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HOMOTOPY ANALYSIS METHOD FOR SOLVING RATIO-DEPENDENT PREDATOR-PREY SYSTEM WITH CONSTANT EFFORT HARVESTING BY USING TWO PARAMETERS H_1 AND H_2

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Abstract. In this paper, we apply the homotopy analysis method (HAM) to obtain approximate solution for the Ratio-dependent predator-prey system with constant effort harvesting. We optimize the values of h_1 and h_2 by an Euclidean residual for the system of equations. The validity of this method is verified, because it agrees with Runge-Kutta (RKF78) in figures. The comparison between the results of the proposed method and homotopy perturbation method(HPM) as well as Adomian decomposition method (ADM), reveals this method is very effective and convenient.

2000 *Mathematics Subject Classification*: 37M20, 37N25, .

1. Introduction

One of the most interesting applications of systems of differential equations is the predator-prey problem [1-15]. It was developed independently by Alfred Lotka and Vito Volterra in the 1920's, and is characterized by oscillations in the population size of both predator and prey, with the peak of the predator's oscillation lagging slightly behind the peak of the prey's oscillation. From then on, the dynamic relationship between predators and their prey has long been and will continue to be one of dominant themes in both ecology and mathematical ecology due to its universal existence and importance. The model makes several simplifying assumptions.
1) The prey population will grow exponentially when the predator is absent.
2) The predator population will starve in the absence of the prey population (as opposed to switching to another type of prey).
3) Predators can consume infinite quantities of prey.
4) There is no environmental complexity (in other words, both populations are

moving randomly through a homogeneous environment).
The general case of predator-prey system is as follows:

$$\begin{cases} \frac{dx}{dt} = xg(x) - yf(x,y) - \mu_x(x)x, \\ \frac{dy}{dt} = \gamma f(x,y)y - \mu_y(y)y, \end{cases} \tag{1}$$

which $x(t)$ and $y(t)$ represent the size of the prey and predator population respectively at time t. $g(x)$ is the per capita prey growth rate in the absence of the predator, μ_x and μ_y are natural mortalities of prey and predator, respectively. $f(x,y)$ is the functional response, and $\gamma f(x,y)$ is the per capita production of predator due to predation, which is often called the numerical response. The functional response plays a main role in system (1). The knowledge of this function determines the dynamics of the whole system and the transfer of the biomass in the predation because it is proportional to the numerical response.

Now by letting $f(x,y) = x/(x+y)$, $\mu_x(x) = r, \mu_y(y) = e$ and $g(x) = (1-x)$ we obtain the special case of predator-prey system called predator-prey system with constant effort harvesting which formulated as follows:

$$\begin{cases} \frac{dx}{dt} = x(t)(1-x(t)) - \frac{bx(t)y(t)}{x(t)+y(t)} - rx(t), \\ \frac{dy}{dt} = \frac{cx(t)y(t)}{x(t)+y(t)} - ey(t), \end{cases} \tag{2}$$

$$x(0) = x_0, \quad y(0) = y_0,$$

where x_0 and y_o are the initial size of the prey and predator population.
It is obvious that the harvesting activity decreases the predator population indirectly by reducing the availability of the prey to the predator.

We rewrite the expanding system equations (2) as follows:

$$\begin{cases} \frac{dx}{dt} = x(t)(1-x(t)) - bf(t) - rx(t), \\ \frac{dy}{dt} = cf(t) - ey(t), \\ f(t) = \frac{x(t)y(t)}{x(t)+y(t)}, \end{cases} \tag{3}$$

$$x(0) = x_0, \quad y(0) = y_0, \quad f(0) = \frac{x_0 y_0}{x_0 + y_0}. \tag{4}$$

Now we want to approximate these equations system by HAM. This method firstly was developed by S. J. Liao in 1992 and was used by him and other scientist [16-20], its results have negligible differences with Runge-Kutta (RKF78) in comparison figures. In addition, the authors use an Euclidean residual and determine the auxiliary parameters h_1 and h_2 so that the residual approaches to zero. Here, HAM is applied on (3) and the results of this method are compared with the results of homotopy perturbation method (HPM) and Adomian decomposition method (ADM) [21-23].

2.Homotopy Analysis Solution

For convenience of the readers, we will first present a brief description of the standard HAM. To achieve our goal, let us assume the nonlinear differential equations be in the form of

$$N_j[u_1(t), u_2(t), ..., u_s(t)] = 0, \quad j = 1...n, \tag{5}$$

where N_j are nonlinear operators, t is an independent variable, $u_i(t)$ are unknown functions. By means of generalizing the traditional homotopy method, Liao's so called zeroth-order deformation equation will be

$$(1-q)L_j[\phi_i(t,q) - u_{i0}(t)] = qh_kH(t)N_j[\phi_1(t,q), \phi_2(t,q), ..., \phi_s(t,q)], \tag{6}$$
$$i = 1...s, j = 1, ...n, k \leq j$$

where $q \in [0,1]$ is an embedding parameter, $H(t)$ is an auxiliary function, h_k are nonzero auxiliary parameters, L_j is an linear operator, $u_{i,0}(t)$ are initial guesses of $u_i(t)$ and $\phi_i(t;q)$ are unknown functions. It is important to note that, one has great freedom to choose auxiliary objects such as $H(t)$ and L_j in HAM; this freedom plays an important role in establishing the keyston of validity and flexibility of HAM as shown in this paper. Obviously, when $q = 0$ and $q = 1$, both

$$\phi_i(t,0) = u_{i,0}(t) \quad and \quad \phi_i(t,1) = u_i(t), \tag{7}$$

hold. Thus as q increases from 0 to 1, the solutions $\phi_i(t;q)$ changes from the initial guesses $u_{i,0}(t)$ to the solutions $u_i(t)$. Expanding $\phi_i(t;q)$ in Taylor series with respect to q, one has

$$\phi_i(t,q) \quad = \quad u_{i,0}(t) + \sum_{s=1}^{+\infty} u_{i,m}(t)q^m, \qquad i = 1, ..., s. \tag{8}$$

where

$$u_{i,m}(t) = \frac{1}{m!}\frac{\partial^m \phi_i(t,q)}{\partial q^m}|_{q=0}, \qquad i = 1, ...s. \tag{9}$$

If the auxiliary linear operator, the initial guesses, the auxiliary parameters h_k, and the auxiliary functions are so properly chosen, then the series (8) converges at $q = 1$, one has

$$\phi_i(t,1) = u_{i,0}(t) + \sum_{s=1}^{+\infty} u_{i,s}(t), \qquad i = 1, ...s, \tag{10}$$

which must be one of the solutions of the original nonlinear equations, as proved by Liao.

Define the vectors

$$\vec{u}_{i,n}(t) = \{u_{i,0}(t), u_{i,1}(t), ..., u_{i,n}(t)\}, \qquad i = 1, ..., s. \tag{11}$$

Differentiating (16), m times with respect to the embedding parameter q and then setting $q = 0$ and finally dividing them by $m!$, we have the so-called N^{th}-order deformation equation:

$$L[u_{i,m}(t) - \chi_m u_{i,m-1}(t)] = h_i R_{i,m}(\vec{u}_{i,m-1}), \quad i = 1, ..., s. \tag{12}$$

Where

$$R_{i,m}(\vec{u}_{i,m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N_j[\phi_i(t,q)]}{\partial q^{m-1}}|_{q=0}, \qquad j = 1, ...n, i = 1, ..., s. \tag{13}$$

and

$$\chi_{\rm m} = \begin{cases} 0 & m \leq 1, \\ 1 & m > 1. \end{cases} \tag{14}$$

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To achieve our goal, let us assume the nonlinear differential equations be in the form of

$$N_j[u_1(t), u_2(t), ..., u_s(t)] = 0, \quad j = 1...n, \tag{15}$$

where N_j are nonlinear operators, t is an independent variable, $u_i(t)$ are unknown functions. By means of generalizing the traditional homotopy method, Liao's so called zeroth-order deformation equation will be

$$(1-q)L_j[\phi_i(t,q) - u_{i0}(t)] = q h_k H(t) N_j[\phi_1(t,q), \phi_2(t,q), ..., \phi_s(t,q)], \tag{16}$$
$$i = 1...s, j = 1, ...n, k \leq j$$

where $q \in [0,1]$ is an embedding parameter, $H(t)$ is an auxiliary function, h_k are nonzero auxiliary parameters, L_j is an linear operator, $u_{i,0}(t)$ are initial guesses of $u_i(t)$ and $\phi_i(t;q)$ are unknown functions. It is important to note that, one has great freedom to choose auxiliary objects such as $H(t)$ and L_j in HAM; this freedom plays an important role in establishing the keyston of validity and flexibility of HAM as shown in this paper. Obviously, when $q = 0$ and $q = 1$, both

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$$\phi_i(t,q) \quad = \quad u_{i,0}(t) + \sum_{s=1}^{+\infty} u_{i,m}(t)q^m, \qquad i = 1, ..., s, \tag{18}$$

where

$$u_{i,m}(t) = \frac{1}{m!}\frac{\partial^m \phi_i(t,q)}{\partial q^m}|_{q=0}, \qquad i = 1, ...s. \tag{19}$$

If the auxiliary linear operator, the initial guesses, the auxiliary parameters h_k, and the auxiliary functions are so properly chosen, then the series (8) converges at $q = 1$, one has

$$\phi_i(t,1) = u_{i,0}(t) + \sum_{s=1}^{+\infty} u_{i,s}(t), \qquad i = 1, ...s, \tag{20}$$

which must be one of the solutions of the original nonlinear equations, as proved by Liao.Define the vectors

$$\vec{u}_{i,n}(t) = \{u_{i,0}(t), u_{i,1}(t), ..., u_{i,n}(t)\}, \qquad i = 1, ..., s. \tag{21}$$

Differentiating (16), m times with respect to the embedding parameter q and then setting $q = 0$ and finally dividing them by $m!$, we have the so-called N^{th}-order deformation equation:

$$L[u_{i,m}(t) - \chi_m u_{i,m-1}(t)] = h_i R_{i,m}(\vec{u}_{i,m-1}), \quad i = 1, ..., s, \tag{22}$$

where

$$R_{i,m}(\vec{u}_{i,m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N_j[\phi_i(t,q)]}{\partial q^{m-1}}|_{q=0}, \qquad j = 1, ...n, i = 1, ..., s. \tag{23}$$

and

$$\chi_{\mathrm{m}} = \begin{cases} 0 & m \leq 1, \\ 1 & m > 1. \end{cases} \tag{24}$$

It should be emphasized that $u_{i,m}(t)$ is governed by the linear equations (16) and (20) with the linear boundary conditions that come from the original problem. These equations can be easily solved by sym be emphasized that $u_{i,m}(t)$ is governed by the linear equations (16) and (20) with the linear boundary conditions that come from the original problem. These equations can be easily solved by symbolic computation softwares such as Maple and Mathematica.

3.Applications

In this section, we will apply the HAM on (3). The solution of $x(t)$, $y(t)$ and $f(t)$ can be expressed by a set of base function

$$\{t^n, \quad n = 0, 1, ...\} \tag{25}$$

as following:

$$v_i(t) \quad = \quad \sum_{n=0}^{+\infty} c_{i,n}(h_1, h_2)t^n, \qquad i = 1, 2, 3, \tag{26}$$

where $c_{i,n}(h_1, h_2)$ are coefficient to be determined and $v_i(t)$ for $i = 1...3$, represent the solution of $x(t)$, $y(t)$ and $f(t)$. In HAM, we have the so-called rule of solution expression, i.e. the solution of (15) must be expressed in the same form as (26). Furthermore, under the first rule of solution expression and according to the condition in (3), it is straightforward to choose the initial approximation of $x(t)$, $y(x)$ and $f(t)$ as:

$$x_0(t) = x_0, \qquad y_0(t) = y_0, \quad f(0) = \frac{x_0 y_0}{x_0 + y_0} \tag{27}$$

and the auxiliary linear operator :

$$L[\phi_i(t, q)] = \frac{\partial \phi_i(t, q)}{\partial t}, \qquad i = 1, 2 \tag{28}$$

as well as:

$$L[\phi_3(t, q)] = \phi_3(t, q), \tag{29}$$

it possesses the property:

$$L[c_i] = 0, \tag{30}$$

where c_i $(i = 1, 2)$ are integral constants. Moreover, Eqs.(3) suggest to define the nonlinear operators

$$\begin{aligned}
N_1[\phi_1(t, q), \phi_3(t, q)] \quad &= \quad \frac{\partial \phi_1(t, q)}{\partial t} - \phi_1(t, q)[1 - \phi_1(t, q)] + b\phi_3(t, q) + r\phi_1(t, q), \\
N_2[\phi_2(t, q), \phi_3(t, q)] \quad &= \quad \frac{\partial \phi_2(t, q)}{\partial t} - c\phi_3 + e\phi_2(t, q), \\
N_3[\phi_1(t, q), \phi_2(t, q)] \quad &= \quad \frac{\phi_1(t, q)\phi_2(t, q)}{\phi_1(t, q) + \phi_2(t, q)}.
\end{aligned} \tag{31}$$

Using the above definition, we construct the zeroth-order deformation equations

$$\begin{aligned} (1-q)L[\phi_1(t,q)-x_0(t)] &= qh_1N_1[\phi_1(t,q),\phi_3(t,q)], \\ (1-q)L[\phi_2(t,q)-y_0(t)] &= qh_1N_2[\phi_2(t,q),\phi_3(t,q)], \\ (1-q)L[\phi_3(t,q)-f_0(t)] &= qh_2N_3[\phi_1(t,q),\phi_2(t,q)]. \end{aligned} \tag{32}$$

Obviously, when $q=0$ and $q=1$,we have

$$\phi_1(t,0) = x_0(t), \qquad \phi_1(t,1)=x(t), \tag{33}$$

$$\phi_2(t,0) = y_0(t), \qquad \phi_2(t,1)=y(t), \tag{34}$$

$$\phi_3(t,0) = f_0(t), \qquad \phi_3(t,1)=f(t). \tag{35}$$

Therefore, as the embedding parameter q increases from 0 to 1, $\phi_i(t,q)$ varies from the initial guesses $x_0(t)$, $y_0(t)$ and $f_0(t)$ to the solution $x(t)$, $y(t)$, and $f(t)$ respectively. Expanding $\phi_i(t,q)$ in Taylor series with respect to q one has

$$\begin{aligned} \phi_1(t,q) &= x_0(t)+\sum_{m=1}^{+\infty} x_m(t)q^m, \\ \phi_2(t,q) &= y_0(t)+\sum_{m=1}^{+\infty} y_m(t)q^m, \\ \phi_3(t,q) &= f_0(t)+\sum_{m=1}^{+\infty} f_m(t)q^m, \end{aligned} \tag{36}$$

where

$$\begin{aligned} x_m(t) &= \frac{1}{m!}\frac{\partial^m\phi_1(t,q)}{\partial q^m}|_{q=0}, \\ y_m(t) &= \frac{1}{m!}\frac{\partial^m\phi_2(t,q)}{\partial q^m}|_{q=0}, \\ f_m(t) &= \frac{1}{m!}\frac{\partial^m\phi_3(t,q)}{\partial q^m}|_{q=0}. \end{aligned} \tag{37}$$

Define the vectors:

$$\begin{aligned} \vec{x}_n(t) &= \{x_0(t),x_1(t),...,x_n(t)\}, \\ \vec{y}_n(t) &= \{y_0(t),y_1(t),...,y_n(t)\}, \\ \vec{f}_n(t) &= \{f_0(t),f_1(t),...,f_n(t)\}. \end{aligned} \tag{38}$$

So the N^{th}-order deformation equations are:

$$\begin{aligned} L[x_m(t)-\chi_m x_{m-1}(t)] &= h_1R_{1,m}(\vec{x}_{m-1}(t)), \\ L[y_m(t)-\chi_m y_{m-1}(t)] &= h_1R_{2,m}(\vec{y}_{m-1}(t)), \\ L[f_m(t)-\chi_m f_{m-1}(t)] &= h_2R_{3,m}(\vec{f}_{m-1}(t)), \end{aligned} \tag{39}$$

where

$$\begin{aligned} R_{1,m}(\vec{x}_{m-1}(t)) &= \frac{1}{(m-1)!}\frac{\partial^{m-1}N_1[\phi_1(t,q),\phi_3(t,q)]}{\partial q^{m-1}}|_{q=0}, \\ R_{2,m}(\vec{y}_{m-1}(t)) &= \frac{1}{(m-1)!}\frac{\partial^{m-1}N_2[\phi_2(t,q),\phi_3(t,q)}{\partial q^{m-1}}|_{q=0}, \\ R_{3,m}(\vec{f}_{m-1}(t)) &= \frac{1}{(m-1)!}\frac{\partial^{m-1}N_1[\phi_1(t,q),\phi_2(t,q)}{\partial q^{m-1}}|_{q=0}. \end{aligned} \tag{40}$$

Now, the solution of the N^{th}-order deformation equations (39) for $m \geq 1$ becomes

$$\begin{aligned} x_m(t) &= \chi_m x_{m-1}(t) + h_1\{\int_0^t [x'_{m-1}(\tau) - x_{m-1}(\tau) \\ &+ \sum_{k=0}^{m-1} x_k(\tau)x_{m-1-k}(\tau) + f_{m-1}(\tau) + rx_{m-1}(\tau)]d\tau\} + c_1, \end{aligned} \tag{41}$$

$$y_m(t) = \chi_m y_{m-1}(t) + h_1\int_0^t [y'_{m-1}(\tau) - cf_{m-1}(\tau) + ey_{m-1}(\tau)]d\tau + c_2, \tag{42}$$

$$f_m(t) = \chi_m f_{m-1}(t) + h_2(f_{m-1}(t) - \frac{\partial^{m-1}[\frac{\phi_1(t,q)\phi_2(t,q)}{\phi_1(t,q)+\phi_2(t,q)}]}{\partial q^{m-1}}|_{q=0}), \tag{43}$$

where the integration constants $c_i (i = 1, 2)$ are determined by the initial conditions (4). Then we obtain some sequences, for approximating the problem :

$$\begin{aligned} x_0(t) &= 0.5, \\ x_1(t) &= -0.05h_1t, \\ x_2(t) &= -0.05h_1t + 0.05h_1^2t - 0.0025h_1^2t^2, \\ &\vdots \end{aligned} \tag{44}$$

$$, \tag{45}$$

$$\begin{aligned} y_0(t) &= 0.3, \\ y_1(t) &= 0.1125h_1t, \\ y_2(t) &= 0.1125h_1t + 0.1125h_1^2t + 0.028125h_1^2t^2, \\ &\vdots \end{aligned} \tag{46}$$

$$and \tag{47}$$

$$\begin{aligned} f_0(t) &= 0.1875, \\ f_1(t) &= 0, \\ f_2(t) &= -0.0369140625h_1h_2t. \end{aligned} \tag{48}$$

Finally, we obtain the exact solution in the form of :

$$\begin{aligned} x(t) &= \sum_{m=0}^{\infty} x_m(t), \\ y(t) &= \sum_{m=0}^{\infty} y_m(t), \\ f(t) &= \sum_{m=0}^{\infty} f_m(t). \end{aligned} \tag{49}$$

4. Optimization method of the homotopy parameters and discussions

Some researchers like Kazuki Yabushita, Mariko Yamashita and Kazuhiro Tsuboi investigate the suitable values for h_1, h_2 by defining a residual for their nonlinear system and in this section we follow this way [24]. Here, we show the optimization method of the homotopy parameters h_1, h_2 for the order of approximation N. The homotopy parameter is an arbitrary constant when N is infinite; however, the optimum values of homotopy parameter should be found under the finite number of N. This method can be applied to the problem without the exact solution. The N^{th}-order approximate solutions of (32) are defined as follows:

$$\begin{aligned} x_N(t) &= \sum_{m=0}^{N} x_m(t), \\ y_N(t) &= \sum_{m=0}^{N} y_m(t), \\ f_N(t) &= \sum_{m=0}^{N} f_m(t). \end{aligned} \tag{50}$$

Now ,We consider a residual of the Nth-order approximate solutions for (3). The residual is expressed as follows:

$$\begin{aligned} \epsilon_N(h_1,h_2) = \quad & [\{\int_0^2 \frac{dx_N(t)}{dt} - x_N(t)(1-x_N(t)) + bf_N(t) + rx_N(t)\}^2 \\ & +\{\int_0^2 \frac{dy_N(t)}{dt} - cf_N(t) + ey_N(t)\}^2]^{\frac{1}{2}}. \end{aligned} \tag{51}$$

Ultimately, we consider two cases of Ratio-dependent predator-prey system with constant effort harvesting :
case 1: $b = 0.8$, $c = 0.2$, $e = 0.5$, $r = 0.1$ and initial conditions $x_0 = 0.5$, $y_0 = 0.3$,
case 2: $b = 0.5$, $c = 0.5$, $e = 0.1$, $r = 0.2$ and initial conditions $x_0 = 0.5, y_0 = 0.2$.

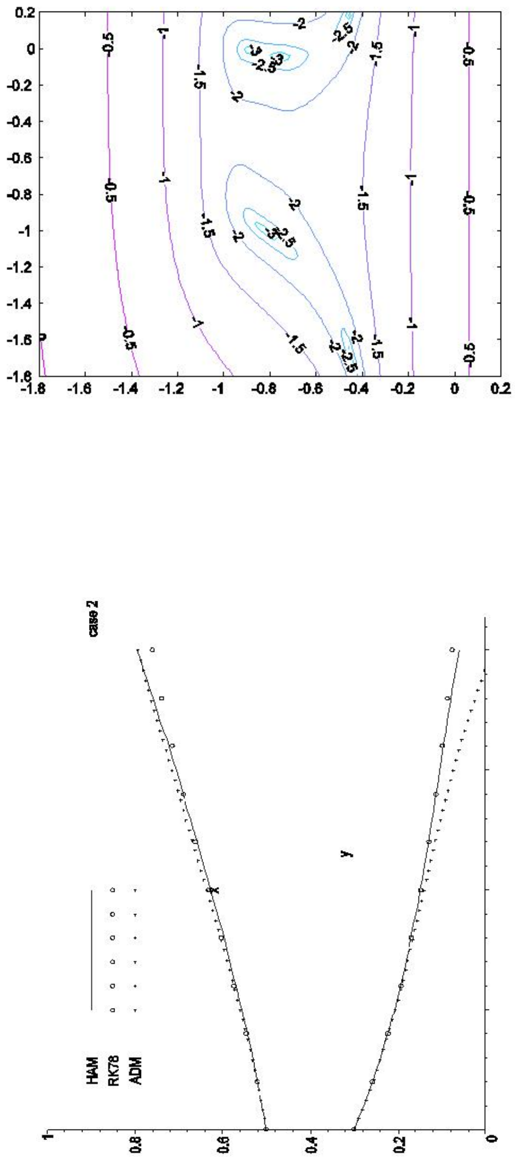
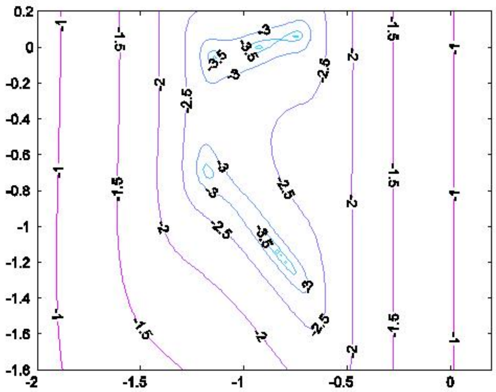


Figure 1: (Top):Contours of $log_{10}\epsilon_3(h_1, h_2)$, (Bottom): Variations of $x(t)$ and $y(t)$ $(x_0 = 0.5, y_0 = 0.3)$.



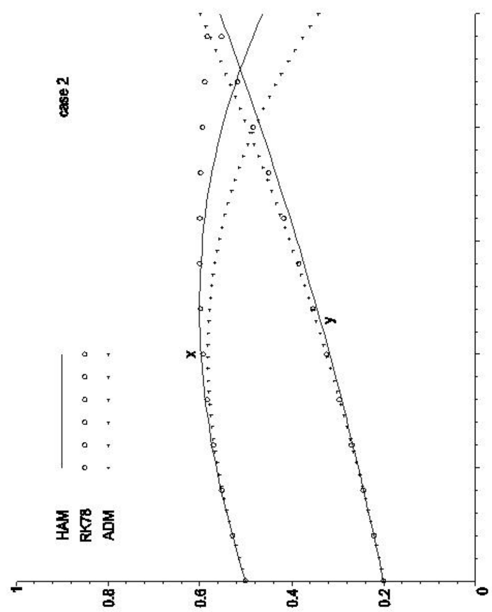


Figure 2: (Top):Contours of $log_{10}\epsilon_3(h_1, h_2)$, (Bottom): Variations of $x(t)$ and $y(t)$ $(x_0 = 0.5, y_0 = 0.2)$.

The figures for each cases are drawn and the solutions are represented and compared with other methods. First we consider the case 1 and draw $log_{10}\epsilon_3(h_1, h_2)$ to obtain suitable values for h_1 and h_2. according to this diagram by choosing $h_1 = -.81$, $h_2 = -1.02$, we can accelerate our accuracy. Now we show our results in this diagram: Similarly, the process of case 2, $h_1 = -.8$, $h_2 = -1.17$ are found likely.

comparing the results with HPM and ADM shows that this method is more effective, moreover its validity verified by approaching to $(RKF78)$.

Conclusion

In this letter, we use the HAM to approximate system of the Ratio-dependent Predator-prey equations. We defined an Euclidean residual for the mentioned system and obtained optimal values for h_1 and h_2.The optimum values of h_1 and h_2 for the order of approximation N are determined successfully. Ultimately, we show the present solution has sufficient validity because it agrees with RKF78 solution and has high accuracy respect to HPM and ADM methods.

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