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ON NONSINGULARITY OF LINEAR COMBINATIONS OF TRIPOTENT MATRICES

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Abstract. Let $\mathbf{T}_1$ and $\mathbf{T}_2$ be two commuting $n \times n$ tripotent matrices and c_1, c_2 two nonzero complex numbers. The problem of when a linear combination of the form $\mathbf{T} = c_1\mathbf{T}_1 + c_2\mathbf{T}_2$ is nonsingular is considered. Some other nonsingularity-type relationships for tripotent matrices are also established. Moreover, a statistical interpretation of the results is pointed out.

Keywords: Idempotent matrix, Tripotent matrix, Linear combination, Nonsingularity, Chi-square distribution.

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1. Introduction

Let $\mathcal{C}$ be the field of complex numbers and $\mathcal{C}^* = \mathcal{C} \setminus \{0\}$. For a positive integer n, let $\mathcal{M}_n$ be the set of all $n \times n$ complex matrices over $\mathcal{C}$. Moreover, let $\mathcal{P}$ and $\mathcal{F} \subset \mathcal{M}_n$ be the set of all $n \times n$ complex idempotent matrices $\left(\mathcal{P} = \left\{\mathbf{P} \in \mathcal{M}_n : \mathbf{P} = \mathbf{P}^2\right\}\right)$ and the set of all $n \times n$ complex tripotent matrices $\left(\mathcal{F} = \left\{\mathbf{T} \in \mathcal{M}_n : \mathbf{T} = \mathbf{T}^3\right\}\right)$, respectively. The symbols $\mathcal{R}(\mathbf{A})$ and $\mathcal{N}(\mathbf{A})$ stand for the range and null space of $\mathbf{A} \in \mathcal{M}_n$.

Groß and Trenkler gave characterizations of the nonsingularity of the difference $\mathbf{P}_1 - \mathbf{P}_2$ and the sum $\mathbf{P}_1 + \mathbf{P}_2$ in [1, Corollaries 1 and 4] in terms of their ranks, and range and null spaces of either $\mathbf{P}_1$ and $\mathbf{P}_2$ directly or matrices being their functions. These characterizations were alternatively established by Koliha et al. [2] using the fact that

$$\mathbf{A} \in \mathcal{M}_n \text{ is nonsingular} \Leftrightarrow \mathcal{N}(\mathbf{A}) = \{\mathbf{0}\}.$$

They also gave explicitly a condition, which combined with the nonsingularity of $\mathbf{P}_1 + \mathbf{P}_2$ leads to the nonsingularity of $\mathbf{P}_1 - \mathbf{P}_2$ [2, Theorem 2.1]. The observation in [1, p. 393] was strengthened by this result.

Baksalary and Baksalary showed that if a linear combination $\tilde{c}_1\mathbf{P}_1 + \tilde{c}_2\mathbf{P}_2$ is nonsingular for some $\tilde{c}_1, \tilde{c}_2 \in \mathcal{C}^*$ satisfying $\tilde{c}_1 + \tilde{c}_2 \neq 0$, then $c_1\mathbf{P}_1 + c_2\mathbf{P}_2$ is nonsingular for all $c_1,\ c_2 \in \mathcal{C}^*$ satisfying $c_1 +\ c_2 \neq 0$ [3, Theorem 1]. Of course, this result is stronger than above. They also gave some other results concerning the nonsingularity of linear combinations of $\mathbf{P}_1$ and $\mathbf{P}_2$.

In this study, some results about the nonsingularity of linear combinations of tripotent matrices are obtained which are similar to the ones obtained by Baksalary and Baksalary for idempotent matrices in [3]. In other words, it is concerned with nontrivial linear combinations of $\mathbf{T}_1, \mathbf{T}_2 \in \mathcal{F}$, i.e., with matrices of the form

$$\mathbf{T} = c_1\mathbf{T}_1 + c_2\mathbf{T}_2, \quad c_1, c_2 \in \mathcal{C}^*. \tag{1}$$

Naturally, the results given involve some additional conditions since an idempotent matrix is always a tripotent matrix but a tripotent matrix may not be idempotent. Special types of matrices, as idempotent, tripotent, and etc., and their functions are very useful in many contexts and have been extensively studied in the literature. For example, it is well known that quadratic forms with idempotent matrices are used extensively in statistical theory. So, it is worth to stress that the results given are also useful in statistical theory because of the fact that a symmetric tripotent matrix can be written as the difference of two disjoint idempotent matrices.

2. Main Results

As we already pointed out, the main results we obtained, which are similar to the ones obtained by Baksalary and Baksalary for idempotent matrices in [3], deal with the nonsingularity of linear combinations of tripotent matrices. Since an idempotent matrix is already a tripotent matrix, we think that it is suitable to start with repeating the observation in [3]:

Even when all matrices $\mathbf{T}$ of the form (1), excluding merely $\mathbf{T} = c_1\mathbf{T}_1 - c_1\mathbf{T}_2$ are nonsingular, the matrix $\mathbf{T}_1 - \mathbf{T}_2$ need not be so. For example, if

$$\mathbf{T}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\mathbf{T}_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$$

then

$$\mathbf{T} = c_1\mathbf{T}_1 + c_2\mathbf{T}_2 = \begin{pmatrix} c_1 + c_2 & c_2 \\ 0 & c_1 \end{pmatrix}$$

is nonsingular for all $c_1, c_2 \in \mathcal{C}^*$ such that $c_1+c_2 \neq 0$, but $\mathbf{T}_1-\mathbf{T}_2$ is clearly a singular matrix. Again, as in [3], it must be of interest to ask if there is any relationship between members of the family $\{c_1\mathbf{T}_1 + c_2\mathbf{T}_2 : c_1, c_2 \in \mathcal{C}^*\}$ and the subfamily of that being $\{c_1\mathbf{T}_1 + c_2\mathbf{T}_2 : c_1, c_2 \in \mathcal{C}^* \text{ and } c_1 + c_2 \neq 0\}$. This is the context of the theorem below.

Theorem 2.1. *Let $\mathbf{T}_1, \mathbf{T}_2 \in \mathcal{F}$ be two tripotent matrices such that $\mathbf{T}_1^2\mathbf{T}_2 = \mathbf{T}_2^2\mathbf{T}_1$. If a linear combination $\tilde{c}_1\mathbf{T}_1 + \tilde{c}_2\mathbf{T}_2$ is nonsingular for some $\tilde{c}_1, \tilde{c}_2 \in \mathcal{C}^*$ satisfying $\tilde{c}_1 + \tilde{c}_2 \neq 0$, then $c_1\mathbf{T}_1 + c_2\mathbf{T}_2$ is nonsingular for all $c_1, c_2 \in \mathcal{C}^*$ satisfying $c_1 + c_2 \neq 0$.*

Proof. Let $c_1, c_2 \in \mathcal{C}^*$ satisfying $c_1 + c_2 \neq 0$. It is then possible to write

$$c_1\mathbf{T}_1\mathbf{x} = -c_2\mathbf{T}_2\mathbf{x} = c_1\mathbf{T}_2^2\mathbf{T}_1\mathbf{x} = -c_2\mathbf{T}_1^2\mathbf{T}_2\mathbf{x}, \tag{2}$$

if we consider $\mathbf{x} \in \mathcal{N}(c_1\mathbf{T}_1 + c_2\mathbf{T}_2)$, and then, taking into account $\mathbf{T}_1^2\mathbf{T}_2 = \mathbf{T}_2^2\mathbf{T}_1$, it is obtained

$$\mathbf{T}_1\mathbf{x} = \mathbf{T}_2^2\mathbf{T}_1\mathbf{x} = \mathbf{0} = \mathbf{T}_1^2\mathbf{T}_2\mathbf{x} = \mathbf{T}_2\mathbf{x}. \tag{3}$$

From (3) under the assumption that $\tilde{c}_1 + \tilde{c}_2 \neq 0$, we get

$$(\tilde{c}_1\mathbf{T}_1 + \tilde{c}_2\mathbf{T}_2)\,\mathbf{x} = (\tilde{c}_1\mathbf{T}_1 + \tilde{c}_2\mathbf{T}_2)^2\,\mathbf{x}$$

and thus, under the condition that $\tilde{c}_1\mathbf{T}_1 + \tilde{c}_2\mathbf{T}_2$ is nonsingular,

$$\mathbf{x} = (\tilde{c}_1\mathbf{T}_1 + \tilde{c}_2\mathbf{T}_2)\,\mathbf{x}. \tag{4}$$

Consequently, (4) implies $\mathbf{x} = \mathbf{0}$. Hence, $\mathcal{N}(c_1\mathbf{T}_1 + c_2\mathbf{T}_2) = \{\mathbf{0}\}$ is obtained which completes the proof.

When $\mathbf{P}_1$ and $\mathbf{P}_2$ are idempotent, a necessary and sufficient condition for $\mathbf{P}_1-\mathbf{P}_2$ to be nonsingular is that $\mathbf{P}_1 + \mathbf{P}_2$ and $\mathbf{I} - \mathbf{P}_1\mathbf{P}_2$ are nonsingular [2, Theorem 2.1]. Baksalary and Baksalary strengthened this result by taking any linear combination of the form $c_1\mathbf{P}_1+c_2\mathbf{P}_2$ instead of $\mathbf{P}_1+\mathbf{P}_2$, where $c_1+c_2 \neq 0$ [3, Theorem 2]. In the following theorem, a similar result is established for tripotent matrices satisfying an additional condition.

Theorem 2.2. *Let $\mathbf{T}_1, \mathbf{T}_2 \in \mathcal{F}$ be two commuting tripotent matrices and $c_1, c_2 \in \mathcal{C}^*$ be any scalars. If $c_1\mathbf{T}_1 + c_2\mathbf{T}_2$ and $\mathbf{I} - \mathbf{T}_1\mathbf{T}_2$ are nonsingular, then $\mathbf{T}_1 - \mathbf{T}_2$ is nonsingular. Moreover, when the condition of $\mathbf{T}_1^2\mathbf{T}_2 = \mathbf{T}_2^2\mathbf{T}_1$ is fulfilled the converse of this implication is also true.*

Proof. Let $\mathbf{x} \in \mathcal{N}(\mathbf{T}_1 - \mathbf{T}_2)$, then

$$\mathbf{T}_1\mathbf{x} = \mathbf{T}_2\mathbf{x}. \tag{5}$$

Premultiplying both sides of (5) first by $c_1\mathbf{T}_1^2$ and then by $c_2\mathbf{T}_2^2$ and combining the equations obtained leads to

$$(c_1\mathbf{T}_1 + c_2\mathbf{T}_2)(\mathbf{I} - \mathbf{T}_1\mathbf{T}_2)\,\mathbf{x} = \mathbf{0}. \tag{6}$$

Under the assumption that $c_1\mathbf{T}_1 + c_2\mathbf{T}_2$ and $\mathbf{I} - \mathbf{T}_1\mathbf{T}_2$ are nonsingular, (6) yields $\mathbf{x} = \mathbf{0}$. This means that $\mathcal{N}(\mathbf{T}_1 - \mathbf{T}_2) = \{\mathbf{0}\}$, thus establishing the first part of the proof.

For the second part of the proof, from the proof of the Theorem 2.1 it is known that if $\mathbf{x} \in \mathcal{N}(c_1\mathbf{T}_1 + c_2\mathbf{T}_2)$, then $\mathbf{x}$ satisfies the equalities in (2). Hence, taking into account $\mathbf{T}_1^2\mathbf{T}_2 = \mathbf{T}_2^2\mathbf{T}_1$, it is seen that $(\mathbf{T}_1 - \mathbf{T}_2)\,\mathbf{x} = \mathbf{0}$. Thus, $\mathbf{x} = \mathbf{0}$ is obtained since $\mathbf{T}_1 - \mathbf{T}_2$ is nonsingular. Moreover, if $\mathbf{x} \in \mathcal{N}(\mathbf{I} - \mathbf{T}_1\mathbf{T}_2)$, then

$$\mathbf{x} = \mathbf{T}_1\mathbf{T}_2\mathbf{x}. \tag{7}$$

Premultiplying both sides of (7) first by $\mathbf{T}_1$ and then by $\mathbf{T}_2$ and combining the equalities obtained yields $(\mathbf{T}_1 - \mathbf{T}_2)\,\mathbf{x} = \mathbf{0}$, i.e., $\mathbf{x} = \mathbf{0}$. This completes the proof.

In case $\mathbf{P}_1,\ \mathbf{P}_2 \in \mathcal{P}$, a necessary and sufficient condition for $\mathbf{P}_1 + \mathbf{P}_2$ to be nonsingular is that $\mathcal{R}(\mathbf{P}_1) \cap \mathcal{R}[\mathbf{P}_2(\mathbf{I} - \mathbf{P}_1)] = \{\mathbf{0}\}$ and $\mathcal{N}(\mathbf{P}_1) \cap \mathcal{N}(\mathbf{P}_2) = \{\mathbf{0}\}$ [2, Corollary 4]. In the more general case, a necessary and sufficient condition for $c_1\mathbf{P}_1 + c_2\mathbf{P}_2$ is nonsingular is that $\mathcal{N}(\mathbf{P}_1) \cap \mathcal{N}(\mathbf{P}_2) = \{\mathbf{0}\}$ and $\mathcal{R}[\mathbf{P}_1(\mathbf{I} - \mathbf{P}_2)] \cap \mathcal{R}[\mathbf{P}_2(\mathbf{I} - \mathbf{P}_1)] = \{\mathbf{0}\}$ [3, Theorem 3], where $c_1, c_2 \in \mathcal{C}^*$. For tripotent matrices a similar results are given by the following.

Theorem 2.3. *Let $\mathbf{T}_1, \mathbf{T}_2 \in \mathcal{F}$ be two commuting tripotent matrices and $c_1, c_2 \in \mathcal{C}^*$ be any scalars. If $c_1\mathbf{T}_1 + c_2\mathbf{T}_2$ is nonsingular, then $\mathcal{N}(\mathbf{T}_1) \cap \mathcal{N}(\mathbf{T}_2) = \{\mathbf{0}\}$. Moreover, if $\mathcal{N}(\mathbf{T}_1) \cap \mathcal{N}(\mathbf{T}_2) = \{\mathbf{0}\}$ and $\mathcal{R}[\mathbf{T}_1(\mathbf{I} - \mathbf{T}_1\mathbf{T}_2)] \cap \mathcal{R}[\mathbf{T}_2(\mathbf{I} - \mathbf{T}_1\mathbf{T}_2)] = \{\mathbf{0}\}$, then $c_1\mathbf{T}_1 + c_2\mathbf{T}_2$ is nonsingular.*

Proof. If $\mathbf{x} \in \mathcal{N}(\mathbf{T}_1) \cap \mathcal{N}(\mathbf{T}_2)$, then $\mathbf{T}_1\mathbf{x} = \mathbf{0} = \mathbf{T}_2\mathbf{x}$ and hence

$$(c_1\mathbf{T}_1 + c_2\mathbf{T}_2)\,\mathbf{x} = \mathbf{0}. \tag{8}$$

Under the assumption that $c_1\mathbf{T}_1 + c_2\mathbf{T}_2$ is nonsingular, $\mathbf{x} = \mathbf{0}$.

For the other part of the proof notice that if $\mathbf{x} \in \mathcal{N}(c_1\mathbf{T}_1 + c_2\mathbf{T}_2)$, then from the equation (2)

$$(c_1 + c_2)\,\mathbf{T}_1\mathbf{x} = c_2\mathbf{T}_1(\mathbf{I} - \mathbf{T}_1\mathbf{T}_2)\,\mathbf{x} = -c_2\mathbf{T}_2(\mathbf{I} - \mathbf{T}_1\mathbf{T}_2)\,\mathbf{x} \tag{9}$$

which shows that $\mathbf{T}_1\mathbf{x} \in \mathcal{R}[\mathbf{T}_1(\mathbf{I} - \mathbf{T}_1\mathbf{T}_2)] \cap \mathcal{R}[\mathbf{T}_2(\mathbf{I} - \mathbf{T}_1\mathbf{T}_2)]$. In view of the assumption that $\mathcal{R}[\mathbf{T}_1(\mathbf{I} - \mathbf{T}_1\mathbf{T}_2)] \cap \mathcal{R}[\mathbf{T}_2(\mathbf{I} - \mathbf{T}_1\mathbf{T}_2)] = \{\mathbf{0}\}$, $\mathbf{T}_1\mathbf{x} = \mathbf{0}$ is obtained.

Combining this with equation (2) leads to $\mathbf{T}_2\mathbf{x} = \mathbf{0}$. So, $\mathbf{x} \in \mathcal{N}(\mathbf{T}_1) \cap \mathcal{N}(\mathbf{T}_2)$, and thus $\mathbf{x} = \mathbf{0}$. Hence the proof is complete.

The following theorem shows that the nonsingularity of a linear combination $\mathbf{T} = c_1\mathbf{T}_1 + c_2\mathbf{T}_2$ is also related to the nonsingularity of the same linear combinations of $\mathbf{T}_1\mathbf{T}_2^2$ and $\mathbf{T}_2\mathbf{T}_1^2$.

Theorem 2.4. *Let* $\mathbf{T}_1, \mathbf{T}_2 \in \mathcal{F}$ *and* $c_1, c_2 \in \mathcal{C}^*$. *The following statements are equivalent:*

(a) $c_1\mathbf{T}_1\mathbf{T}_2^2 + c_2\mathbf{T}_2\mathbf{T}_1^2$ *is nonsingular,*

(b) $c_1\mathbf{T}_1 + c_2\mathbf{T}_2$ *and* $\mathbf{I} - \mathbf{T}_1^2 - \mathbf{T}_2^2$ *are nonsingular.*

Proof. The result follows quite easily from the equality

$$(c_1\mathbf{T}_1 + c_2\mathbf{T}_2)\left(\mathbf{I} - \mathbf{T}_1^2 - \mathbf{T}_2^2\right) = -\left(c_1\mathbf{T}_1\mathbf{T}_2^2 + c_2\mathbf{T}_2\mathbf{T}_1^2\right).$$

We want to conclude this study by giving a statistical interpretation. In addition to purely algebraic aspects, if the problems considered in this note and [4] which deal with the nonsingularity and tripotency of linear combinations of tripotent matrices of the form (1) are taken into account together, then it can be given an interpretation from a statistical point of view. A possibility of such an interpretation follows from the fact that if $\mathbf{A}$ is $n \times n$ real symmetric matrix and $\mathbf{x}$ is an $n \times 1$ real random vector having the multivariate normal distribution $N_n(\mathbf{0}, \mathbf{I})$, where $\mathbf{I}$ stands for the identity matrix of order n and $\mathbf{0}$ stands for the zero vector, then necessary and sufficient conditions for the quadratic form $\mathbf{x}'\mathbf{A}\mathbf{x}$ to be distributed as a difference of two independent χ^2-variables with the degree of freedom n_1 and n_2, respectively, satisfying $n_1 + n_2 = n$ are the nonsingularity and tripotency of $\mathbf{A}$. See, for example, [4], [5], and [6] for similar interpretations and details.

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