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ON A CLASS OF FAMILY OF BAZILEVIC FUNCTIONS

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Abstract.In this paper the author introduce and study a more generalized class of the family of Bazilevic functions by using derivative operator under which we consider coefficient inequalities, inclusion relation, extremal problem, and coefficient bounds. The consequences of parametrics are also discussed.

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1. Introduction and preliminaries

Let A be the class of function $f(z)$ analytic in the unit disk $U = \{z : |z| < 1\}$ and of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1}$$

For function $f \in A$, we consider the differential operator $I^m(\lambda, l)$ introduced and studied by Cătaş et al in [9].
We define according to [9] the following derivative operator $I^m(\lambda, l) : A \to A$ as follows

$$I^0(\lambda, l)f(z) = f(z)$$

$$I^1(\lambda, l)f(z) = I(\lambda, l)f(z) = I^0(\lambda, l)f(z)\frac{(1-\lambda+l)}{1+l} + (I^0(\lambda, l)f(z))'\frac{\lambda z}{1+l} = \tag{2}$$

$$z + \sum_{k=2}^{\infty} \left(\frac{1+\lambda(k-1)+l}{1+l}\right) a_k z^k$$

$$I^2(\lambda,l)f(z) = I(\lambda,l)f(z)\frac{(1-\lambda+l)}{1+l} + (I(\lambda,l)f(z))'\,\frac{\lambda z}{1+l} = \tag{3}$$

$$z + \sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^2 a_k z^k$$

and in general,

$$I^m(\lambda,l)f(z) = I(\lambda,l)(I^{m-1}(\lambda,l)f(z)) = z + \sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m a_k z^k,\ m\in N_0, \lambda\geq 0, l\geq 0. \tag{4}$$

Using the operator above we give the definition of a more larger and generalized class of family of Bazilevic functions as follows:

Definition 1.1. *Let $T_m^{\alpha}(\lambda,\beta,\gamma,l)$ denote the subclass of A consisting of functions $f(z)$ which satisfy the inequality*

$$Re\frac{I^m(\lambda,l)f(z)^{\alpha}}{\left(\frac{1+\lambda(\alpha-1)+l}{1+l}\right)^m z^{\alpha}} > \gamma\left|\frac{I^m(\lambda,l)\,(f(z))^{\alpha}}{\left(\frac{1+\lambda(\alpha-1)+l}{1+l}\right)^m z^{\alpha}} - 1\right| + \beta \tag{5}$$

for some $\lambda\geq 0, l\geq 0, \alpha>0, 0\leq\beta<1, \gamma\geq 0, m\in N_0=\{0,1,2,...\}$ and all the index meant principal determination only.

Base on the above Definition 1.1, we have the following remark to make.

Remark A: (i). For $\gamma=0$ in (5) we have

$$Re\frac{I^m(\lambda,l)f(z)^{\alpha}}{\left(\frac{1+\lambda(\alpha-1)+l}{1+l}\right)^m z^{\alpha}} > \beta$$

that is, $f\in T_m^{\alpha}(\lambda,\beta,0,l)$ which is a complete new class of Bazilevic functions and it shall be included as a corollary in the result presented in this paper.

(ii). For $\gamma=0, l=0$ in (5) we have

$$Re\frac{I^m(\lambda,0)f(z)^{\alpha}}{(1+\lambda(\alpha-1))^m z^{\alpha}} > \beta \Leftrightarrow Re\frac{D_{\lambda}^m f(z)^{\alpha}}{(1+\lambda(\alpha-1))^m z^{\alpha}} > \beta$$

where D_{λ}^m is the Al-Oboudi derivative operator and $f\in T_m^{\alpha}(\lambda,\beta,0,0)$ which is also new and it shall be treated as a corollary in this work.

iii.For $\gamma=0, l=0, \lambda=1$ in (5)we have

$$Re\frac{I^m(1,0)f(z)^{\alpha}}{\alpha^m z^{\alpha}} > \beta \Leftrightarrow Re\frac{D^m f(z)^{\alpha}}{\alpha^m z^{\alpha}} > \beta$$

which is the class of functions studied in [2,3]. That is,$f\in T_m^{\alpha}(1,\beta,0,0)\equiv T_m^{\alpha}(\beta)$

(iv). For $\gamma=0, l=0, \lambda=1, \beta=0$ in (5) we have

$$Re\frac{I^m(1,0)f(z)^{\alpha}}{z^{\alpha}} > 0 \Leftrightarrow Re\frac{D^m f(z)^{\alpha}}{z^{\alpha}} > 0$$

which is the class of functions studied by Abduhalim in [1] and D^m is the Sălăgean derivative operator see [5]. That is $f \in T_m^\alpha(1,0,0,0) = T_m^\alpha(0) \equiv B_n(\alpha)$
(v). For $\gamma = 0, l = 0, \lambda = 1, \beta = 0, m = 0, \alpha = 1$ in (5) we have

$$Re\frac{I^0(1,0)f(z)}{z} > 0 \Leftrightarrow Re\frac{f(z)}{z} > 0$$

which is the class of functions studied by Yamaguchi in [6].That is, $f \in T_0^1(1,0,0,0) = T_0^1(0)$.
(vi). For $\gamma = 0, l = 0, \lambda = 1, \beta = 0, m = 1$, in (5) we have

$$Re\frac{I^1(1,0)f(z)^\alpha}{z^\alpha} > 0 \Rightarrow Re\frac{D^1 f(z)^\alpha}{z^\alpha} > 0 \Leftrightarrow Re\left\{\frac{\alpha z f' f^{\alpha-1}}{z^\alpha}\right\} > 0$$

The functions with this property belongs in the class $B_1(\alpha)$.
The class of Bazilevic functions that was studied by Singh in [4].
(vii). For $\gamma = 0, l = 0, \lambda = 1, 0 \leq \beta < 1, m = 0, \alpha = 1$ in (5) we have

$$Re\frac{I^0(1,0)f(z)}{z} > \beta,$$

the class of functions studied in [11].
(viii). For $\gamma = 0, l = 0, \lambda = 1, \beta = 0, m = 1, \alpha = 1$ in (5) we have

$$Re\frac{I^1(1,0)f(z)}{z} > 0,$$

the class of functions studied in [10].
The motivation for this present paper is that many works have been done on the various classes of Bazilevic functions with several different perspectives of study, but it is surprising that authors are not really interested in the aspect of their coefficient inequalities and coefficient bounds, our thinking of reasoning for this, is likely to be associated with the problem the index alpha may likely pose, especially, when $\alpha > 1$. This paper is designed to address this problem and to add up to the few existing literatures in this direction.
For the purpose of simplicity and clearity we wish to state the following, that is from (1) we can write that

$$(f(z))^\alpha = \left(z + \sum_{k=2}^{\infty} a_k z^k\right)^\alpha. \tag{6}$$

Using binomial expansion on (6) we have

$$(f(z))^\alpha = z^\alpha + \sum_{k=2}^{\infty} a_k(\alpha) z^{\alpha+k-1} \tag{7}$$

We know from Definition 1.1 that all the index are meant principal determination only. Therefore, the coefficients a_k shall depend so much on the parameter α. On applying Catas et al derivative operator on (7) we obtain

$$I^m(\lambda,l)f(z)^\alpha = \left(\frac{1+\lambda(\alpha-1)+l}{1+l}\right)^m z^\alpha + \sum_{k=2}^{\infty}\left(\frac{1+\lambda(\alpha+k-2)+l}{1+l}\right)^m a_k(\alpha) z^{\alpha+k-1} \tag{8}$$

where all the paprameters are as earlier defined.

2. Coefficient inequalities for the class $T_m^\alpha(\lambda,\beta,\gamma,l)$

Theorem 2.1 *If $f(z)\in A$ satifies*

$$\sum_{k=2}^{\infty}\Omega(m,\alpha,\beta,\lambda,\gamma,l,k)|a_k(\alpha)|\leq 2(1-\beta)\left(\frac{1+\lambda(\alpha-1)+l}{1+l}\right)^m \tag{9}$$

where
$\Omega(m,\alpha,\beta,\lambda,\gamma,l,k)=\left|\left(\frac{1+\lambda(\alpha+k-2)+l}{1+l}\right)^m\right|+\left(\frac{1+\lambda(\alpha+k-2)+l}{1+l}\right)^m+2\gamma\left|\left(\frac{1+\lambda(\alpha+k-2)+l}{1+l}\right)^m\right|$
for some $\alpha>0,\lambda\geq 0,l\geq 0,\gamma\geq 0,0\leq\beta<1,k\geq 2,m\in N_0$. Then $f(z)\in T_m^\alpha(\lambda,\beta,\gamma,l)$.
Proof: Suppose that (9) is true $\alpha>0,\lambda\geq 0,l\geq 0,\gamma\geq 0,0\leq\beta<1,k\geq 2,m\in N_0$. For $f(z)\in A$ and applying (5), let us define the function $F(z)^\alpha$ by

$$F(z)^\alpha=\frac{I^m(\lambda,l)f(z)^\alpha}{\left(\frac{1+\lambda(\alpha-1)+l}{1+l}\right)^m z^\alpha}-\gamma\left|\frac{I^m(\lambda,l)f(z)^\alpha}{\left(\frac{1+\lambda(\alpha-1)+l}{1+l}\right)^m z^\alpha}-1\right|-\beta. \tag{10}$$

It is suffices to show that

$$\left|\frac{F(z)^\alpha-1}{F(z)^\alpha+1}\right|<1,\ \alpha>0,\ z\in U. \tag{11}$$

We note that
$$\left|\frac{F(z)^\alpha-1}{F(z)^\alpha+1}\right|=\left|\frac{I^m(\lambda,l)f(z)^\alpha-\gamma e^{i\theta}\left|I^m(\lambda,l)f(z)^\alpha-\left(\frac{1+\lambda(\alpha-1)+l}{1+l}\right)^m z^\alpha\right|-\left(\frac{1+\lambda(\alpha-1)+l}{1+l}\right)^m z^\alpha-\beta\left(\frac{1+\lambda(\alpha-1)+l}{1+l}\right)^m z^\alpha}{I^m(\lambda,l)f(z)^\alpha-\gamma e^{i\theta}\left|I^m(\lambda,l)f(z)^\alpha-\left(\frac{1+\lambda(\alpha-1)+l}{1+l}\right)^m z^\alpha\right|+\left(\frac{1+\lambda(\alpha-1)+l}{1+l}\right)^m z^\alpha-\beta\left(\frac{1+\lambda(\alpha-1)+l}{1+l}\right)^m z^\alpha}\right|$$
By making use of (8) we have
$$\left|\frac{F(z)^\alpha-1}{F(z)^\alpha+1}\right|=\left|\frac{-\beta\left(\frac{1+\lambda(\alpha-1)+l}{1+l}\right)^m+\sum_{k=2}^{\infty}\left(\frac{1+\lambda(\alpha+k-2)+l}{1+l}\right)^m a_k(\alpha)z^{k-1}-\gamma e^{i\theta}\left|\left(\frac{1+\lambda(\alpha+k-2)+l}{1+l}\right)^m a_k(\alpha)z^{k-1}\right|}{(2-\beta)\left(\frac{1+\lambda(\alpha-1)+l}{1+l}\right)^m+\sum_{k=2}^{\infty}\left(\frac{1+\lambda(\alpha+k-2)+l}{1+l}\right)^m a_k(\alpha)z^{k-1}-\gamma e^{i\theta}\left|\left(\frac{1+\lambda(\alpha+k-2)+l}{1+l}\right)^m a_k(\alpha)z^{k-1}\right|}\right|$$
$$\leq\frac{\beta\left(\frac{1+\lambda(\alpha-1)+l}{1+l}\right)^m+\sum_{k=2}^{\infty}\left|\left(\frac{1+\lambda(\alpha+k-2)+l}{1+l}\right)^m\right||a_k(\alpha)||z^{k-1}|+\gamma|e^{i\theta}|\sum_{k=2}^{\infty}\left|\left(\frac{1+\lambda(\alpha+k-2)+l}{1+l}\right)^m\right||a_k(\alpha)||z^{k-1}|}{(2-\beta)\left(\frac{1+\lambda(\alpha-1)+l}{1+l}\right)^m-\sum_{k=2}^{\infty}\left(\frac{1+\lambda(\alpha+k-2)+l}{1+l}\right)^m|a_k(\alpha)||z^{k-1}|-\gamma\sum_{k=2}^{\infty}\left(\frac{1+\lambda(\alpha+k-2)+l}{1+l}\right)^m|a_k(\alpha)||z^{k-1}|}$$
$$\leq\frac{\beta\left(\frac{1+\lambda(\alpha-1)+l}{1+l}\right)^m+\sum_{k=2}^{\infty}\left|\left(\frac{1+\lambda(\alpha+k-2)+l}{1+l}\right)^m\right||a_k(\alpha)|+\gamma\sum_{k=2}^{\infty}\left|\left(\frac{1+\lambda(\alpha+k-2)+l}{1+l}\right)^m\right||a_k(\alpha)|}{(2-\beta)\left(\frac{1+\lambda(\alpha-1)+l}{1+l}\right)^m-\sum_{k=2}^{\infty}\left(\frac{1+\lambda(\alpha+k-2)+l}{1+l}\right)^m|a_k(\alpha)|-\gamma\sum_{k=2}^{\infty}\left(\frac{1+\lambda(\alpha+k-2)+l}{1+l}\right)^m|a_k(\alpha)|}$$
The last expression is bounded above by 1, if
$\beta\left(\frac{1+\lambda(\alpha-1)+l}{1+l}\right)^m+\sum_{k=2}^{\infty}\left|\left(\frac{1+\lambda(\alpha+k-2)+l}{1+l}\right)^m\right||a_k(\alpha)+\gamma\sum_{k=2}^{\infty}\left|\left(\frac{1+\lambda(\alpha+k-2)+l}{1+l}\right)^m\right||a_k(\alpha)|\leq$
$(2-\beta)\left(\frac{1+\lambda(\alpha-1)+l}{1+l}\right)^m-\sum_{k=2}^{\infty}\left(\frac{1+\lambda(\alpha+k-2)+l}{1+l}\right)^m|a_k(\alpha)|$
$-\gamma\sum_{k=2}^{\infty}\left(\frac{1+\lambda(\alpha+k-2)+l}{1+l}\right)^m|a_k(\alpha)|$
which is equivalent to condition (9).

setting $\lambda = 1$ in Theorem 2.1 we have the following
Corollary 2.1 *If $f(z) \in A$ satisfies the inequality*

$$\sum_{k=2}^{\infty} \Omega(m, \alpha, \beta, 1, \gamma, l, k)|a_k(\alpha)| \leq 2(1-\beta)\left(\frac{\alpha + l}{1+l}\right)^m$$

where

$$\Omega(m, \alpha, \beta, 1, \gamma, l, k) = \left|\left(\frac{\alpha + k - 1 + l}{1+l}\right)^m\right| + \left(\frac{\alpha + k - 1 + l}{1+l}\right)^m + 2\gamma\left|\left(\frac{\alpha + k - 1 + l}{1+l}\right)^m\right|$$

Then $f(z) \in T_m^{\alpha}(1, \beta, \gamma, l) = T_m^{\alpha}(\beta, \gamma, l)$.
Setting $\lambda = 1, l = 0$ in Theorem 2.1 we have
Corollary 2.2 *If $f(z) \in A$ satisfies*

$$\sum_{k=2}^{\infty} \Omega(m, \alpha, \beta, 1, \gamma, 0, k)|a_k(\alpha)| \leq 2\alpha^m(1-\beta)$$

where

$$\Omega(m, \alpha, \beta, 1, \gamma, 0, k) = |(\alpha + k - 1)^m| + (\alpha + k - 1)^m + 2\gamma|(\alpha + k - 1)^m|$$

then $f(z) \in T_m^{\alpha}(1, \beta, \gamma, 0) = T_m^{\alpha}(\beta, \gamma)$.
Also on setting $\lambda = 1, l = 0, \gamma = 0$ in Theorem 2.1 we have
Corollary 2.3 *If $f(z) \in A$ satisfies*

$$\sum_{k=2}^{\infty} \Omega(m, \alpha, \beta, 1, 0, 0, k)|a_k(\alpha)| \leq 2\alpha^m(1-\beta)$$

where

$$\Omega(m, \alpha, \beta, 1, 0, 0, k) = |(\alpha + k - 1)^m| + (\alpha + k - 1)^m$$

then $f(z) \in T_m^{\alpha}(\beta)$ which is the class of functions studied in [2,3].
Furthermore, setting $\lambda = 1, l = 0, \gamma = 0, \beta = 0$ in Theorem 2.1 we have
Corollary 2.4.*If $f(z) \in A$ satisfies*

$$\sum_{k=2}^{\infty} \Omega(m, \alpha, 0, 1, 0, 0, k)|a_k(\alpha)| \leq 2\alpha^m$$

where

$$\Omega(m, \alpha, 0, 1, 0, 0, k) = |(\alpha + k - 1)^m| + (\alpha + k - 1)^m$$

then $f(z) \in T_m^{\alpha}(0) \equiv B_n(\alpha)$ which is the class of functions studied by Abduhalim in [1].
On putting $m = 1$ in Corollary 2.4 we have
Corollary 2.5 *If $f(z) \in A$ satisfies the inequality*

$$\sum_{k=2}^{\infty} \Omega(1, \alpha, 0, 1, 0, 0, k)|a_k(\alpha)| \leq 2\alpha$$

where

$$\Omega(1,\alpha,0,1,0,0,k) = |(\alpha+k-1)| + (\alpha+k-1)$$

then $f(z) \in T_1^{\alpha}(0) \equiv B_1(\alpha)$ *which denote the class of Bazilevic functions with logarithmic growth see [4,7].*
Also setting $\alpha = 1$ in the last corollary we have the class $T_1^1(0,0,0,0) \equiv B_1(1)$ which is the class of close-to-convex functions. Selected choices of parameters involved give some new results and several exixting ones.
Theorem 2.2. $T_{m+1}^{\alpha}(\lambda,\beta,\gamma_1,l) \subset T_m^{\alpha}(\lambda,\beta,\gamma_2,l)$ *for some* γ_1 *and* γ_2*,* $0 \le \gamma_1 \le \gamma_2$ *and* $m \in N_0$*.*
Proof. For $0 \le \gamma_1 \le \gamma_2$*, we obtain*

$$\sum_{k=2}^{\infty}\Omega(m+1,\alpha,\lambda,\beta,\gamma_1,l,k)a_k(\alpha) \le \sum_{k=2}^{\infty}\Omega(m,\alpha,\lambda,\beta,\gamma_2,l,k)a_k(\alpha).$$

Therefore, if $f(z) \in T_m^{\alpha}(\lambda,\beta,\gamma_2,l)$ *then* $f(z) \in T_{m+1}^{\alpha}(\lambda,\beta,\gamma_1,l)$*. Hence we get the required result.*
For our next result we shall first state and proof the following:
Lemma 2.3. *If* $f(z) \in T_m^{\alpha}(\lambda,\beta,\gamma,l)$*, then we have*

$$\sum_{k=p+1}^{\infty} a_k(\alpha) \le \frac{2(1-\beta)\left(\frac{1+\lambda(\alpha-1)+l}{1+l}\right)^m - \sum_{k=2}^{p}\Omega(m,\lambda,\alpha,\beta,\gamma,k,l)a_k(\alpha)}{\Omega(m,\lambda,\alpha,\beta,\gamma,p+1,l)} \tag{12}$$

Proof: In view of Theorem 2.1, we can write

$$\sum_{k=p+1}^{\infty}\Omega(m,\lambda,\alpha,\beta,\gamma,k,l)a_k(\alpha) \le 2(1-\beta)\left(\frac{1+\lambda(\alpha-1)+l}{1+l}\right)^m - \sum_{k=2}^{p}\Omega(m,\lambda,\alpha,\beta,\gamma,k,l)a_k(\alpha)$$

Clearly $\Omega(m,\lambda,\alpha,\beta,\gamma,k,l)a_k(\alpha)$ is an increasing function for k. Then we have

$$\Omega(m,\lambda,\alpha,\beta,\gamma,p+1,l)\sum_{k=p+1}^{\infty} a_k(\alpha) \le 2(1-\beta)\left(\frac{1+\lambda(\alpha-1)+l}{1+l}\right)^m - \sum_{k=2}^{p}\Omega(m,\lambda,\alpha,\beta,\gamma,k,l)a_k(\alpha)$$

Thus we obtain

$$\sum_{k=p+1}^{\infty} a_k(\alpha) \le \frac{2(1-\beta)\left(\frac{1+\lambda(\alpha-1)+l}{1+l}\right)^m - \sum_{k=2}^{p}\Omega(m,\lambda,\alpha,\beta,\gamma,k,l)a_k(\alpha)}{\Omega(m,\lambda,\alpha,\beta,\gamma,p+1,l)} = A_k$$

With various choices of the parameters involved, the subclasses earlier mentioned in the cited literatures could be derived.
Theorem 2.3. Let $f(z) \in T_m^{\alpha}(\lambda,\beta,\gamma,l)$. Then for $|z| = r < 1$

$$r^{\alpha} - \sum_{k=2}^{\infty} a_k(\alpha)|z|^{\alpha+k-1} - A_k r^{\alpha+p} \le |f(z)^{\alpha}| \le r^{\alpha} + \sum_{k=2}^{\infty} a_k(\alpha)|z|^{\alpha+k-1} + A_k r^{\alpha+p} \tag{13}$$

where A_k is as given in Lemma 2.3. (13)

Proof. Let $f(z)$ be given by (1) and transform as in (7). For $|z| = r < 1$, using Lemma 2.3, we have

$$\begin{aligned}|f(z)^{\alpha}| &\leq |z|^{\alpha} + \sum_{k=2}^{\infty} a_k(\alpha)|z|^{\alpha+k-1} + \sum_{k=2}^{\infty} a_k(\alpha)|z|^{\alpha+k-1}\\ &\leq |z|^{\alpha} + \sum_{k=2}^{\infty} a_k(\alpha)|z|^{\alpha+k-1} + |z|^{\alpha+p}\sum_{k=2}^{\infty} a_k(\alpha)\\ &\leq r^{\alpha} + \sum_{k=2}^{\infty} a_k(\alpha)|z|^{\alpha+k-1} + A_k r^{\alpha+p}\end{aligned}$$

and

$$\begin{aligned}|f(z)^{\alpha}| &\geq |z|^{\alpha} - \sum_{k=2}^{\infty} a_k(\alpha)|z|^{\alpha+k-1} - \sum_{k=2}^{\infty} a_k(\alpha)|z|^{\alpha+k-1}\\ &\geq |z|^{\alpha} - \sum_{k=2}^{\infty} a_k(\alpha)|z|^{\alpha+k-1} - |z|^{\alpha+p}\sum_{k=2}^{\infty} a_k(\alpha)\\ &\geq r^{\alpha} - \sum_{k=2}^{\infty} a_k(\alpha)|z|^{\alpha+k-1} - A_k r^{\alpha+p}\end{aligned}$$

which completes the assertion of Theorem 2.3.

3. Coefficient bounds for functions in the class $T_m^{\alpha}(\lambda, \beta, \gamma, l)$

In this section we consider the coefficient bound for the functions in the class $T_m^{\alpha}(\lambda, \beta, \gamma, l)$ as we proof the following

Theorem 3.1. *If $f(z) \in T_m^{\alpha}(\lambda, \beta, \gamma, l)$, then*

$$|a_2| \leq \frac{2(1-\beta)}{\alpha|1-\gamma|\Psi_1^m} \tag{14}$$

$$|a_3| \leq \begin{cases} \frac{2(1-\beta)}{\alpha|1-\gamma|\Psi_2^m} - \frac{2(\alpha-1)(1-\beta)^2}{\alpha^2|1-\gamma|^2\Psi_1^{2m}}, 0 < \alpha < 1\\ \frac{2(1-\beta)}{\alpha|1-\gamma|\Psi_2^m},\ \alpha \geq 1\end{cases}$$

$$|a_4| \leq \begin{cases} \Omega_1 + \Omega_2 + \Omega_3, \alpha \in (0,1)\\ \Omega_1 + \Omega_3 + \Omega_4, \alpha \in [1,2)\\ \Omega_1 + \Omega_3, \alpha \in [2,\infty)\end{cases}$$

where

$$\Omega_1 = \frac{2(1-\beta)}{\alpha|1-\gamma|\Psi_3^m}$$

$$\Omega_2 = \frac{4(\alpha-1)(1-\beta)^2}{\alpha^2|1-\gamma|^2\Psi_2^m\Psi_1^m}$$

$$\Omega_3 = \frac{a(\alpha-1)^2(1-\beta)^2}{\alpha^3|1-\gamma|^3\Psi_1^{3m}}$$

$$\Omega_4 = \frac{4(1-\beta)^3(\alpha-1)(2-\alpha)}{3\alpha^3|1-\gamma|\Psi_1^{3m}}$$

$$|a_5| \leq \begin{cases} \Omega_1+\Omega_2+\Omega_4+\Omega_5+\Omega_6+\Omega_8, \alpha\in(0,1) \\ \Omega_1+\Omega_2+\Omega_7, \alpha\in[1,2) \\ \Omega_1+\Omega_2+\Omega_3+\Omega_8, \alpha\in[2,3) \\ \Omega_1+\Omega_2+\Omega_8, \alpha\in[3,\infty) \end{cases}$$

where

$$\Omega_1 = \frac{2(1-\beta)}{\alpha|1-\gamma|\Psi_4^m}-, \ \Omega_2 = \frac{12(\alpha-1)^2(1-\beta)^3}{\alpha^3|1-\gamma|^3\Psi_2^m\Psi_1^{2m}}$$

$$\Omega_3 = \frac{20(\alpha-1)^2(2-\alpha)(1-\beta)^4}{3\alpha^4|1-\gamma|^{42}\Psi_1^{4m}}, \ \Omega_4 = \frac{10(1-\alpha)^3(1-\beta)^4}{\alpha^4|1-\gamma|^4\Psi_1^{4m}}$$

$$\Omega_5 = \frac{4(1-\alpha)(1-\beta)^2}{\alpha^2|1-\gamma|^2\Psi_1^m\Psi_3^m}, \ \Omega_6 = \frac{2(1-\alpha)(1-\beta)^2}{\alpha^2|1-\gamma|\Psi_2^{2m}}$$

$$\Omega_7 = \frac{4(\alpha-1)(2-\alpha)(1-\beta)^3}{\alpha^3|1-\gamma|^3\Psi_1^{2m}\Psi_2^m}, \ \Omega_8 = \frac{2(\alpha-1)(\alpha-2)(3-\alpha)(1-\beta)^4}{3\alpha^4|1-\gamma|^2\Psi_1^{4m}}$$

and

$$\Psi_1^m = \left(\frac{1+\lambda\alpha+l}{1+\lambda(\alpha-1)+l}\right)^m$$
$$\Psi_2^m = \left(\frac{1+\lambda(\alpha+1)+l}{1+\lambda(\alpha-1)+l}\right)^m$$
$$\Psi_3^m = \left(\frac{1+\lambda(\alpha+2)+l}{1+\lambda(\alpha-1)+l}\right)^m$$
$$\Psi_4^m = \left(\frac{1+\lambda(\alpha+3)+l}{1+\lambda(\alpha-1)+l}\right)^m$$

Proof. *Note that, for* $f(z)\in T_m^\alpha(\lambda,\beta,\gamma,l)$,

$$Re\frac{I^m(\lambda,l)f(z)^\alpha}{\left(\frac{1+\lambda(\alpha-1)+l}{1+l}\right)^m z^\alpha} > \frac{\beta-\gamma}{1-\gamma}, \ (z\in U)$$

and all the papr ameters are as earlier defined.
If we define the function $p(z)$ by

$$p(z) = \frac{(1-\gamma)\frac{I^m(\lambda,l)f(z)^\alpha}{\left(\frac{1+\lambda(\alpha-1)+l}{1+l}\right)^m z^\alpha} - (\beta-\gamma)}{1-\beta} = 1 + p_1 z + p_2 z^2 + ... \tag{15}$$

Then $p(z)$ is analytic in U with $p(0) = 1$ and $Rep(z) > 0$, $z \in U$.
For the sake of clearity we let

$$f(z)^\alpha = z^\alpha[1 + \sum_{j=1}^{\infty} \alpha_j (a_2 z + a_3 z^2 + ...)^j] \tag{16}$$

where for convinience in the above we let

$$\alpha_j = \binom{\alpha}{j}, \; j = 1, 2, 3, ... \tag{17}$$

hence from (16) and (17) we have

$$p(z) = 1 + \alpha_1 \frac{\sigma_1}{B} a_2 z + (\alpha_1 a_3 + \alpha_2 a_2^2) \frac{\sigma_2}{B} z^2 + (\alpha_1 a_4 + 2\alpha_2 a_2 a_3 + \alpha_3 a_2^3) \frac{\sigma_3}{B} z^3 \\ + (\alpha_1 a_5 + \alpha_2(2a_2 a_4 + a_3^2) + 3\alpha_3 a_2^2 a_3 + \alpha_4 a_2^4) \frac{\sigma_4}{B} z^4 \tag{18}$$

where $\alpha_j (j = 1, 2, 3, ...)$ is as earlier defined in (17),

$$\Psi_j^m = \left(\frac{1 + \lambda(\alpha + j - 1) + l}{1 + \lambda(\alpha - 1) + l}\right)^m, \; j = 1, 2, 3, ...$$

$$B = \frac{1-\beta}{|1-\gamma|}$$

On comparing coefficients in (19) and using the fact the $|p_k| \le 2$, $k \ge 1$ the results follow and the proof is complete. On setting $\lambda = 1, l = 0$ in Theorem 3.1 we have

Corollary 3.1. *If $f(z) \in T_m^\alpha(1, \gamma, \beta, 0) \equiv T_m^\alpha(\gamma, \beta)$, for $\alpha > 0$, $0 \le \beta < 1$, $0 \le \gamma \le \beta$, or $\gamma > \frac{1+\beta}{2}$, $n = 0, 1, 2, ...$ then,*

$$|a_2| \le \frac{2(1-\beta)\alpha^{n-1}}{(\alpha+1)^n |1-\gamma|}$$

$$|a_3| \le \begin{cases} \frac{2(1-\beta)\alpha^{n-1}}{(\alpha+2)^n|1-\gamma|} - \frac{2(\alpha-1)(1-\beta)^2\alpha^{2n-2}}{(\alpha+1)^{2n}|1-\gamma|^2}, 0 < \alpha < 1 \\ \frac{2(1-\beta)\alpha^{n-1}}{(\alpha+2)^n|1-\gamma|}, \alpha \ge 1 \end{cases}$$

$$|a_4| \le \begin{cases} \Omega_1 + \Omega_2 + \Omega_3, \alpha \in (0, 1) \\ \Omega_1 + \Omega_3 + \Omega_4, \alpha \in [1, 2) \\ \Omega_1 + \Omega_3, \alpha \in [2, \infty) \end{cases}$$

where

$$\Omega_1 = \frac{2(1-\beta)\alpha^{m-1}}{(\alpha+3)^m|1-\gamma|}$$

$$\Omega_2 = \frac{4(\alpha-1)(1-\beta)^2\alpha^{2m-2}}{(\alpha-1)^m(\alpha+2)^m|1-\gamma|^2}$$

$$\Omega_3 = \frac{4(\alpha-1)^2(1-\beta)^3\alpha^{3m-3}}{(\alpha+1)^{3m}|1-\gamma|^3}$$

$$\Omega_4 = \frac{4(\alpha-1)(2-\alpha)(1-\beta)^3\alpha^{3m-3}}{3(\alpha+1)^{3m}|1-\gamma|^3}$$

$$|a_5| \leq \begin{cases} \Omega_1+\Omega_2+\Omega_4+\Omega_5+\Omega_6+\Omega_8, \alpha\in(0,1) \\ \Omega_1+\Omega_2+\Omega_7, \alpha\in[1,2) \\ \Omega_1+\Omega_2+\Omega_3+\Omega_8, \alpha\in[2,3) \\ \Omega_1+\Omega_2+\Omega_8, \alpha\in[3,\infty) \end{cases}$$

where

$$\Omega_1 = \frac{2(1-\beta)\alpha^{m-1}}{(\alpha+4)^m|1-\gamma|},\ \Omega_2 = \frac{12(\alpha-1)^2(1-\beta)^3\alpha^{3m-3}}{(\alpha+1)^{2m}(\alpha+2)^m|1-\gamma|^3}$$

$$\Omega_3 = \frac{20(\alpha-1)^2(2-\alpha)(1-\beta)^4\alpha^{4m-4}}{3(\alpha+1)^{4m}|1-\gamma|^4},\ \Omega_4 = \frac{10(1-\alpha)^3(1-\beta)^4\alpha^{4m-4}}{(\alpha+1)^{4m}|1-\gamma|^4}$$

$$\Omega_5 = \frac{4(1-\alpha)(1-\beta)^2\alpha^{2m-2}}{(\alpha+1)^m(\alpha+3)^m|1-\gamma|^2},\ \Omega_6 = \frac{2(1-\alpha)(1-\beta)^2\alpha^{2m-2}}{(\alpha+2)^{2m}|1-\gamma|^2}$$

$$\Omega_7 = \frac{4(\alpha-1)(2-\alpha)(1-\beta)^3\alpha^{3m-3}}{(\alpha+1)^{2m}(\alpha+2)^m|1-\gamma|^3},\ \Omega_8 = \frac{2(\alpha-1)(\alpha-2)(3-\alpha)(1-\beta)^4\alpha^{4m-4}}{3(\alpha+1)^{4m}|1-\gamma|^4},$$

On setting $\gamma = 0$ in Corollary 3.1, we have

Corollary 3.2. *If* $f(z) \in T_m^\alpha(1,0,\beta,0) \equiv T_m^\alpha(\beta)$,*for* $\alpha > 0,\ 0 \leq \beta < 1, 0 \leq \gamma \leq \beta,$ *or* $\gamma > \frac{1+\beta}{2}, n = 0,1,2,...$ *then,*

$$|a_2| \leq \frac{2(1-\beta)\alpha^{n-1}}{(\alpha+1)^n}$$

$$|a_3| \leq \begin{cases} \frac{2(1-\beta)\alpha^{n-1}}{(\alpha+2)^n} - \frac{2(\alpha-1)(1-\beta)^2\alpha^{2n-2}}{(\alpha+1)^{2n}}, 0<\alpha<1 \\ \frac{2(1-\beta)\alpha^{n-1}}{(\alpha+2)^n}, \alpha \geq 1 \end{cases}$$

$$|a_4| \leq \begin{cases} \Omega_1 + \Omega_2 + \Omega_3, \alpha \in (0,1) \\ \Omega_1 + \Omega_3 + \Omega_4, \alpha \in [1,2) \\ \Omega_1 + \Omega_3, \alpha \in [2,\infty) \end{cases}$$

where

$$\Omega_1 = \frac{2(1-\beta)\alpha^{m-1}}{(\alpha+3)^m}$$

$$\Omega_2 = \frac{4(\alpha-1)(1-\beta)^2\alpha^{2m-2}}{(\alpha-1)^m(\alpha+2)^m}$$

$$\Omega_3 = \frac{4(\alpha-1)^2(1-\beta)^3\alpha^{3m-3}}{(\alpha+1)^{3m}}$$

$$\Omega_4 = \frac{4(\alpha-1)(2-\alpha)(1-\beta)^3\alpha^{3m-3}}{3(\alpha+1)^{3m}}$$

$$|a_5| \leq \begin{cases} \Omega_1 + \Omega_2 + \Omega_4 + \Omega_5 + \Omega_6 + \Omega_8, \alpha \in (0,1) \\ \Omega_1 + \Omega_2 + \Omega_7, \alpha \in [1,2) \\ \Omega_1 + \Omega_2 + \Omega_3 + \Omega_8, \alpha \in [2,3) \\ \Omega_1 + \Omega_2 + \Omega_8, \alpha \in [3,\infty) \end{cases}$$

where

$$\Omega_1 = \frac{2(1-\beta)\alpha^{m-1}}{(\alpha+4)^m},\ \Omega_2 = \frac{12(\alpha-1)^2(1-\beta)^3\alpha^{3m-3}}{(\alpha+1)^{2m}(\alpha+2)^m}$$

$$\Omega_3 = \frac{20(\alpha-1)^2(2-\alpha)(1-\beta)^4\alpha^{4m-4}}{3(\alpha+1)^{4m}|1-\gamma|^4},\ \Omega_4 = \frac{10(1-\alpha)^3(1-\beta)^4\alpha^{4m-4}}{(\alpha+1)^{4m}|1-\gamma|^4}$$

$$\Omega_5 = \frac{4(1-\alpha)(1-\beta)^2\alpha^{2m-2}}{(\alpha+1)^m(\alpha+3)^m},\ \Omega_6 = \frac{2(1-\alpha)(1-\beta)^2\alpha^{2m-2}}{(\alpha+2)^{2m}}$$

$$\Omega_7 = \frac{4(\alpha-1)(2-\alpha)(1-\beta)^3\alpha^{3m-3}}{(\alpha+1)^{2m}(\alpha+2)^m},\ \Omega_8 = \frac{2(\alpha-1)(\alpha-2)(3-\alpha)(1-\beta)^4\alpha^{4m-4}}{3(\alpha+1)^{4m}},$$

With different choices of the parameters involved, various coefficient bounds for the classes of functions in the cited literatures could be derived.

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