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^3 \times U \rightarrow \mathbb{C}$ such that*

$$\phi \left(Me^{i\theta}, \frac{k+p}{p+1} Me^{i\theta}, \frac{L + [(2p+1)k + p^2] Me^{i\theta}}{(p+1)^2}; z \right) \notin \Omega \quad (2.19)$$

whenever $z \in U$, $\theta \in \mathbb{R}$, $\Re(Le^{-i\theta}) \geq (k-1)kM$ for all real θ , $p \in \mathbb{N}$ and $k \geq 1$.

Corollary 5. *Let $\phi \in \Phi_{I,1}[\Omega, M]$. If $f(z) \in A(p)$ satisfies*

$$\phi \left(\frac{I_p^\alpha f(z)}{z^{p-1}}, \frac{I_p^{\alpha-1} f(z)}{z^{p-1}}, \frac{I_p^{\alpha-2} f(z)}{z^{p-1}}; z \right) \in \Omega \quad (\alpha > 2; p \in \mathbb{N}; z \in U),$$

then

$$\left| \frac{I_p^\alpha f(z)}{z^{p-1}} \right| < M \quad (z \in U).$$

In the special case $\Omega = \{\omega : |\omega| < M\}$, the class $\Phi_{I,1}[\Omega, M]$ is simply denoted by $\Phi_{I,1}[M]$.

Corollary 6. *Let $\phi \in \Phi_{I,1}[M]$. If $f(z) \in A(p)$ satisfies*

$$\left| \phi \left(\frac{I_p^\alpha f(z)}{z^{p-1}}, \frac{I_p^{\alpha-1} f(z)}{z^{p-1}}, \frac{I_p^{\alpha-2} f(z)}{z^{p-1}}; z \right) \right| < M \quad (\alpha > 2; p \in \mathbb{N}; z \in U),$$

then

$$\left| \frac{I_p^\alpha f(z)}{z^{p-1}} \right| < M \quad (z \in U).$$

Corollary 7. If $k \geq 1$ and $f(z) \in A(p)$ satisfies

$$\left| \frac{I_p^{\alpha-1} f(z)}{z^{p-1}} \right| < M \quad (\alpha > 1; p \in \mathbb{N}; z \in U).$$

then

$$\left| \frac{I_p^\alpha f(z)}{z^{p-1}} \right| < M \quad (z \in U).$$

Proof. This follows from Corollary 6 by taking $\phi(u, v, w; z) = v = \frac{k+p}{p+1} M e^{i\theta}$.

Definition 7. Let Ω be a set in \mathbb{C} and $q(z) \in Q_1 \cap H$. The class of admissible functions $\Phi_{I,2}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times U \rightarrow \mathbb{C}$ that satisfy the admissibility condition

$$\phi(u, v, w; z) \notin \Omega$$

$$\text{whenever } u = q(\zeta), v = \frac{1}{p+1} \left\{ (p+1)q(\zeta) + \frac{k\zeta q'(\zeta)}{q(\zeta)} \right\},$$

$$\Re \left\{ \frac{(p+1)(vw - 3uv + 2u^2)}{v - u} \right\} \geq k \Re \left\{ 1 + \frac{\zeta q''(\zeta)}{q'(\zeta)} \right\},$$

where $z \in U$, $\zeta \in \partial U \setminus E(q)$, $p \in \mathbb{N}$ and $k \geq 1$.

Theorem 7. Let $\phi \in \Phi_{I,2}[\Omega, q]$ and $I_p^\alpha f(z) \neq 0$. If $f(z) \in A(p)$ satisfies

$$\left\{ \phi \left(\frac{I_p^{\alpha-1} f(z)}{I_p^\alpha f(z)}, \frac{I_p^{\alpha-2} f(z)}{I_p^{\alpha-1} f(z)}, \frac{I_p^{\alpha-3} f(z)}{I_p^{\alpha-2} f(z)}; z \right) : z \in U \right\} \subset \Omega \quad (\alpha > 3; p \in \mathbb{N}), \quad (2.20)$$

then

$$\frac{I_p^{\alpha-1} f(z)}{I_p^\alpha f(z)} \prec q(z).$$

Proof. Define an analytic function $g(z)$ in U by

$$g(z) = \frac{I_p^{\alpha-1} f(z)}{I_p^\alpha f(z)} \quad (\alpha > 3; p \in \mathbb{N}; z \in U). \quad (2.21)$$

Using (2.21), we get

$$\frac{zg'(z)}{g(z)} = \frac{z(I_p^{\alpha-1} f(z))'}{I_p^{\alpha-1} f(z)} - \frac{z(I_p^\alpha f(z))'}{I_p^\alpha f(z)}. \quad (2.22)$$

By making use of the identity (1.3) in (2.22), we get

$$\frac{I_p^{\alpha-2} f(z)}{I_p^{\alpha-1} f(z)} = \frac{1}{p+1} \left\{ (p+1) g(z) + \frac{z g'(z)}{g(z)} \right\}. \quad (2.23)$$

Further computations show that

$$\frac{I_p^{\alpha-3} f(z)}{I_p^{\alpha-2} f(z)} = \frac{1}{p+1} \left\{ (p+1) g(z) + \frac{z g'(z)}{g(z)} + \frac{(p+1) z g'(z) + \frac{z g'(z)}{g(z)} + \frac{z^2 g''(z)}{g(z)} - \left(\frac{z g'(z)}{g(z)} \right)^2}{(p+1) g(z) + \frac{z g'(z)}{g(z)}} \right\}. \quad (2.24)$$

Define the transformations from \mathbb{C}^3 to \mathbb{C} by

$$u = r, v = \frac{1}{p+1} \left\{ (p+1) r + \frac{s}{r} \right\}, w = \frac{1}{p+1} \left\{ (p+1) r + \frac{s}{r} + \frac{(p+1) s + \frac{s}{r} + \frac{t}{r} - \left(\frac{s}{r} \right)^2}{(p+1) r + \frac{s}{r}} \right\}. \quad (2.25)$$

Let

$$\begin{aligned} \psi(r, s, t; z) &= \phi(u, v, w; z) \\ &= \phi \left(r, \frac{1}{p+1} \left\{ (p+1) r + \frac{s}{r} \right\}, \frac{1}{p+1} \left\{ (p+1) r + \frac{s}{r} \right. \right. \\ &\quad \left. \left. + \frac{(p+1) s + \frac{s}{r} + \frac{t}{r} - \left(\frac{s}{r} \right)^2}{(p+1) r + \frac{s}{r}} \right\}; z \right). \end{aligned} \quad (2.26)$$

The proof shall make use of Lemma 1. Using equations (2.21), (2.23) and (2.24), from (2.26), we obtain

$$\psi(g(z), z g'(z), z^2 g''(z); z) = \phi \left(\frac{I_p^{\alpha-1} f(z)}{I_p^{\alpha} f(z)}, \frac{I_p^{\alpha-2} f(z)}{I_p^{\alpha-1} f(z)}, \frac{I_p^{\alpha-3} f(z)}{I_p^{\alpha-2} f(z)}; z \right). \quad (2.27)$$

Hence (2.20) becomes

$$\psi(g(z), z g'(z), z^2 g''(z); z) \in \Omega.$$

The proof is completed if it can be shown that the admissibility condition for $\phi \in \Phi_{I,2}[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1. Note that

$$\frac{t}{s} + 1 = \frac{(p+1)(vw - 3uv + 2u^2)}{v - u},$$

and hence $\psi \in \Psi [\Omega, q]$. By Lemma 1,

$$g(z) \prec q(z) \quad \text{or} \quad \frac{I_p^{\alpha-1} f(z)}{I_p^\alpha f(z)} \prec q(z).$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(U)$, for some conformal mapping $h(z)$ of U onto Ω . In this case the class $\Phi_{I,2} [h(U), q]$ is written as $\Phi_{I,2} [h, q]$. In the particular case $q(z) = Mz$, $M > 0$, the class of admissible functions $\Phi_{I,2} [\Omega, q]$ becomes the class $\Phi_{I,2} [\Omega, M]$.

Proceeding similarly as in the previous section, the following result is an immediate consequence of Theorem 7.

Theorem 8. *Let $\phi \in \Phi_{I,2} [h, q]$. If $f(z) \in A(p)$ satisfies*

$$\phi \left(\frac{I_p^{\alpha-1} f(z)}{I_p^\alpha f(z)}, \frac{I_p^{\alpha-2} f(z)}{I_p^{\alpha-1} f(z)}, \frac{I_p^{\alpha-3} f(z)}{I_p^{\alpha-2} f(z)}; z \right) \prec h(z) \quad (\alpha > 3; p \in \mathbb{N}), \quad (2.28)$$

then

$$\frac{I_p^{\alpha-1} f(z)}{I_p^\alpha f(z)} \prec q(z).$$

Definition 8. *Let Ω be a set in \mathbb{C} and $M > 0$. The class of admissible functions $\Phi_{I,2} [\Omega, M]$ consists of those functions $\phi : \mathbb{C}^3 \times U \rightarrow \mathbb{C}$ such that*

$$\phi \left(Me^{i\theta}, \frac{k+(p+1)Me^{i\theta}}{p+1}, \frac{1}{p+1} \left\{ (p+1)Me^{i\theta} + k + \frac{(p+1)kM^2e^{i\theta} + kM + Le^{-i\theta} - k^2M}{(p+1)M^2e^{i\theta} + kM} \right\}; z \right) \notin \Omega, \quad (2.29)$$

whenever $z \in U$, $\theta \in \mathbb{R}$, $p \in \mathbb{N}$, $\Re (Le^{-i\theta}) \geq (k-1)kM$ for all real θ , $p \in \mathbb{N}$ and $k \geq 1$.

Corollary 8. *Let $\phi \in \Phi_{I,2} [\Omega, M]$. If $f(z) \in A(p)$ satisfies*

$$\phi \left(\frac{I_p^{\alpha-1} f(z)}{I_p^\alpha f(z)}, \frac{I_p^{\alpha-2} f(z)}{I_p^{\alpha-1} f(z)}, \frac{I_p^{\alpha-3} f(z)}{I_p^{\alpha-2} f(z)}; z \right) \in \Omega \quad (\alpha > 3; p \in \mathbb{N}; z \in U),$$

then

$$\left| \frac{I_p^{\alpha-1} f(z)}{I_p^\alpha f(z)} \right| < M \quad (M > 0; z \in U).$$

In the special case $\Omega = q(U) = \{\omega : |\omega| < M\}$, the class $\Phi_{I,2} [\Omega, M]$ is denoted by $\Phi_{I,2} [M]$.

Corollary 9. *Let $\phi \in \Phi_{I,2} [M]$. If $f(z) \in A(p)$ satisfies*

$$\left| \phi \left(\frac{I_p^{\alpha-1} f(z)}{I_p^\alpha f(z)}, \frac{I_p^{\alpha-2} f(z)}{I_p^{\alpha-1} f(z)}, \frac{I_p^{\alpha-3} f(z)}{I_p^{\alpha-2} f(z)}; z \right) \right| < M \quad (\alpha > 3; p \in \mathbb{N}; M > 0; z \in U),$$

then

$$\left| \frac{I_p^{\alpha-1} f(z)}{I_p^\alpha f(z)} \right| < M \quad (z \in U).$$

Remark 3. The result in Corollary 9 is *extension* of the result obtained by Aouf [3, Theorem 3].

3. SUPERORDINATION RESULTS OF THE INTEGRAL OPERATOR

The dual problem of differential subordination, that is, differential superordination of the integral operator I_p^α is investigated in this section. For this purpose the class of admissible functions is given in the following definition.

Definition 9. Let Ω be a set in \mathbb{C} and $q(z) \in H[0, p]$ with $zq'(z) \neq 0$. The class of admissible functions $\Phi_I'[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \bar{U} \rightarrow \mathbb{C}$ that satisfy the admissibility condition:

$$\phi(u, v, w; \zeta) \in \Omega$$

whenever $u = q(z)$, $v = \frac{\zeta q'(z) + mq(z)}{m(p+1)}$,

$$\Re \left\{ \frac{(p+1)^2 w - 2(p+1)v + u}{(p+1)v - u} \right\} \leq \frac{1}{m} \Re \left\{ 1 + \frac{zq''(z)}{q'(z)} \right\},$$

where $z \in U$, $\zeta \in \partial U$ and $m \geq p$.

Theorem 9. Let $\phi \in \Phi_I'[\Omega, q]$. If $f(z) \in A(p)$, $I_p^\alpha f(z) \in Q_0$ and

$$\phi(I_p^\alpha f(z), I_p^{\alpha-1} f(z), I_p^{\alpha-2} f(z); z)$$

is univalent in U , then

$$\Omega \subset \left\{ \phi(I_p^\alpha f(z), I_p^{\alpha-1} f(z), I_p^{\alpha-2} f(z); z) : z \in U \right\} \quad (\alpha > 2; p \in \mathbb{N}), \quad (3.1)$$

implies

$$q(z) \prec I_p^\alpha f(z).$$

Proof. From (2.7) and (3.1), we have

$$\Omega \subset \left\{ \psi(g(z), zg'(z), z^2g''(z); z) : z \in U \right\}.$$

From (2.5), we see that the admissibility condition for $\phi \in \Phi_I'[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 2. Hence $\psi \in \Psi_p'[\Omega, q]$, and by Lemma 2,

$$q(z) \prec g(z) \quad \text{or} \quad q(z) \prec I_p^\alpha f(z).$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping $h(z)$ of U onto Ω . In this case the class $\Phi'_I[h(U), q]$ is written as $\Phi'_I[h, q]$.

Proceeding similarly as in the previous section, the following result is an immediate consequence of Theorem 9.

Theorem 10. *Let $h(z)$ is analytic on U and $\phi \in \Phi'_I[h, q]$. If $f(z) \in A(p)$, $I_p^\alpha f(z) \in Q_0$ and*

$$\phi(I_p^\alpha f(z), I_p^{\alpha-1} f(z), I_p^{\alpha-2} f(z); z)$$

is univalent in U , then

$$h(z) \prec \phi(I_p^\alpha f(z), I_p^{\alpha-1} f(z), I_p^{\alpha-2} f(z); z) \quad (\alpha > 2; p \in \mathbb{N}), \quad (3.2)$$

implies

$$q(z) \prec I_p^\alpha f(z).$$

Theorems 9 and 10 can only be used to obtain subordinants of differential superordination of the form (3.1) or (3.2). The following theorem proves the existence of the best subordinant of (3.2) for certain ϕ .

Theorem 11. *Let $h(z)$ be analytic in U and $\phi : \mathbb{C}^3 \times \bar{U} \rightarrow \mathbb{C}$. Suppose that the differential equation*

$$\phi(q(z), zq'(z), z^2q''(z); z) = h(z)$$

has a solution $q(z) \in Q_0$. If $\phi \in \Phi'_I[h, q]$, $f(z) \in A(p)$, $I_p^\alpha f(z) \in Q_0$ and

$$\phi(I_p^\alpha f(z), I_p^{\alpha-1} f(z), I_p^{\alpha-2} f(z); z)$$

is univalent in U , then

$$h(z) \prec \phi(I_p^\alpha f(z), I_p^{\alpha-1} f(z), I_p^{\alpha-2} f(z); z) \quad (\alpha > 2; p \in \mathbb{N}; z \in U)$$

implies

$$q(z) \prec I_p^\alpha f(z)$$

and $q(z)$ is the best subordinant.

Proof. The proof is similar to the proof of Theorem 4 and is therefore omitted. Combining Theorems 2 and 10, we obtain the following sandwich-type theorem.

Corollary 10. *Let $h_1(z)$ and $q_1(z)$ be analytic functions in U , $h_2(z)$ be univalent function in U , $q_2(z) \in Q_0$ with $q_1(0) = q_2(0) = 0$ and $\phi \in \Phi_I[h_2, q_2] \cap \Phi'_I[h_1, q_1]$. If $f(z) \in A(p)$, $I_p^\alpha f(z) \in H[0, p] \cap Q_0$ and*

$$\phi(I_p^\alpha f(z), I_p^{\alpha-1} f(z), I_p^{\alpha-2} f(z); z) \quad (\alpha > 2; p \in \mathbb{N}; z \in U)$$

is univalent in U , then

$$h_1(z) \prec \phi \left(I_p^\alpha f(z), I_p^{\alpha-1} f(z), I_p^{\alpha-2} f(z); z \right) \prec h_2(z) \quad (\alpha > 2; p \in \mathbb{N})$$

implies

$$q_1(z) \prec I_p^\alpha f(z) \prec q_2(z).$$

Definition 10. Let Ω be a set in \mathbb{C} and $q(z) \in H_0$ with $zq'(z) \neq 0$. The class of admissible functions $\Phi'_{I,1}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \bar{U} \rightarrow \mathbb{C}$ that satisfy the admissibility condition:

$$\phi(u, v, w; \zeta) \in \Omega \tag{3.3}$$

whenever $u = q(z), v = \frac{zq'(z) + mpq(z)}{m(p+1)},$

$$\Re \left\{ \frac{(p+1)^2 w - 2p(p+1)v + p^2 u}{(p+1)v - pu} \right\} \leq \frac{1}{m} \Re \left\{ 1 + \frac{zq''(z)}{q'(z)} \right\},$$

where $z \in U, \zeta \in \partial U$ and $m \geq 1$.

Now we will give the dual result of Theorem 5 for differential superordination.

Theorem 12. Let $\phi \in \Phi'_{I,1}[\Omega, q]$. If $f(z) \in A(p), \frac{I_p^\alpha f(z)}{z^{p-1}} \in Q_0$ and

$$\phi \left(\frac{I_p^\alpha f(z)}{z^{p-1}}, \frac{I_p^{\alpha-1} f(z)}{z^{p-1}}, \frac{I_p^{\alpha-2} f(z)}{z^{p-1}}; z \right)$$

is univalent in U , then

$$\Omega \subset \left\{ \phi \left(\frac{I_p^\alpha f(z)}{z^{p-1}}, \frac{I_p^{\alpha-1} f(z)}{z^{p-1}}, \frac{I_p^{\alpha-2} f(z)}{z^{p-1}}; z \right) : z \in U \right\} \quad (\alpha > 2; p \in \mathbb{N}) \tag{3.4}$$

implies

$$q(z) \prec \frac{I_p^\alpha f(z)}{z^{p-1}}.$$

Proof From (2.17) and (3.4), we have

$$\Omega \subset \left\{ \psi \left(g(z), zg'(z), z^2g''(z); z \right) : z \in U \right\} \quad (\alpha > 2; p \in \mathbb{N}).$$

From (2.15), we see that the admissibility condition for $\phi \in \Phi'_{I,1}[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 2. Hence $\psi \in \Psi'[\Omega, q]$, and by Lemma 2,

$$q(z) \prec g(z) \quad \text{or} \quad q(z) \prec \frac{I_p^\alpha f(z)}{z^{p-1}}.$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, and $\Omega = h(U)$ for some conformal mapping $h(z)$ of U onto Ω and the class $\Phi'_{I,1}[h(U), q]$ is written as $\Phi'_{I,1}[h, q]$.

Proceeding similarly as in the previous section, the following result is an immediate consequence of Theorem 12.

Theorem 13. *Let $q(z) \in H_0$, $h(z)$ is analytic on U and $\phi \in \Phi'_{I,1}[h, q]$. If $f(z) \in A(p)$, $\frac{I_p^\alpha f(z)}{z^{p-1}} \in Q_0$ and*

$$\phi \left(\frac{I_p^\alpha f(z)}{z^{p-1}}, \frac{I_p^{\alpha-1} f(z)}{z^{p-1}}, \frac{I_p^{\alpha-2} f(z)}{z^{p-1}}; z \right)$$

is univalent in U , then

$$h(z) \prec \phi \left(\frac{I_p^\alpha f(z)}{z^{p-1}}, \frac{I_p^{\alpha-1} f(z)}{z^{p-1}}, \frac{I_p^{\alpha-2} f(z)}{z^{p-1}}; z \right) \quad (\alpha > 2; p \in \mathbb{N}), \quad (3.5)$$

implies

$$q(z) \prec \frac{I_p^\alpha f(z)}{z^{p-1}}.$$

Combining Theorems 6 and 13, we obtain the following sandwich-type theorem.

Corollary 11. *Let $h_1(z)$ and $q_1(z)$ be analytic functions in U , $h_2(z)$ be univalent function in U , $q_2(z) \in Q_0$ with $q_1(0) = q_2(0) = 0$ and $\phi \in \Phi_{I,1}[h_2, q_2] \cap \Phi'_{I,1}[h_1, q_1]$. If $f(z) \in A(p)$, $\frac{I_p^\alpha f(z)}{z^{p-1}} \in H_0 \cap Q_0$ and*

$$\phi \left(\frac{I_p^\alpha f(z)}{z^{p-1}}, \frac{I_p^{\alpha-1} f(z)}{z^{p-1}}, \frac{I_p^{\alpha-2} f(z)}{z^{p-1}}; z \right)$$

is univalent in U , then

$$h_1(z) \prec \phi \left(\frac{I_p^\alpha f(z)}{z^{p-1}}, \frac{I_p^{\alpha-1} f(z)}{z^{p-1}}, \frac{I_p^{\alpha-2} f(z)}{z^{p-1}}; z \right) \prec h_2(z) \quad (\alpha > 2; p \in \mathbb{N})$$

implies

$$q_1(z) \prec \frac{I_p^\alpha f(z)}{z^{p-1}} \prec q_2(z).$$

Definition 11. *Let Ω be a set in \mathbb{C} , $q(z) \neq 0$, $zq'(z) \neq 0$ and $q(z) \in H$. The class of admissible functions $\phi \in \Phi'_{I,2}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \bar{U} \rightarrow \mathbb{C}$ that satisfy the admissibility condition:*

$$\phi(u, v, w; \zeta) \in \Omega$$

whenever $u = q(z)$, $v = \frac{1}{p+1} \left\{ (p+1)q(z) + \frac{zq'(z)}{mq(z)} \right\}$,

$$\Re \left\{ \frac{(p+1)(vw - 3uv + 2u^2)}{v - u} \right\} \leq \frac{1}{m} \Re \left\{ 1 + \frac{zq''(z)}{q'(z)} \right\},$$

where $z \in U$, $\zeta \in \partial U$, $p \in \mathbb{N}$ and $m \geq 1$.

Now we will give the dual result of Theorem 7 for the differential superordination.

Theorem 14. Let $\phi \in \Phi'_{I,2}[\Omega, q]$. If $f(z) \in A(p)$, $\frac{I_p^{\alpha-1}f(z)}{I_p^\alpha f(z)} \in Q_1$ and

$$\phi \left(\frac{I_p^{\alpha-1}f(z)}{I_p^\alpha f(z)}, \frac{I_p^{\alpha-2}f(z)}{I_p^{\alpha-1}f(z)}, \frac{I_p^{\alpha-3}f(z)}{I_p^{\alpha-2}f(z)}; z \right)$$

is univalent in U , then

$$\Omega \subset \left\{ \phi \left(\frac{I_p^{\alpha-1}f(z)}{I_p^\alpha f(z)}, \frac{I_p^{\alpha-2}f(z)}{I_p^{\alpha-1}f(z)}, \frac{I_p^{\alpha-3}f(z)}{I_p^{\alpha-2}f(z)}; z \right) : z \in U \right\} \quad (\alpha > 3; p \in \mathbb{N}) \quad (3.6)$$

implies

$$q(z) \prec \frac{I_p^{\alpha-1}f(z)}{I_p^\alpha f(z)}.$$

Proof From (2.27) and (3.6), we have

$$\Omega \subset \left\{ \psi \left(g(z), zg'(z), z^2g''(z); z \right) : z \in U \right\}.$$

In view of (2.25), the admissibility condition for $\phi \in \Phi'_{I,2}[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 2. Hence $\psi \in \Psi'[\Omega, q]$, and by Lemma 2,

$$q(z) \prec g(z) \quad \text{or} \quad q(z) \prec \frac{I_p^{\alpha-1}f(z)}{I_p^\alpha f(z)}.$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping $h(z)$ of U onto Ω . In this case the class $\Phi'_{I,2}[h(U), q]$ is written as $\Phi'_{I,2}[h, q]$.

Proceeding similarly as in the previous section, The following result is an immediate consequence of Theorem 14.

Theorem 15. Let $q(z) \in H$, $h(z)$ be analytic in U and $\phi \in \Phi'_{I,2}[h, q]$. If $f(z) \in A(p)$, $\frac{I_p^{\alpha-1}f(z)}{I_p^\alpha f(z)} \in Q_1$ and

$$\phi \left(\frac{I_p^{\alpha-1}f(z)}{I_p^\alpha f(z)}, \frac{I_p^{\alpha-2}f(z)}{I_p^{\alpha-1}f(z)}, \frac{I_p^{\alpha-3}f(z)}{I_p^{\alpha-2}f(z)}; z \right)$$

is univalent in U , then

$$h(z) \prec \phi \left(\frac{I_p^{\alpha-1} f(z)}{I_p^\alpha f(z)}, \frac{I_p^{\alpha-2} f(z)}{I_p^{\alpha-1} f(z)}, \frac{I_p^{\alpha-3} f(z)}{I_p^{\alpha-2} f(z)}; z \right) \quad (\alpha > 3; p \in \mathbb{N}), \quad (3.7)$$

implies

$$q(z) \prec \frac{I_p^{\alpha-1} f(z)}{I_p^\alpha f(z)}.$$

Combining Theorems 8 and 15, we obtain the following sandwich-type theorem.

Corollary 12. Let $h_1(z)$ and $q_1(z)$ be analytic functions in U , $h_2(z)$ be univalent function in U , $q_2(z) \in Q_1$ with $q_1(0) = q_2(0) = 1$ and $\phi \in \Phi_{I,2}[h_2, q_2] \cap \Phi'_{I,2}[h_1, q_1]$.

If $f(z) \in A(p)$, $\frac{I_p^{\alpha-1} f(z)}{I_p^\alpha f(z)} \in H \cap Q_1$ and

$$\phi \left(\frac{I_p^{\alpha-1} f(z)}{I_p^\alpha f(z)}, \frac{I_p^{\alpha-2} f(z)}{I_p^{\alpha-1} f(z)}, \frac{I_p^{\alpha-3} f(z)}{I_p^{\alpha-2} f(z)}; z \right)$$

is univalent in U , then

$$h_1(z) \prec \phi \left(\frac{I_p^{\alpha-1} f(z)}{I_p^\alpha f(z)}, \frac{I_p^{\alpha-2} f(z)}{I_p^{\alpha-1} f(z)}, \frac{I_p^{\alpha-3} f(z)}{I_p^{\alpha-2} f(z)}; z \right) \prec h_2(z) \quad (\alpha > 3; p \in \mathbb{N})$$

implies

$$q_1(z) \prec \frac{I_p^{\alpha-1} f(z)}{I_p^\alpha f(z)} \prec q_2(z).$$

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