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THE SEMI ORLICZ SPACE OF Λ^2

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Abstract.Let χ^2 denote the space of all double gai sequences. Let Λ^2 denote the space of all double analytic sequences. This paper is to introduce a new class of sequence spaces namely the semi difference Orlicz space of Λ^2. It is shown that the intersection of all semi difference Orlicz space of Λ^2 is $I \subset \eta^2$ and $\Lambda^2 \subset I$.

Keywords: double sequence spaces, analytic sequence, gai sequences, semi Orlicz of Λ^2, duals.

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1. Introduction

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively.
We write w^2 for the set of all complex sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces is found in Bromwich[4]. Later on, they were investigated by Hardy[8], Moricz[12], Moricz and Rhoades[13], Basarir and Solankan[2], Tripathy[20], Colak and Turkmenoglu[6], Turkmenoglu[22], and many others.

Let us define the following sets of double sequences:

$$\mathcal{M}_u(t) := \left\{ (x_{mn}) \in w^2 : sup_{m,n\in N} |x_{mn}|^{t_{mn}} < \infty \right\},$$

$$\mathcal{C}_p(t) := \left\{ (x_{mn}) \in w^2 : p - lim_{m,n\to\infty} |x_{mn} - l|^{t_{mn}} = 1 \, for\, some\, l \in \mathbb{C} \right\},$$

$$\mathcal{C}_{0p}(t) := \left\{ (x_{mn}) \in w^2 : p - lim_{m,n\to\infty} |x_{mn}|^{t_{mn}} = 1 \right\},$$

$$\mathcal{L}_u(t) := \left\{ (x_{mn}) \in w^2 : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty \right\},$$

$$\mathcal{C}_{bp}(t) := \mathcal{C}_p(t) \bigcap \mathcal{M}_u(t) \text{ and } \mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \bigcap \mathcal{M}_u(t);$$

where $t = (t_{mn})$ is the sequence of strictly positive reals t_{mn} for all $m, n \in \mathbb{N}$ and $p - lim_{m,n\to\infty}$ denotes the limit in the Pringsheim's sense. In the case $t_{mn} = 1$ for all $m, n \in \mathbb{N}; \mathcal{M}_u(t), \mathcal{C}_p(t), \mathcal{C}_{0p}(t), \mathcal{L}_u(t), \mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets $\mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{0p}, \mathcal{L}_u, \mathcal{C}_{bp}$ and $\mathcal{C}_{0bp}$, respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Colak [27,28] have proved that $\mathcal{M}_u(t)$ and $\mathcal{C}_p(t), \mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and gave the $\alpha-, \beta-, \gamma-$ duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, in her PhD thesis, Zelter [29] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [30] have recently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Nextly, Mursaleen [31] and Mursaleen and Edely [32] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the $M-$core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{jk})$ into one whose core is a subset of the $M-$core of x. More recently, Altay and Basar [33] have defined the spaces $\mathcal{BS}, \mathcal{BS}(t), \mathcal{CS}_p, \mathcal{CS}_{bp}, \mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u, \mathcal{M}_u(t), \mathcal{C}_p, \mathcal{C}_{bp}, \mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also examined some properties of those sequence spaces and determined the $\alpha-$ duals of the spaces $\mathcal{BS}, \mathcal{BV}, \mathcal{CS}_{bp}$ and the $\beta(\vartheta)-$ duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Quite recently Basar and Sever [34] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space ℓ_q of single sequences and examined some properties of the space $\mathcal{L}_q$. Quite recently Subramanian and Misra [35] have studied the space $\chi^2_M(p, q, u)$ of double sequences and gave some inclusion relations.

We need the following inequality in the sequel of the paper. For $a, b, \geq 0$ and $0 < p < 1$, we have

$$(a+b)^p \leq a^p + b^p \tag{1}$$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence (s_{mn}) is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m, n \in \mathbb{N})$ (see[1]).

A sequence $x = (x_{mn})$is said to be double analytic if $sup_{mn} |x_{mn}|^{1/m+n} < \infty$. The vector space of all double analytic sequences will be denoted by Λ^2. A sequence $x = (x_{mn})$ is called double gai sequence if $((m+n)! |x_{mn}|)^{1/m+n} \to 0$ as $m, n \to \infty$. The double gai sequences will be denoted by χ^2. Let $\phi = \{all finite sequences\}$.

Consider a double sequence $x = (x_{ij})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij} \Im_{ij}$ for all $m, n \in \mathbb{N}$; where $\Im_{ij}$ denotes the double sequence whose only non zero term is a $\frac{1}{(i+j)!}$ in the $(i,j)^{th}$ place for each $i, j \in \mathbb{N}$.

An FK-space(or a metric space)X is said to have AK property if $(\Im_{mn})$ is a Schauder basis for X. Or equivalently $x^{[m,n]} \to x$.

An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{mn})(m, n \in \mathbb{N})$ are also continuous.

Orlicz[16] used the idea of Orlicz function to construct the space (L^M). Lindenstrauss and Tzafriri [10] investigated Orlicz sequence spaces in more detail, and they proved that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p\,(1 \leq p < \infty)$. subsequently, different classes of sequence spaces were defined by Parashar and Choudhary [17], Mursaleen et al. [14], Bektas and Altin [3], Tripathy et al. [21], Rao and Subramanian [18], and many others. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in [9].

Recalling [16] and [9], an Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing, and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by subadditivity of M, then this function is called modulus function, defined by Nakano [15] and further discussed by Ruckle [19] and Maddox [11], and many others.

An Orlicz function M is said to satisfy the Δ_2- condition for all values of u if there exists a constant $K > 0$ such that $M(2u) \leq KM(u)\,(u \geq 0)$. The Δ_2- condition is equivalent to $M(\ell u) \leq K\ell M(u)$, for all values of u and for

$\ell > 1$.

Lindenstrauss and Tzafriri [10] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty,\, for\, some\, \rho > 0 \right\},$$

The space ℓ_M with the norm

$$\|x\| = inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\},$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p\,(1 \leq p < \infty)$, the spaces ℓ_M coincide with the classical sequence space ℓ_p. If X is a sequence space, we give the following definitions:

(i)X'= the continuous dual of X;

(ii)$X^{\alpha} = \left\{ a = (a_{mn}) : \sum_{m,n=1}^{\infty} |a_{mn}x_{mn}| < \infty,\, for\, each\, x \in X \right\}$;

(iii)$X^{\beta} = \left\{ a = (a_{mn}) : \sum_{m,n=1}^{\infty} a_{mn}x_{mn}\, is\, convegent,\, foreach\, x \in X \right\}$;

(iv)$X^{\gamma} = \left\{ a = (a_{mn}) : sup_{mn} \geq 1 \left| \sum_{m,n=1}^{M,N} a_{mn}x_{mn} \right| < \infty, foreachx \in X \right\}$;

(v)$let\, X\, beanFK - space \supset \phi;\, then\, X^f = \left\{ f(\Im_{mn}) : f \in X' \right\}$;

(vi)$X^{\delta} = \left\{ a = (a_{mn}) : sup_{mn} |a_{mn}x_{mn}|^{1/m+n} < \infty,\, foreach\, x \in X \right\}$;

$X^{\alpha}.X^{\beta}, X^{\gamma}$ are called $\alpha -(or K\ddot{o}the - Toeplitz)$dual of X, $\beta -(or\, generalized - K\ddot{o}the - Toeplitz)\, dual\, of\, X, \gamma -\, dual\, of\, X,\, \delta\, -\, dual\, of\, X\, respectively. X^{\alpha}$ is defined by Gupta and Kamptan [24]. It is clear that $x^{\alpha} \subset X^{\beta}$ and $X^{\alpha} \subset X^{\gamma}$, but $X^{\alpha} \subset X^{\gamma}$ does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [36] as follows

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\}$$

for $Z = c, c_0$ and ℓ_{∞}, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$. Here w, c, c_0 and ℓ_{∞} denote the classes of all, convergent,null and bounded sclar valued single sequences respectively. The above spaces are Banach spaces normed by

$$\|x\| = |x_1| + sup_{k\geq 1} |\Delta x_k|$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \{x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z\}$$

where $Z = \Lambda^2, \chi^2$ and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbb{N}$.
As in single sequences $(see\, [23, Theorem 7.2.7])$
(i) $X^{\gamma} \subset X^f$ (ii)If X has AD, $X^{\beta} = X^f$; (iii) If X has AD, $X^{\beta} = X^f$.

2. Definitions and Preliminaries

Let w^2 denote the set of all complex double sequences $x = (x_{mn})_{m,n=1}^{\infty}$ and $M : [0, \infty) \to [0, \infty)$ be an Orlicz function, or a modulus function.

$$\chi_M^2 = \left\{x \in w^2 : \left(M\left(\frac{((m+n)!|x_{mn}|)^{1/m+n}}{\rho}\right)\right) \to 0\, as\, m, n \to \infty\, for\, some\, \rho > 0\right\} \text{ and}$$

$$\Lambda_M^2 = \left\{x \in w^2 : sup_{m,n\geq 1}\left(M\left(\frac{|x_{mn}|^{1/m+n}}{\rho}\right)\right) < \infty\, for\, some\, \rho > 0\right\}.$$

Define the sets $\chi_M^2(\Delta) = \{x \in w^2 : \Delta x \in \chi_M^2\}$ and $\Lambda_M^2(\Delta) = \{x \in w^2 : \Delta x \in \Lambda_M^2\}$,

The space $\Lambda_M^2(\Delta)$ is a metric space with the metric

$$d(x, y) = inf\left\{\rho > 0 : sup_{m,n\geq 1}\left(M\left(\frac{|\Delta x_{mn} - \Delta y_{mn}|}{\rho}\right)\right)^{1/m+n} \leq 1\right\} \quad (2)$$

The space $\chi_M^2(\Delta)$ is a metric space with the metric

$$d(x, y) = inf\left\{\rho > 0 : sup_{m,n\geq 1}\left(M\left(\frac{(m+n)!\, |\Delta x_{mn} - \Delta y_{mn}|}{\rho}\right)\right)^{1/m+n} \leq 1\right\} \quad (3)$$

Because of the historical roots of summability in convergence, conservative space and matrices play a special role in its theory. However, the results seem mainly to depend on a weaker assumption, that the spaces be semi conservative. (See [23]). Snyder and Wilansky [37] introduced the concept of semi conservative spaces. Snyder [38] studied the properties of semi conservative spaces.

In the year 1996 the semi replete spaces were introduced by Chandrasekhara Rao and Srinivasalu [39]. K.Chandrasekhara Rao and N.Subramanian [40] and [41] introduced the concept of semi analytic spaces and the semi Orlicz space of analytic sequences. Recently N.Subramanian, B.C.Tripathy and C.Murugesan has [42] introduced the concept of the semi Orlicz space of $cs \bigcap d_1$.

In a similar way, in this paper we define semi difference Orlicz spaces of Λ^2, and show that semi difference Orlicz space of Λ^2 is $I \subset \eta^2$ and $\Lambda^2 \subset I$.

3. Main Results

Proposition 1. $\chi^2_M(\Delta)$ *has AK-property*

Proof: Let $x = (x_{mn}) \in \chi^2_M(\Delta)$ and take $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij} \Im_{ij}$ for all $m, n \in \mathbb{N}$. Hence

$d\left(x, x^{[r,s]}\right) = inf\left\{sup_{mn}\left\{\left((m+n)!\left|\Delta x_{mn}\right|\right)^{1/m+n} : m \geq r+1, n \geq s+1\right\} \leq 1\right\} \rightarrow 0$ as $m, n \rightarrow \infty$. Therefore, $x^{[r,s]} \rightarrow x$ as $r, s \rightarrow \infty$ in $\chi^2_M(\Delta)$. Thus $\chi^2_M(\Delta)$ has AK. This completes the proof.

Proposition 2. $\chi^2_M \subset \chi^2_M(\Delta)$

Proof: Let $x \in \chi^2_M$. Then we have the following implications

$\left(M\left(\frac{((m+n)!|x_{mn}|)^{1/m+n}}{\rho}\right)\right) \rightarrow 0$ as $m, n \rightarrow \infty$.

$\Rightarrow \left(M\left(\frac{((m+n)!|(x_{mn}-x_{mn+1})-(x_{m+1n}-x_{m+1n+1})|)^{1/m+n}}{\rho}\right)\right) \leq \left(M\left(\frac{((m+n)!|x_{mn}|)^{1/m+n}}{\rho}\right)\right) +$
$\left(M\left(\frac{((m+n+1)!|x_{mn+1}|)^{1/m+n+1}}{\rho}\right)\right) + \left(M\left(\frac{((m+n+1)!|x_{m+1n}|)^{1/m+n+1}}{\rho}\right)\right)$
$+ \left(M\left(\frac{((m+n+2)!|x_{m+1n+1}|)^{1/m+n+2}}{\rho}\right)\right)$
$\rightarrow 0$ as $m, n \rightarrow \infty$

$\Rightarrow \left(M\left(\frac{((m+n)!|\Delta x_{mn}|)^{1/m+n}}{\rho}\right)\right) \rightarrow 0$ as $m, n \rightarrow \infty$.

$\Rightarrow x \in \chi^2_M(\Delta) \Rightarrow \chi^2_M \subset \chi^2_M(\Delta)$

Now take

$$\text{If } \left(M\left(\frac{(m+n)!x_{mn}}{\rho}\right)\right) = 1^* = \begin{pmatrix} 1, & 1, & ...1 \\ 1, & 1, & ...1 \\ . & & \\ . & & \\ . & & \\ 1, & 1, & ...1 \end{pmatrix}$$

Then $1^* \in \chi^2_M(\Delta)$. but $1^* \notin \chi^2_M$. Hence the inclusion $\chi^2_M \subset \chi^2_M(\Delta)$ is strict. This completes the proof.

Proposition 3. $\left(\chi^2_M(\Delta)\right)^\beta = \Lambda^2$

Proof: **Step 1.** $\chi^2_M \subset \chi^2_M(\Delta)$, by Proposition 4.2
$\Rightarrow (\chi^2_M(\Delta))^\beta \subset (\chi^2_M)^\beta$. But $(\chi^2_M)^\beta = \Lambda^2$

$$\left(\chi^2_M(\Delta)\right)^\beta \subset \Lambda^2 \tag{4}$$

Step2. We observe that $\chi^2_M(\Delta) \subset \Gamma^2_M(\Delta)$.
$\Rightarrow (\Gamma^2_M(\Delta))^\beta \subset (\chi^2_M(\Delta))^\beta$. But $(\Gamma^2_M(\Delta))^\beta \overset{\subset}{\neq} \Lambda^2$,

$$\Lambda^2 \subset \left(\chi^2_M(\Delta)\right)^\beta \tag{5}$$

From (4) and (5) we get $(\chi^2_M(\Delta))^\beta = \Lambda^2$. This completes the proof.
Proposition 4.$\chi^2_M(\Delta)$ *is solid*

*Proof:*Let $|x_{mn}| \le |y_{mn}|$ and $y = (y_{mn}) \in \chi^2_M(\Delta)$.
Then $\left\{M\left(\frac{((m+n)!|\Delta x_{mn}|)^{1/m+n}}{\rho}\right)\right\} \le \left\{M\left(\frac{((m+n)!|\Delta y_{mn}|)^{1/m+n}}{\rho}\right)\right\}$, becuase M is non-decreasing.
But $\left\{M\left(\frac{(k!|\Delta y_{mn}|)^{1/m+n}}{\rho}\right)\right\} \in \chi^2$, because $y \in \chi^2_M(\Delta)$.
That is $\left\{M\left(\frac{((m+n)!|\Delta y_{mn}|)^{1/m+n}}{\rho}\right)\right\} \to 0\, as\, mn \to \infty$, and $\left\{M\left(\frac{((m+n)!|\Delta x_{mn}|)^{1/m+n}}{\rho}\right)\right\} \to 0\, as\, mn \to \infty$. Therefore $x = \{x_{mn}\} \in \chi^2_M(\Delta)$. This completes the proof.
Proposition 5.$(\chi^2_M(\Delta))^\mu = \Lambda^2$ for $\mu = \alpha, \beta, \gamma, f$
Step 1: $(\chi^2_M(\Delta))$ has AK by Proposition 4.1. Hence by Lemma 2 (i) we get $(\chi^2_M(\Delta))^\beta = (\chi^2_M(\Delta))^f$. But $(\chi_M(\Delta))^\beta = \Lambda^2$ Hence

$$\left(\chi^2_M(\Delta)\right)^f = \Lambda^2. \tag{6}$$

Step 2: Since AK implies AD. Hence by Lemma 2(iii) we get $(\chi^2_M(\Delta))^\beta = (\chi^2_M(\Delta))^\gamma$. Therefore

$$\left(\chi^2_M(\Delta)\right)^\gamma = \Lambda^2. \tag{7}$$

Step 3:$(\chi^2_M(\Delta))$ is normal by Proposition 4.4. Hence by Proposition 2.7 [24]. We get

$$\left(\chi^2_M(\Delta)\right)^\alpha = \left(\chi^2_M(\Delta)\right)^\beta = \Lambda^2 \tag{8}$$

From (6),(7) and (8) we have $(\chi^2_M(\Delta))^\alpha = (\chi^2_M(\Delta))^\beta = (\chi^2_M(\Delta))^\gamma = (\chi^2_M(\Delta))^f = \Lambda^2$.
Lemma 1.[23, Theorem 8.6.1] $Y \supset X \Leftrightarrow Y^f \subset X^f$ *where* X *is an AD-space*

and Y an FK-space.

Proposition 6. *Let Y be any FK-space $\supset \phi$. Then $Y \supset \chi^2_M(\Delta)$ if and only if the sequence $\Im^{(mn)}$ is weakly Λ^2.*

Proof: The following implications establish the result $Y \supset \chi^2_M(\Delta) \Leftrightarrow Y^f \subset (\chi^2_M(\Delta))^f$, since $\chi^2_M(\Delta)$ has AD by Lemma 4.6.

$\Leftrightarrow Y^f \subset \Lambda^2$, since $(\chi^2_M(\Delta))^f = \Lambda^2$.

$\Leftrightarrow$ for each $f \in Y'$, the topological dual of $Y.f\left(\Im^{(mn)}\right) \in \Lambda^2$.

$\Leftrightarrow f\left(\Im^{(mn)}\right)$ is Λ^2.

$\Leftrightarrow \Im^{(mn)}$ is weakly Λ^2. This completes the proof.

Proposition 7. *For every $p = (p_{mn})$, $[\Lambda^2_M(p)]^\beta = [\Lambda^2_M(p)]^\alpha = [\Lambda^2_M(p)]^\gamma = \eta^2_M(p)$,*

where $\eta^2_M(p) = \bigcap_{N \in N-\{1\}} \left\{x = x_{mn} : \sum_{m,n}\left(M\left(\frac{|x_{mn}|N^{m+n/p_{mn}}}{\rho}\right)\right) < \infty\right\}.$

Proof: (1) First we show that $\eta^2_M(p) \subset [\Lambda^2_M(p)]^\beta$.

Let $x \in \eta^2_M(p)$ and $y \in \Lambda^2_M(p)$. Then we can find a positive integer N such that $\left(|y_{mn}|^{1/m+n}\right)^{p_{mn}} < max\left(1, sup_{m,n\geq 1}\left(|y_{mn}|^{1/m+n}\right)^{p_{mn}}\right) < N$, for all m,n.

Hence we may write

$\left|\sum_{m,n} x_{mn}y_{mn}\right| \leq \sum_{m,n}|x_{mn}y_{mn}| \leq \sum_{mn}\left(M\left(\frac{|x_{mn}y_{mn}|}{\rho}\right)\right)$

$\leq \sum_{m,n}\left(M\left(\frac{|x_{mn}|N^{m+n/p_{mn}}}{\rho}\right)\right).$

Since $x \in \eta^2_M(p)$. the series on the right side of the above inequality is convergent, whence $x \in [\Lambda^2_M(p)]^\beta$. Hence $\eta^2_M(p) \subset [\Lambda^2_M(p)]^\beta$.

Now we show that $[\Lambda^2_M(p)]^\beta \subset \eta^2_M(p)$.

For this, let $x \in [\Lambda^2_M(p)]^\beta$, and suppose that $x \notin \Lambda^2_M(p)$. Then there exists a positive integer $N > 1$ such that $\sum_{m,n}\left(M\left(\frac{|x_{mn}|N^{m+n/p_{mn}}}{\rho}\right)\right) = \infty$.

If we define $y_{mn} = N^{m+n/p_{mn}} Sgn x_{mn}\, m,n = 1,2,\cdots$, then $y \in \Lambda^2_M(p)$.

But, since

$\left|\sum_{m,n} x_{mn}y_{mn}\right| = \sum_{mn}\left(M\left(\frac{|x_{mn}y_{mn}|}{\rho}\right)\right) = \sum_{m,n}\left(M\left(\frac{|x_{mn}|N^{m+n/p_{mn}}}{\rho}\right)\right) = \infty,$

we get $x \notin [\Lambda^2_M(p)]^\beta$, which contradicts to the assumption $x \in [\Lambda^2_M(p)]^\beta$. Therefore $x \in \eta^2_M(p)$. Therefore $[\Lambda^2_M(p)]^\beta = \eta^2_M(p)$.

(ii)and (iii) can be shown in a similar way of (i). Therefore we omit it.

4. Properties of semi difference Orlicz space of Λ^2

Definition 1. *An FK-space ΔX is called "semi difference Orlicz space of Λ^2" if its dual $(\Delta X)^f \subset \Lambda^2$. In other words ΔX is semi difference Orlicz space of Λ^2 if $f\left(\Im^{(mn)}\right) \in \Lambda^2 \forall f \in (\Delta X)^{'}$ for each fixed m, n*

Example:$\chi^2_M(\Delta)$ is semi difference Orlicz space of Λ^2 . Indeed, if $\chi^2_M(\Delta)$ is the space of all difference Orlicz sequence of double gai sequences, then by Lemma 5.3 $(\chi^2_M(\Delta))^f = \Lambda^2$.

Lemma 1.$(\chi^2_M(\Delta))^f = \Lambda^2$.

Proof: $(\chi^2_M(\Delta))^\beta = \Lambda^2$ by Proposition 4.3. But $(\chi^2_M(\Delta))$ has AK by Proposition 4.1. Hence $(\chi^2_M(\Delta))^\beta = (\chi^2_M(\Delta))^f$. Therefore $(\chi^2_M(\Delta))^f = \Lambda^2$ This completes the proof. We recall

Lemma 2. (See 23, Theorem 4.3.7)*Let z be a sequence. Then $\left(z^\beta, P\right)$ is an AK space with $P = (P_k : k = 0, 1, 2, \ldots)$, where $P_0(x) = \overset{sup}{m} \left|\sum_{k=1}^m z_k x_k\right|, P_n(x) = |x_n|$. For any k such that $z_k \neq 0, P_k$ may be omitted. If $z \in \phi, P_0$ may be omitted.*

Proposition 1. *Let z be a sequence z^β is semi difference Orlicz space of Λ^2 if and only if z is Λ^2.*

Proof: **Step 1**. Suppose that z^β is semi difference Orlicz space of Λ^2. z^β has AK by Lemma 5.4. Therefore $Z^{\beta\beta} = \left(z^\beta\right)^f$ by Theorem 7.2.7 of Wilansky [23]. So Z^β is semi difference Orlicz space of Λ^2 if and only if $z^{\beta\beta} \subset \Lambda^2$. But then $z \in z^{\beta\beta} \subset \Lambda^2$. Hence z is Λ^2.

Step2: Conversely, suppose that z is Λ^2. Then $z^\beta \supset \{\Lambda^2\}^\beta$and $z^{\beta\beta} \subset \{\Lambda^2\}^{\beta\beta} = \Lambda^2$. But$\left(z^\beta\right)^f = z^{\beta\beta}$. Hence $\left(z^\beta\right)^f \subset \Lambda^2$. Therefore z^β is semi difference Orlicz space of Λ^2. This completes the proof.

Proposition 2. *Every semi difference Orlicz space of Λ^2 contains χ^2_M.*

Proof: Let ΔX be any semi difference Orlicz space of Λ^2. Hence $(\Delta X)^f \subset \Lambda^2$. Therefore $f\left(\Im^{(mn)}\right) \in \Lambda^2 \forall f \in (\Delta X)^{'}$. So, $\left\{\Im^{(mn)}\right\}$ is weakly Λ^2 with respect to ΔX. Hence $\Delta X \supset \chi^2_M(\Delta)$ by Proposition 4.7. But $\chi^2_M(\Delta) \supset \chi^2_M$. Hence $\Delta X \supset \chi^2_M$. This completes the proof.

Proposition 3. *The intersection of all semi difference Orlicz space of Λ^2.* $\{\Delta X_{mn} : m, n = 1, 2, \ldots\}$ is semi differene Orlicz space of Λ^2.

Proof: Let $\Delta X = \bigcap_{m,n=1}^\infty \Delta X_{mn}$. Then ΔX is an FK-space which contains

ϕ. Also every $f \in (\Delta X)^{'}$ can be written as $f = g_{11} + \ldots + g_{mn}$, where $g_{mn} \in (\Delta X_{mn})^{'}$ for some mn and for $1 \leq mn \leq i, j$. But then $f(\Im^{mn}) = g_1(\Im^{mn}) + \ldots + g_{mn}(\Im^{mn})$. Since $\Delta X_{mn}(m, n = 1, 2, \ldots)$ are semi difference Orlicz space of Λ^2, it follows that $g_{ii}(\Im^{mn}) \in \Lambda^2$.. for all $i = 1, 2, \ldots mn$. Therefore $f(\Im^{mn}) \in \Lambda^2 \forall mn \, and \forall f$. Hence ΔX is semi difference Orlicz space of Λ^2.This completes the proof.

Proposition 4. *The intersection of all semi difference Orlicz space Λ^2 is $I \subset \eta^2$ and $\Lambda^2 \subset I$.*

Proof: Let I be the intersection of all semi difference Orlicz space of Λ^2. By Proposition 5.5 we see that the intersection

$$I \subset \bigcap \left\{ z^{\beta} : z \in \Lambda^2 \right\} = \left\{ \Lambda^2 \right\}^{\beta} = \eta^2. \tag{9}$$

By Proposition 5.7 it follows that I is semi difference Orlicz space of Λ^2. By Proposition 5.6 consequently

$$\chi^2_M(\Delta) = \Lambda^2 \subset I \tag{10}$$

From (9) and (10) we get $I \subset \eta^2$ and $\Lambda^2 \subset I$. This completes the proof.

Corollary: The smallest semi difference Orlicz space of Λ^2 is $I \subset \eta^2$ and $\Lambda^2 \subset I$.

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