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A NEW APPROACH TO INTUITIONISTIC COMPACTNESS

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Abstract. The basic concepts of the theory of intuitionistic fuzzy topological spaces are defined by D.Coker and coworkers. In a recent paper, we define the notion of prefilters in intuitionistic fuzzy sets and obtain some of its properties. The purpose of this paper is to introduce a new concept of compactness for intuitionistic fuzzy topological spaces.

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1.Introduction

The notion of intuitionistic fuzzy set is defined by K.T.Atanassov [1] and the theory has been developed by various authors. In particular D.Coker [4] has defined the intuitionistic fuzzy topological spaces. Blasco Mardones et al [3] introduced a new process of compactification for a fuzzy topological space .In this paper we define prime prefilter and prove some of its relationship with prefilters and ultra filters .Next, we use these ideas to study compactness in intuitionistic fuzzy topological spaces. Here we give a brief review of some preliminaries.

Definition 1.1[1]*Let X be a nonempty set. An intuitionistic fuzzy set(IFS for short) A is an object having the form $A : \{\langle x, \mu(x), \nu(x) : x \in X\rangle\}$ where the functions $\mu_A : X \to I$ and $\nu_A : X \to I$ denote the degree of membership(namely $\mu_A(x)$) and the degree of non membership (namely $\nu_A(x)$)of each element $x \in X$ to the set A , respectively, and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$.*

Definition 1.2[2]*Let X be a nonempty set and let A, B be two IFSs of X.Then*

(i) $A \subseteq B$ *iff* $\mu_A(x) \leq \mu_A(x)$ *and* $\nu_A(x) \geq \nu_B(x)$ *for each* $x \in X$

(ii) $A = B$ *iff* $A \subseteq B$ *and* $B \subseteq A$

(iii) $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)\rangle : x \in X\}$

(iv) $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)\rangle : x \in X\}$

(v) $\bar{A} = \{\langle x, \nu_A(x)), \mu_A(x)\rangle : x \in X\}$

Definition 1.3[4]*Let* $\{A_i : i \in J\}$ *be an arbitrary family of IFSs in* X *Then*

(i) $\bigcap A_i = \{\langle x, \bigwedge \mu_{A_i}, \bigvee \nu_{A_i}\rangle : x \in X\}$

(ii) $\bigcap A_i = \{\langle x, \bigvee \mu_{A_i}, \bigwedge \nu_{A_i}\rangle : x \in X\}$

Definition 1.4[4] $0_\sim = \{\langle x, 0, 1)\rangle : x \in X\}$ *and* $1_\sim = \{\langle x, 1, 0)\rangle : x \in X\}$

Definition 1.5[4]*An intuitionistic fuzzy topology (IFT for short) on a nonempty set* X *is a family* τ *of IFSs in* X *satisfying the following axioms:*

(i) $0_\sim, 1_\sim \in \tau$

(ii) $A_1 \cap A_2 \in \tau$ *for any* $A_1, A_2 \in \tau$

(iii) $\bigcup A_i \in \tau$ *for any arbitrary family* $\{A_i : i \in J\}$
In this case the pair (X, τ) *is called an intuitionistic fuzzy topological space (IFTS for short) and any IFS in* τ *is known as an intuitionistic fuzzy open set (IFOS for short) in* X *.*

Definition 1.6 [4]*An intuitionistic fuzzy topological space is a pair* (X, τ) *where* (X, τ) *is an IFTS and each IFS in the form* $C_{\alpha,\beta} = \{\langle x, \alpha, \beta)\rangle : x \in X\}$ *where* $\alpha, \beta \in I$ *are arbitrary and* $\alpha + \beta \leq 1$*,belongs to* τ^c*.*

Definition 1.7 [4]*The complement* $\bar{A}$ *of an IFOS A in an IFTS* (X, τ) *is called an intuitionistic fuzzy closed set (IFCS for short) in* X *.*

Definition 1.8 [5]*The support of a fuzzy set* A *is a crisp set that contains all the elements of* X *that have nonzero membership grades in* A *.*

Definition 1.9 [6]*Let* (X, τ) *be an IFTS.Let* $\mathcal{F} \subset \tau^c$ *satisfies*

(i) $\mathcal{F} \neq 0_\sim$ *and* $0_\sim \notin \mathcal{F}$

(ii) $A_1, A_2 \in \tau$ *then* $A_1 \cap A_2 \in \tau$

(iii) If $A \in \mathcal{F}$ *and* $B \in \tau^c$ *with* $A \subseteq B$ *then* $B \in \mathcal{F}$
$\mathcal{F}$ *is called an IF closed filter or a* τ^c *- prefilter on* X*.*

Definition 1.10 [6]*Let* $\mathcal{F}$ *be* τ^c *prefilter.* $\mathcal{F}$ *is an intuitionistic fuzzy* τ^c *ultra filter if* $\mathcal{F}$ *is a maximal element in the set of* τ^c *prefilters ordered by the inclusion relation.*

Theorem 1.11 **[6]***Every τ^c prefilter is contained in some intuitionistic fuzzy τ^c ultra filter.*

Theorem 1.12 [**6**] *Let $\mathcal{F}$ be τ^c prefilter on X.The following statements are equivalent:*

(i) *$\mathcal{F}$ is an intuitionistic fuzzy τ^c ultra filter*

(ii) *If A is an element of τ^c such that $A \cap B \neq 0_\sim$ for each $B \in \mathcal{F}$ then $A \in \mathcal{F}$.*

(ii) *If $A \in \tau^c$ and $A \notin \mathcal{F}$, then there is $B \in \mathcal{F}$ such that $Supp(B) \subseteq X - Supp(A)$.*

Definition 1.13 [**4**] *Let (X, τ) be IFTs.*

(i) *If a family $\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle : i \in J\}$ of IFOSs in X satisfies the condition $\bigcup\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle : i \in J\} = 1_\sim$ then it is called a fuzzy open cover of X. A finite subfamily of a fuzzy open cover $\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle : i \in J\}$ of X , which is also a fuzzy open cover of X, is called a finite sub cover of $\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle : i \in J\}$.*

(ii) *A family $\{\langle x, \mu_{k_i}, \nu_{k_i}\rangle : i \in J\}$ of IFCSs in X satisfies the finite intersection property iff every finite sub family $\{\langle x, \mu_{k_i}, \nu_{k_i}\rangle : i = 1, 2, ..n\}$of the family satisfies the condition $\bigcap\{\langle x, \mu_{k_i}, \nu_{k_i}\rangle : i = 1, 2, ..n\} \neq 0_\sim$.*

Definition 1.14 [**4**] *An IFTS (X, τ) is called fuzzy compact iff every fuzzy open cover of X has a finite subcover.*

Theorem 1.15 [**4**] *An IFTS (X, τ) is fuzzy compact iff every family $\{\langle x, \mu_{k_i}, \nu_{k_i}\rangle : i \in J\}$ of IFCS's in X having the finite intersection property has a non empty intersection.*

2.Intuitionistic Compactness

In this section, intuitionistic prime prefilter is introduced and its relationship with prefilters and ultra filters are studied.

Definition 2.1*Let $\mathcal{F}$ be a τ^c prefilter on X. $\mathcal{F}$ is said to be intuitionistic prime τ^c if given $A, B \in \tau^c$ such that $A \cup B \in \mathcal{F}$,then $A \in \mathcal{F}$ or $B \in \mathcal{F}$.*

Theorem 2.2*Every intuitionistic fuzzy τ^c ultra filter $\mathcal{U}$ on X is an intuitionistic prime τ^c prefilter*

Proof. Let $A, B \in \tau^c$ be such that $A \cup B \in \mathcal{U}$. Suppose both $A, B \notin \mathcal{U}$.
Since $A \notin \mathcal{U}$, by [6] there exists $A^\star \in \mathcal{U}$ such that $Supp(A^\star) \subseteq X - Supp(A)$.
Therefore $Supp(A^\star) \cap Supp(A) = 0_\sim$.
Similarly since $B \notin \mathcal{U}$, there exists $B^\star \in \mathcal{U}$ such that $Supp(B^\star) \subseteq X - Supp(B)$
Therefore $Supp(B^\star) \cap Supp(B) = 0_\sim$.
Since $A \cup B$, $A^\star, B^\star$ belong to $\mathcal{U}$ we get $(A \cup B) \cap A^\star \cap B^\star$ belong to $\mathcal{U}$.
Consider
$Supp[(A \cup B) \cap A^\star \cap B^\star] = [Supp(A \cup B)] \cap Supp(A^\star) \cap Supp(B^\star)$
$\subseteq [(Supp(A) \cup Supp(B)] \cap Supp(A^\star) \cap Supp(B^\star)$
$= [Supp(A) \cap Supp(A^\star) \cap Supp(B^\star)] \cup [Supp(B) \cap Supp(A^\star) \cap Supp(B^\star)]$
$= 0_\sim$.
Thus $0_\sim \in \mathcal{U}$. This is a contradiction. Hence either $A \in \mathcal{U}$ or $B \in \mathcal{U}$.
Therefore $\mathcal{U}$ is an intuitionistic prime τ^c prefilter.

Theorem 2.3 *Let $\mathcal{F}$ be a prefilter. Let $\mathcal{P}(\mathcal{F})$ be the family of all prime prefilters which contain $\mathcal{F}$. Then $\mathcal{F} = \bigcap_{\mathcal{G} \in \mathcal{P}(\mathcal{F})} \mathcal{G}$.*

Proof. Since $\mathcal{F} \subset \mathcal{G}$ for every $\mathcal{G} \in \mathcal{P}(\mathcal{F})$, $\mathcal{F} \subset \bigcap_{\mathcal{G} \in \mathcal{P}(\mathcal{F})} \mathcal{G}$.
To prove the other way inclusion take $A \in \tau^c$ such that $A \notin \mathcal{F}$.
It is enough if we prove $A \notin \bigcap_{\mathcal{G} \in \mathcal{P}(\mathcal{F})} \mathcal{G}$.
Consider the family $\mathcal{S} = \{\mathcal{G} : \mathcal{G}\ is\ a\ \tau^c\ prefilter, \mathcal{F} \subset \mathcal{G}\ and A \notin \mathcal{F}\}$
Let $\mathcal{U}$ be a maximal element in $\mathcal{S}$. Next we prove $\mathcal{U}$ is an intuionistic prime τ^c prefilter.
Let $D_1, D_2 \in \tau^c$ with $D_1 \cup D_2 \in U$. We have to show that either $D_1 \in U$ or $D_2 \in U$.
Assume that both $D_1 and D_2 \notin \mathcal{U}$. To get the contradiction we need to prove the following:
Consider the family $\mathcal{L} = \{B \in \tau^c : B \cup D_2 \in U\}$
Then

(i) $\mathcal{L}$ is a prefilter
Since $D_1 \cup D_2 \in U$, $D_1 \in \mathcal{L}$. Hence $\mathcal{L} \neq 0_\sim$.
Consider $0_\sim \cup D_2 = D_2 \notin \mathcal{U}$. Hence $0_\sim \notin \mathcal{L}$.
Take $B_1, B_2 \in \mathcal{L}$. Then $B_1 \cup D_2 \in \mathcal{U}$ and $B_2 \cup D_2 \in \mathcal{U}$.
Then $(B_1 \cup D_2) \cap (B_2 \cup D_2) \in \mathcal{U}$. That is $(B_1 \cap B_2) \cup D_2 \in \mathcal{U}$. Hence $(B_1 \cap B_2) \in \mathcal{L}$.
Take $B \in \mathcal{L}$ and $D \in \tau^c$ be such that $B \subset D$.
Consider $(B \cup D_2) \subset (D \cup D_2)$
Since $(B \cup D_2) \in \mathcal{U}$ and $\mathcal{U}$ is a τ^c prefilter, $(D \cup D_2) \in \mathcal{U}$.
Therefore $D \in \mathcal{L}$. Hence $\mathcal{L}$ is a prefilter.

(ii) $\mathcal{U} \subset \mathcal{L}$
Take $B \in \mathcal{U}$. $B \subset (B \cup D_2)$ we get $B \cup D_2 \in \mathcal{U}$.By definition $B \in \mathcal{L}$.Therefore $\mathcal{U} \subset \mathcal{L}$.

(iii) $\mathcal{U} \neq \mathcal{L}$
Since $D_1 \in \mathcal{L}$ and $D_1 \notin \mathcal{U}$ we get $\mathcal{U} \neq \mathcal{L}$.
Let $\mathcal{K} = \{D \in \tau^c : A \cup D \in U\}$

(iv) $\mathcal{K}$ is prefilter
Since $A \cup 1_\sim = 1_\sim \in \mathcal{U}$,we get $1_\sim \in \mathcal{K}$.Therefore $\mathcal{K} \neq 0_\sim$.
Since $\mathcal{U} \in \mathcal{S}, A \notin \mathcal{U} \Rightarrow A \cup 0_\sim \notin \mathcal{U} \Rightarrow 0_\sim \notin \mathcal{K}$ Take $D_1, D_2 \in \mathcal{K}$.
Then $A \cup D_1 \in \mathcal{U}$ and $A \cup D_2 \in \mathcal{U}$.
$\Rightarrow (A \cup D_1) \cap (A \cup D_2) \in \mathcal{U}.\Rightarrow A \cup (D_1 \cap D_2) \in \mathcal{U}.\Rightarrow (D_1 \cup D_2) \in \mathcal{U}$
Take $D \in \mathcal{K}$ and $B \in \tau^c$ be such that $D \subset B$
Consider $(A \cup D) \subset (A \cup B) \in \mathcal{U}$.Since $A \cup D \in \mathcal{U}$,$\mathcal{U}$ is a prefilter $A \cup B \in \mathcal{U}$.Therefore $B \in \mathcal{K}$.

(v) $\mathcal{F} \subset \mathcal{K}$
Take $D \in \mathcal{U}$.Since $D \subset A \cup D \in \mathcal{U}$ we get $D \in \mathcal{K}$.Hence $\mathcal{U} \subset \mathcal{K}$.
Since $\mathcal{F} \in \mathcal{S}$ and $\mathcal{U}$ is a maximal element in $\mathcal{S}$,$\mathcal{F} \subset \mathcal{U}$.
Combining both the inclusions we get $\mathcal{F} \subset \mathcal{K}$.

(vi) $A \notin \mathcal{K}$
Since $A \cup A = A \notin \mathcal{U}, A \notin \mathcal{K}$

By (iv), (v), (vi) we get $\mathcal{K} \in \mathcal{S}$.
Hence $\mathcal{K} \subset \mathcal{U}$.But $\mathcal{U} \subset \mathcal{K}$.
This implies $\mathcal{U} = \mathcal{K}$
Claim 1:$A \notin \mathcal{L}$
Suppose not,then $A \cup D_2 \in \mathcal{U}$.By definition of $\mathcal{K}$ this implies that $D_2 \in \mathcal{K} = \mathcal{U}$.
This is a contradiction.
Claim 2:$\mathcal{F} \subset \mathcal{L}$ Since $\mathcal{F} \in \mathcal{S}, \mathcal{F} \subset \mathcal{U}$.By (ii) $\mathcal{U} \subset \mathcal{L}$.Hence $\mathcal{F} \subset \mathcal{L}$.

By claims 1 and 2 we get $\mathcal{L} \in \mathcal{S}$.Therefore $\mathcal{L} \subset \mathcal{U}$.
By (ii) $\mathcal{U} \subset \mathcal{L}$.This implies $\mathcal{U} = \mathcal{L}$.
But by (iii) $\mathcal{U} \neq \mathcal{L}$.Hence we get a contradiction .
Therfore our assumption that both D_1, D_2 does not belong to $\mathcal{U}$ is wrong.
Therefore $\mathcal{U}$ is an intuionistic prime τ^c, $\mathcal{F} \subset \mathcal{U}$ and $A \notin \mathcal{U}$.
That is $\mathcal{U} \in \mathcal{P}(\mathcal{F})$ and $A \notin \mathcal{U}$.Hence $A \notin \bigcap_{\mathcal{G} \in \mathcal{P}(\mathcal{F})} \mathcal{G}$.
Therefore $\bigcap_{\mathcal{G} \in \mathcal{P}(\mathcal{F})} \mathcal{G} \subset \mathcal{F}$.

Combining both the inclusions we get $\mathcal{F} = \bigcap_{\mathcal{G} \in \mathcal{P}(\mathcal{F})} \mathcal{G}$

Theorem 2.4 *The following are equivalent for an intuitionistic fuzzy topological space (X, τ):*

(i) (X, τ) is is intuitionistic fuzzy compact

(ii) Every τ^c prefilter $\mathcal{F}$ satisfies $\bigcap_{A \in \mathcal{F}} A \neq 0_\sim$

(iii) Every prime τ^c prefilter $\mathcal{F}$ satisfies $\bigcap_{A \in \mathcal{F}} A \neq 0_\sim$

(iv) Every τ^c ultra filter $\mathcal{U}$ satisfies $\bigcap_{A \in \mathcal{U}} A \neq 0_\sim$.

Proof.(i)⇒(ii) Suppose $\bigcap_{A \in \mathcal{F}} \mathcal{A} = 0_\sim$.Then $\bigcup_{A \in \mathcal{F}} A^c = 1_\sim$
Since $A^c \in \tau$ and (X, τ) is intuitionistic fuzzy compact, there is a finite sub collection $\{A_1^c, A_2^c, ...A_n^c\}$ such that $1_\sim = (A_1^c) \cup (A_1^c) \cup ... \cup (A_n^c)$.
That is $A_1 \cap A_2 \cap ... \cap A_n = 0_\sim$. Since $A_1 \cap A_2 \cap ... \cap A_n \in \mathcal{F}$ we get $0_\sim \in \mathcal{F}$which is a contradiction.Hence $\bigcap_{A \in \mathcal{F}} A \neq 0_\sim$.
(ii)⇒(iii) and (iii)⇒(iv) are obvious.
(iv) ⇒(i) Let $\mathcal{K}$ be a family of intuitionistic fuzzy closed sets on X having finite intersection property.
For each $B \in \mathcal{F}$ define $\mathcal{G}_B = \{A \in \tau^c : B \subset A\}$.
Obviously $B \in \mathcal{G}_B$. Let $\mathcal{G} = \bigcup_{B \in \mathcal{K}} \mathcal{G}_B$. Since $\mathcal{K}$ has the finite intersection property, we get that $\mathcal{G}$ also has the finite intersection property. Hence there exists an intuitionistic fuzzy τ^c ultra filter $\mathcal{U}$ such that $\mathcal{G} \subset \mathcal{U}$.
Since $B \in \mathcal{K}$ implies $B \in \mathcal{G}_B$ we get $\mathcal{K} \subset \mathcal{U}$.Hence $\mathcal{K} \subset \mathcal{G} \subset \mathcal{U}$.
Therefore $\bigcap_{A \in \mathcal{U}} A \subset \bigcap_{A \in \mathcal{G}} A \bigcap_{A \in \mathcal{K}} A$.
By (iv) $\bigcap_{A \in \mathcal{U}} A \neq 0_\sim$. Therefore $\bigcap_{A \in \mathcal{K}} A \neq 0_\sim$.
Hence $\mathcal{K}$ has a non empty intersection.
Hence by Theorem1.15 we get(X, τ) is intuitionistic fuzzy compact.

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