Acta Universitatis Apulensis No. 25/2011
ISSN: 1582-5329 pp. 229-233

COINCIDENCE THE SETS OF NORM AND NUMERICAL RADIUS ATTAINING HOLOMORPHIC FUNCTIONS ON FINITE-DIMENSIONAL SPACES

Ramazan Zarghami

Abstract. Let X be a complex Banach space having property (β) with constant $\rho = 0$ and $A_\infty(B_X; X)$ be the space of bounded functions from B_X to X that are holomorphic on the open unit ball. In this paper we prove that in $A_\infty(B_X; X)$, the set of norm attaining elements contains numerical radius attaining elements, and also when X is a finite-dimensional space those are coincide.

2000 *Mathematics Subject Classification*: 46B25, 46B20, 46B03.

1. Introduction

A Banach space X with topological dual space X^* has property (β), if there exists a system $\{x_i, x_i^*\}_{i\in I} \subset X \times X^*$ and a real number $0 \leq \rho < 1$ satisfying
1) $\|x_i\| = \|x_i^*\| = x_i^*(x_i) = 1$ for all $i \in I$;
2) $|x_i^*(x_j)| \leq \rho$ for all $i, j \in I$, $i \neq j$;
3) $\|x\| = \sup\{|x_i^*(x)|;\ i \in I\}$ for every $x \in X$.

Property (β) which was introduced by J. Lindenstrauss [5] in the study of norm attaining operators, is a sufficient condition for denseness of norm attaining operators from any Banach space into X, in set of all bounded operators. c_0 , l_∞ and $C(K)$ (for K scattered) are the memorable spaces which have property (β). J. Partington [6] showed that every Banach space can be equivalently renormed to have property (β). A. S. Granero, M. J. Sevilla and J. P. Moreno [3] proved that every Banach space is isometric to a quotient of a Banach space with property (β). They also presented examples of some non-reflexive classical Banach spaces whose usual norm can be approximated by norms with property (β), namely $(C(K), \|.\|_\infty)$(where K is separable compact Hausdorff space) and $(L_1[0, 1], \|.\|_1)$.

If the supremum is replaced with maximum in the third condition of property (β), we say that the complex Banach space X has property Q. It is known that the space c_0 has property Q. In this paper for a complex Banach space X we will denote by B_X, S_X and X^* its closed unit ball , its sphere and its topological dual, respectively. We will denote by $A_\infty(B_X; X)$ the set of all bounded functions from B_X to X whose restriction to the open unit ball are holomorphic functions. The concepts of numerical range and numerical radius for holomorphic functions introduced by L.

Harris [4]. For $h \in A_\infty(B_X; X)$, the numerical radius is defined by

$$v(h) = \sup\{|x^*(h(x))| : (x, x^*) \in \Pi(X)\}$$

where

$$\Pi(X) = \{(x, x^*) \in X \times X^* : \|x\| = \|x^*\| = x^*(x) = 1\}.$$

We say that h attains its numerical radius when there exists (x_0, x_0^*) in $\Pi(X)$ such that $v(h) = |x_0^*(h(x_0))|$. We denote by $NRA(A_\infty(B_X; X))$ the subset of numerical attaining elements in $A_\infty(B_X; X)$.

It is said $h \in A_\infty(B_X; X)$ attains its norm if there exists $x_0 \in S_X$ such that $\|h\| = \|h(x_0)\|$. Moreover $NA(A_\infty(B_X; X))$ denotes the subset of norm attaining functions in $A_\infty(B_X; X)$.

M. D. Acosta and S. G. Kim [1], [2] obtained the sufficient condition for denseness of $NA(A_\infty(B_X, X))$ in $A_\infty(B_X, X)$ when X has property (β). They also proved that for a complex Banach space X which has property (β) with constant $\rho = 0$, if X has property Q then

$$NA(A_\infty(B_X, X)) = NRA(A_\infty(B_X, X)),$$

consequently

$$NA(A_\infty(B_{c_0}, c_0)) = NRA(A_\infty(B_{c_0}, c_0)).$$

They posed the question: Does the above theorem remain true in case of omitting hypothesis $\rho = 0$ or property Q?

In this paper we prove that when X has property (β) with constant $\rho = 0$, in $A_\infty(B_X, X)$, the set of all the numerical radius attaining elements is the subset of the set of all the norm attaining elements. Further if X is a finite-dimensional space then the two sets are coincide, and in this case we give an affirmative answer to the afore-mentioned question.

2. MAIN RESULTS

Lemma 1 *Let X be a complex Banach space satisfying property (β) with constant $\rho = 0$. If $h \in A_\infty(B_X; X)$ then $v(h) = \|h\|$.*

Proof. Let (x_i, x_i^*); $i \in I$ be the set in the definition of property (β) and $\epsilon > 0$. For each $x^* \in X^*$ and $h \in A_\infty(B_X; X)$ we define $h^*x^* \in A_\infty(B_X)$ by

$$h^*x^*(x) = x^*(h(x)), \quad (x \in B_X).$$

Since X has property (β), so

$$\|h\| = sup_{i \in I}\|h^*x_i^*\|.$$

By the Characteristic property of the supremum, there exists $i_0 \in I$ such that

$$\|h^* x_{i_0}^*\| \geq \|h\| - \epsilon.$$

And also there exists $x_0 \in B_X$ such that

$$\|h^* x_{i_0}^*(x_0)\| \geq \|h^* x_{i_0}^*\| - \epsilon.$$

Define

$$\psi(\lambda) = x_{i_0}^*(h[x_0 + (\lambda - x_{i_0}^*(x_0))]) \quad (\lambda \in \mathbb{C}, |\lambda| \leq 1).$$

By the Maximum Modulus Principle, there exists $\lambda_1 \in \mathbb{C}$ such that $|\lambda_1| = 1$ and

$$|\psi(\lambda_1)| \geq |\psi(x_{i_0}^*(x_0))| = |x_{i_0}^*(h(x_0))| \geq \|h\| - 2\epsilon.$$

Note that $(x_0 + (\lambda_1 - x_{i_0}^*(x_0))x_{i_0}, \overline{\lambda_1} x_{i_0}^*) \in \pi(X)$ and so

$$\upsilon(h) \geq |x_{i_0}^*(h[x_0 + (\lambda_1 - x_{i_0}^*(x_0))x_{i_0}])| = |\psi(\lambda_1)| \geq \|h\| - 2\epsilon.$$

Thus $\upsilon(h) \geq \|h\| - 2\epsilon$. Since $\|h\| - 2\epsilon \leq \upsilon(h) \leq \|h\|$ for every $\epsilon > 0$, therefore $\upsilon(h) = \|h\|$.

Theorem 2 *Let X be a complex Banach space satisfying property (β) with constant $\rho = 0$. Then*

$$NRA(A_\infty(B_X; X)) \subseteq NA(A_\infty(B_X; X)).$$

Proof. Let $h \in NRA(A_\infty(B_X; X))$. Then there exists $(x_0, x_0^*) \in \pi(X)$ such that

$$\upsilon(h) = |x_0^*(h(x_0))|.$$

By Lemma 1, $\upsilon(h) = \|h\|$. Therefore,

$$\begin{aligned} \|h(x_0)\| &= sup\{|x^*(h(x_0))|; \|x^*\| = 1\} \\ &\geq sup\{|x^*(h(x_0))|; \|x^*\| = x^*(x_0) = 1\} \\ &= |x_0^*(h(x_0))| \\ &= \upsilon(h) = \|h\| \end{aligned}$$

That is $h \in NA(A_\infty(B_X; X))$. Hence

$$NRA(A_\infty(B_X; X)) \subseteq NA(A_\infty(B_X; X)).$$

Theorem 3 *Let X be a finite-dimensional space. Then*

$$NRA(A_\infty(B_X; X)) = NA(A_\infty(B_X; X)).$$

Proof. Since the finite-dimensional spaces, endowed with maximum norm have property β with constant $\rho = 0$, at the Theorem 2, it is sufficient we prove that

$$NA(A_\infty(B_X; X)) \subseteq NRA(A_\infty(B_X; X)).$$

Let $g \in NA(A_\infty(B_X; X))$ and $n \in \mathbb{N}$. Then there is an $x_0 \in S_X$ such that $\|g(x_0)\| = \|g\|$. By the Characteristic property of supremum there exists an $i_n \in I$ such that $|x^*_{i_n}(g(x_0))| \geq \|g\| - \frac{1}{n}$. Now similar to proof of Lemma 2.2, there exists $\lambda_n \in \mathbb{C}$ with $|\lambda_n| = 1$ such that

$$|x^*_{i_n}(g([x_0 + (\lambda_n - x^*_{i_n}(x_0))x_{i_n}])| \geq \|g\| - \frac{2}{n}.$$

By putting $t_n = x_0 + (\lambda_n - x^*_{i_n}(x_0))x_{i_n}$ and $t^*_n = \overline{\lambda_n} x^*_{i_n}$ we have $(t_n, t^*_n) \in \pi(X)$ and

$$\upsilon(g) \geq |t^*_n(g(t_n)| \geq \upsilon(g) - \frac{2}{n}.$$

That is $|t^*_n(g(t_n)| \to \upsilon(g)$. By reflexivity of X, the bounded sequence $\{t_n\}$ has a convergence subsequence $\{t_{nk}\}$. Suppose t^*_0 be the w^*-cluster point of the sequence $\{t^*_{nk}\}$ in B_{X^*} and $t_0 = \lim_{k\to\infty} t_{nk}$. Then for all $k \in \mathbb{N}$ we have

$$|1 - t^*_0(t_0)| = |t^*_{nk}(t_{nk}) - t^*_0(t_0)| \quad \leq \quad |t^*_{nk}(t_{nk}) - t^*_{nk}(t_0)| + |t^*_{nk}(t_0) - t^*_0(t_0)| \to 0.$$

Thus $(t_0, t^*_0) \in \pi(X)$. Also we get

$$\big|\upsilon(g) - |t^*_0(g(t_0))|\big| \leq \big|\upsilon(g) - |t^*_{nk}(g(t_{nk}))|\big| + \big||t^*_{nk}(g(t_{nk})) - |t^*_0(g(t_0))|\big| \to 0.$$

That is g attains its numerical radius at $(t_0, t^*_0) \in \pi(X)$.

Remark 4 The finite-dimensional spaces are only reflexive spaces which have property (β) with constant $\rho = 0$, because in the infinite-dimensional case, if we choice a countable (and infinite) number of the set of vectors appearing in the definition of property (β), then this set is a Schauder basis of a copy of c_0 in the space, so the space cannot be reflexive.

References

[1] M. D. Acosta and S. G. kim, *Denseness of holomorphic functions attaining their numerical radii*, Israel Journal of Mathematics, 161, (2007), 373-386.

[2] M. D. Acosta, *Boundaries for spaces of holomorphic functions on C(K)*, Publications of the Research Institute for mathematic Sciences, 42, (2006), 27-44.

[3] A. S. Granero, M. Jimenez Sevilla and J. P. Moreno, *Geometry of Banach spaces with property β*, Israel Journal of Mathematics, 111, (1999), 263-273.

[4] L. Harris, *The numerical range of holomorphic functios in Banach spaces*, American Journal of Mathematics, 93, (1971) 7-19.

[5] J. Lindenstrauss, *On oprators which attain their norm*, Israel Journal of Mathematics, 1, (1963), 139-148.

[6] J. Partigton, *Norm attaining operators*, Israel Journal of Mathematics, 43, (1982), 273-276.

Ramazan Zarghami
Department of Civil, Marand Faculty of Engineering
University of Tabriz
Marand Faculty of Engineering
Mrand - Iran
email:*zarghamir@gmail.com*