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SHARP FUNCTION ESTIMATE AND BOUNDEDNESS ON L^P FOR MULTILINEAR COMMUTATOR OF PSEUDO-DIFFERENTIAL OPERATORS

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Abstract. In this paper, we establish a sharp estimate for the multilinear commutator associated to a class of pseudo-differential operators. By using the sharp estimate, we obtain the $L^p(1 < p < \infty)$ norm inequality for the multilinear commutator.

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1. Introduction and Theorems

As the development of the Calderón-Zygmund singular integral operators, their commutators and multilinear operators have been well studied(see [4][6-8]). In [4], Hu and Yang proved a variant sharp function estimate for the multilinear singular integral operators. In [6-8], C. Pérez, G. Pradolini and R. Trujillo-Gonzalez obtained a sharp weighted estimates for the singular integral operators and their commutators. The boundedness of the pseudo-differential operators was studied by many authors(see [1-3][5][9-12]). In [9], the boundedness of the commutators associated to the pseudo-differential operators was obtained.

The main purpose of this paper is to prove the sharp function inequality for the Multilinear Commutator associated to a class of pseudo-differential operators with symbols $\delta(x,\xi)$ in the class $S^{-na/2}_{1-a,\sigma}, 0 \le \sigma < 1-a, 0 < a < 1$. By using the sharp inequality, we obtain the $L^p(1 < p < \infty)$ norm inequality for the multilinear commutator. In order to state our results, we begin by introducing the relevant notions and definitions.

We say b belongs to $BMO(R^n)$, if $b^{\#} \in L^\infty(R^n)$, and we define $||b||_{BMO} = ||b^{\#}||_{L^\infty}$.

Given some function $b_j, 1 \le j \le m$, we denote by C^m_j the family of all finite subsets $\sigma = \{\sigma(1), \cdots, \sigma(j)\}$ of j different elements in $\{1, \cdots, m\}$. For $\gamma \in C^m_j$, we denote $\gamma^c = \{1, \cdots, m\} \backslash \gamma$. For $\vec{b} = \{b_1, \cdots, b_m\}$ and $\gamma = \{\gamma(1), \cdots, \gamma(j)\} \in C_j^m$, we denote $\vec{b_\gamma} = \{b_{\gamma(1)}, \cdots, b_{\gamma(j)}\}$, and $b_\gamma = b_{\gamma(1)} \cdots b_{\gamma(j)}$, and $||b_\gamma||_{BMO} = ||b_{\gamma(1)}||_{BMO} \cdots ||b_{\gamma(j)}||_{BMO}$.

We set $M_r(f)(x) = \sup_{Q \ni x} \left(\frac{1}{|Q|} \int_Q |f(y)|^r dy\right)^{1/r}$, where Q is a cube with sides parallel to the coordinate axes. $M_r(f)$ is the generalized Hardy-Littlewood maximal function of f.

We also need the sharp maximal function $f^\#$ of f, which is given by

$$f^\#(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where $f_Q = \frac{1}{|Q|} \int_Q f(y) dy$.

And it is well-known that

$$f^\#(x) \approx \sup_{Q \ni x} \inf_{c \in C} \frac{1}{|Q|} \int_Q |f(y) - c| dy.$$

We say $\delta(x, \xi) \in S^m_{\epsilon,\sigma}$, if for $x, \xi \in R^n$, $\left|\frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial \xi^\beta} \delta(x, \xi)\right| \le C_{\alpha,\beta}(1 + |\xi|)^{m - \epsilon|\beta| + \sigma|\alpha|}$.

The pseudo-differential operators $\psi \cdot d \cdot o \cdot$ with symbols $\delta(x, \xi) \in S^m_{\epsilon,\sigma}$ is given by

$$T(f)(x) = \int_{R^n} e^{2\pi i(x,\xi)} \delta(x, \xi) \hat{f}(\xi) d\xi,$$

where f is a Schwartz function and $\hat{f}$ denotes the Fourier transform of f.

The pseudo-differential operators $\psi \cdot d \cdot o \cdot$ also have another expression

$$T(f)(x) = \int_{R^n} K(x, x - y) f(y) dy,$$

where $K(x, x - y) = \int_{R^n} e^{2\pi i(x-y)\cdot\xi} \delta(x, \xi) d\xi$.

Let b_j $(j = 1, \cdots, m)$ be the fixed locally integrable functions on R^n. The multilinear commutator associated to the pseudo-differential operator is defined by

$$T_{\vec{b}}(f)(x) = \int_{R^n} \prod_{j=1}^m (b_j(x) - b_j(y)) K(x, x - y) f(y) dy.$$

Now we state the main results as follows.

Theorem 1. *Let T be a $\psi \cdot d \cdot o \cdot$ with symbol $\delta(x, \xi) \in S^{-na/2}_{1-a,\sigma}, 0 \le \sigma < 1 - a, 0 < a < 1, b_j \in BMO(R^n), 1 \le j \le m$. Then there exists a constant $C > 0$ such that, for any $f \in C_0^\infty(R^n)$, $\tilde{x} \in R^n$ and $2 < r < \infty$,*

$$(T_{\vec{b}}(f))^\#(\tilde{x}) \le C||b||_{BMO} \left(M_r(f)(\tilde{x}) + \sum_{j=1}^m \sum_{\gamma \in C_j^m} M_r(T_{\vec{b}_{\gamma^c}}(f))(\tilde{x}) \right).$$

Theorem 2. *Let T be a $\psi \cdot d \cdot o \cdot$ with symbol $\delta(x,\xi) \in S^{-na/2}_{1-a,\sigma}, 0 \leq \sigma < 1-a, 0 < a < 1, b_j \in BMO(R^n), 1 \leq j \leq m$. Then $T_{\vec{b}}$ is bounded on $L^p(R^n)$ for $2 < p < \infty$.*

2. Preliminary Lemmas

Lemma 1. *Let Q be a cube and $b_j \in BMO(R^n), 1 \leq j \leq m, m \geq 1$, then for $1 < q < \infty$,*

$$\frac{1}{|Q|}\int_Q \prod_{j=1}^m |b_j(y) - (b_j)_Q| dy \leq C \prod_{j=1}^m ||b_j||_{BMO}$$

and

$$\left(\frac{1}{|Q|}\int_Q \prod_{j=1}^m |b_j(y) - (b_j)_Q|^q dy\right)^{1/q} \leq C \prod_{j=1}^m ||b_j||_{BMO},$$

where $(b_j)_Q = \frac{1}{|Q|}\int_Q b_j(y)dy$.

Proof. Choose $p_j, 1 \leq j \leq m$ such that $1/p_1 + \cdots + 1/p_m = 1$. By the Hölder's inequality and Corollary of the John-Nirenberg inequality in Chapter 4 in Stein' book(see[1]),

$$\begin{aligned} & \frac{1}{|Q|}\int_Q \prod_{j=1}^m |b_j(y) - (b_j)_Q| dy \\ \leq \; & C \prod_{j=1}^m \left(\frac{1}{|Q|}\int_Q |b_j(y) - (b_j)_Q|^{p_j} dy\right)^{1/p_j} \\ \leq \; & C \prod_{j=1}^m ||b_j||_{BMO} \end{aligned}$$

and

$$\begin{aligned} & \left(\frac{1}{|Q|}\int_Q \prod_{j=1}^m |b_j(y) - (b_j)_Q|^q dy\right)^{1/q} \\ \leq \; & C \prod_{j=1}^m \left(\frac{1}{|Q|}\int_I |b_j(y) - (b_j)_Q|^{qp_j} dy\right)^{1/qp_j} \\ \leq \; & C \prod_{j=1}^m ||b_j||_{BMO}. \end{aligned}$$

The lemma 1 follows.

Lemma 2. **(see [10])** *Let Q be a cube and $b \in BMO(R^n)$, then for $k \geq 1$, there exists a constant C such that $||b - b_{2^kQ}||_{BMO} \leq Ck||b||_{BMO}$, where $b_{2^kQ} = \frac{1}{|2^kQ|}\int_{2^kQ} f(y)dy$.*

Lemma 3. **(see [1])** *Let $\delta(x,\xi) \in S^{-na/2}_{1-a,\sigma}, 0 \leq \sigma < 1-a, 0 < a < 1$, and $K(x,w)$ denote the inverse Fourier transformations in the ξ-variable and in the distribution sense of $\delta(x,\xi)$, that is informally $K(x,w) = \int_{R^n} e^{2\pi i w\cdot\xi}\delta(x,\xi)d\xi$, then for $|x - x_0| \leq d \leq 1/2$ and $N \geq 0$,*

$$\begin{aligned} &\left(\int_{(2^N d)^{1-a} \leq |y-x_0| \leq (2^{N+1}d)^{1-a}} |K(x,x-y) - K(x_0,x_0-y)|^2 dy\right)^{1/2} \\ \leq\ & C|x-x_0|^{(1-a)(m-n/2)}(2^{N+1}d)^{-m(1-a)}, \end{aligned}$$

where m is an integer such that $n/2 < m < n/2 + 1/(1-a)$.

Lemma 4. **(see [1])** *Let $\delta(x,\xi) \in S^0_{\epsilon,\sigma}, 0 < \epsilon < 1$, and as usual $K(x,w) = \int_{R^n} e^{2\pi i w\cdot\xi}\delta(x,\xi)d\xi$, then for $|w| \geq 1/4$ and arbitrarily large M, $|K(x,w)| \leq C_M|w|^{-2M}$.*

Lemma 5. **(see [1])** *Given $\delta(x,\xi) \in S^{-na/2}_{1-a,\sigma}, 0 \leq \sigma < 1-a, 0 < a < 1$, then for $1 < p < \infty$, we have $||T(f)||_p \leq C_p||f||_p$.*

3. Proof of Theorem

Proof of Theorem 1. It suffices to prove, for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds

$$\frac{1}{|Q|}\int_Q |T_{\vec{b}}(x) - C_0|dx \leq C||b||_{BMO}\left(M_r(f)(\tilde{x}) + \sum_{j=1}^m \sum_{\gamma\in C_j^m} M_r(T_{\vec{b}_{\gamma^c}}(f))(\tilde{x})\right).$$

Fix a cube $Q = Q(x_0,d)$ and $\tilde{x} \in Q$.

We first consider the case $m = 1$ and $d \leq 1$.

Let $f(x) = f_1(x) + f_2(x)$, with $f_1(x) = f(x)\chi_Q(x)$ and $f_2(x) = f(x)\chi_{Q^c}(x)$, where J is a cube concentric with Q of side-length d^{1-a}.

$$\begin{aligned} &T_{\vec{b_1}}(f)(x) = \int_{R^n} K(x,x-y)((b_1(x) - (b_1)_J) - (b_1(y) - (b_1)_J))f(y)dy \\ =\ & (b_1(x) - (b_1)_J)\int_{R^n} K(x,x-y)f(y)dy - \int_{R^n} K(x,x-y)(b_1(y) - (b_1)_J)f(y)dy \\ =\ & (b_1(x) - (b_1)_J)\int_{R^n} K(x,x-y)f(y)dy - \int_{R^n} K(x,x-y)(b_1(y) - (b_1)_J)f_1(y)dy \\ & - \int_{R^n} K(x,x-y)(b_1(y) - (b_1)_J)f_2(y)dy \\ =\ & (b_1(x) - (b_1)_J)T(f)(x) - T((b_1 - (b_1)_J)f_1)(x) - T((b_1 - (b_1)_J)f_2)(x), \end{aligned}$$

thus

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q |T_{b_1}(f)(x) - T(((b_1)_J - b_1)f_2)(x_0)|dx \\
= &\frac{1}{|Q|}\int_Q |(b_1(x) - (b_1)_J)T(f)(x) - T((b_1 - (b_1)_J)f_1)(x) \\
- &T((b_1 - (b_1)_J)f_2)(x) - T(((b_1)_J - b_1)f_2)(x_0)|dx \\
\le &\frac{1}{|Q|}\int_Q |b_1(x) - (b_1)_J||T(f)(x)|dx + \frac{1}{|Q|}\int_Q |T((b_1 - (b_1)_J)f_1)(x)|dx \\
&+\frac{1}{|Q|}\int_Q |T((b_1 - (b_1)_J)f_2)(x) - T((b_1 - (b_1)_J)f_2)(x_0)|dx \\
= &A_1 + B_1 + C_1.
\end{aligned}$$

For A_1, by Hölder's inequality with exponent $1/r + 1/r' = 1$ and lemma 1, we get

$$\begin{aligned}
A_1 &\le C\left(\frac{1}{|J|}\int_J |b_1(x) - (b_1)_J|^{r'}dx\right)^{1/r'}\left(\frac{1}{|Q|}\int_Q |T(f)(x)|^r dx\right)^{1/r} \\
&\le C||b_1||_{BMO} M_r(T(f))(\tilde{x}).
\end{aligned}$$

For B_1, choose $s, q, 1 < s, q < \infty$ such that $qs = r$. By Hölder's inequality and the boundedness of T on the $L^q(R^n)$ and lemma 1, we get

$$\begin{aligned}
B_1 &\le C\left(\frac{1}{|Q|}\int_Q |T((b_1 - (b_1)_J)f_1(x)|^q dx\right)^{1/q} \\
&\le C\left(\frac{1}{|Q|}\int_{R^n} |T((b_1 - (b_1)_J)f_1)(x)|^q dx\right)^{1/q} \\
&\le C\left(\frac{1}{|Q|}\int_{R^n} |b_1(x) - (b_1)_J|^q|f_1(x)|^q dx\right)^{1/q} \\
&\le C\left(\frac{1}{|J|}\int_J |b_1(x) - (b_1)_J|^{qs'}\right)^{1/qs'}\left(\frac{1}{|J|}\int_J |f(x)|^{qs}dx\right)^{1/qs} \\
&\le C||b_1||_{BMO} M_r(f)(\tilde{x}).
\end{aligned}$$

For C_1, choose $v, 1 < v < \infty$ such that $1/v + 1/r + 1/2 = 1$. By Hölder's inequality and lemma 2,3 and $n/2 < m < n/2 + 1/(1-a)$, we get

$$\begin{aligned}
C_1(x) &= |T((b_1 - (b_1)_J)f_2)(x) - T((b_1 - (b_1)_J)f_2)(x_0)| \\
&= \left|\int_{|y-x_0|>d^{1-a}} (K(x, x-y) - K(x_0, x_0 - y))(b_1(y) - (b_1)_J)f(y)dy\right| \\
&\le \int_{|y-x_0|>d^{1-a}} |K(x, x-y) - K(x_0, x_0 - y)||b_1(y) - (b_1)_J||f(y)|dy
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{N=0}^{\infty} \int_{(2^N d)^{1-a} \leq |y-x_0| \leq (2^{N+1}d)^{1-a}} |K(x,x-y) - K(x_0,x_0-y)| \\
&\times |b_1(y) - (b_1)_J||f(y)|dy \\
&\leq C \sum_{N=0}^{\infty} \left(\int_{(2^N d)^{1-a} \leq |y-x_0| \leq (2^{N+1}d)^{1-a}} |K(x,x-y) - K(x_0,x_0-y)|^2 dy \right)^{1/2} \\
&\times \left(\int_{|y-x_0| \leq (2^{N+1}d)^{1-a}} |b_1(y) - (b_1)_J|^v dy \right)^{1/v} \left(\int_{|y-x_0| \leq (2^{N+1}d)^{1-a}} |f(y)|^r dy \right)^{1/r} \\
&\leq C \sum_{N=0}^{\infty} |x-x_0|^{(1-a)(m-n/2)} (2^{N+1}d)^{-m(1-a)+n(1-a)/2} \\
&\times \left(\frac{1}{(2^{N+1}d)^{n(1-a)}} \int_{|y-x_0| \leq (2^{N+1}d)^{1-a}} |b_1(y) - (b_1)_J|^v dy \right)^{1/v} \\
&\times \left(\frac{1}{(2^{N+1}d)^{n(1-a)}} \int_{|y-x_0| \leq (2^{N+1}d)^{1-a}} |f(y)|^r dy \right)^{1/r} \\
&\leq C \sum_{N=0}^{\infty} d^{(1-a)(m-n/2)} (2^{N+1}d)^{(1-a)(n/2-m)} N||b_1||_{BMO} M_r(f)(x_0) \\
&\leq C \sum_{N=0}^{\infty} 2^{(N+1)(1-a)(n/2-m)} N||b_1||_{BMO} M_r(f)(x_0) \\
&\leq C||b_1||_{BMO} M_r(f)(\tilde{x}),
\end{aligned}
$$

thus

$$C_1 \leq C||b_1||_{BMO} M_r(f)(\tilde{x}).$$

Combining all the estimates, we finish the case $m = 1$ and $d \leq 1$.

In case $m = 1$ and $d > 1$, we proceeds the case as follows.

Let $f(x) = f_1(x) + f_2(x)$, with $f_1(x) = f(x)\chi_{2Q}(x)$ and $f_2(x) = f(x)\chi_{(2Q)^c}(x)$. We have

$$
\begin{aligned}
& T_{\vec{b_1}}(f)(x) = \int_{R^n} K(x,x-y)((b_1(x) - (b_1)_{2Q}) - (b_1(y) - (b_1)_{2Q}))f(y)dy \\
&= (b_1(x) - (b_1)_{2Q}) \int_{R^n} K(x,x-y)f(y)dy - \int_{R^n} K(x,x-y)(b_1(y) - (b_1)_{2Q})f(y)dy \\
&= (b_1(x) - (b_1)_{2Q}) \int_{R^n} K(x,x-y)f(y)dy - \int_{R^n} K(x,x-y)(b_1(y) - (b_1)_{2Q})f_1(y)dy \\
&\quad - \int_{R^n} K(x,x-y)(b_1(y) - (b_1)_{2Q})f_2(y)dy \\
&= (b_1(x) - (b_1)_{2Q})T(f)(x) - T((b_1 - (b_1)_{2Q})f_1)(x) - T((b_1 - (b_1)_{2Q})f_2)(x),
\end{aligned}
$$

thus

$$\begin{aligned}
& \frac{1}{|Q|}\int_Q |T_{\vec{b_1}}(f)(x)|dx \\
= & \frac{1}{|Q|}\int_Q |(b_1(x)-(b_1)_{2Q})T(f)(x)-T((b_1-(b_1)_{2Q})f_1)(x)-T((b_1-(b_1)_{2Q})f_2)(x)|dx \\
\leq & \frac{1}{|Q|}\int_Q |b_1(x)-(b_1)_{2Q}||T(f)(x)|dx \\
+ & \frac{1}{|Q|}\int_Q |T((b_1-(b_1)_{2Q})f_1)(x)|dx \\
+ & \frac{1}{|Q|}\int_Q |T((b_1-(b_1)_{2I})f_2)(x)|dx \\
= & A_2+B_2+C_2.
\end{aligned}$$

Similar to A_1, $A_2 \leq C||b_1||_{BMO}M_r(T(f))(\tilde{x})$.
Similar to B_2, $B_2 \leq C||b_1||_{BMO}M_r(f)(\tilde{x})$.

For C_2, by Hölder's inequality with exponent $1/r+1/r'=1$ and lemma 2, 4, we get

$$\begin{aligned}
C_2(x) &= |T((b_1-(b_1)_{2Q})f_2)(x)| \\
&= \left|\int_{|y-x_0|>2d} K(x,x-y)(b_1(y)-(b_1)_{2Q})f(y)dy\right| \\
&\leq \int_{|y-x_0|>2d} |K(x,x-y)||b_1(y)-(b_1)_{2Q}||f(y)|dy \\
&\leq C\int_{|y-x_0|>2d} |x-y|^{-2n}|b_1(y)-(b_1)_{2Q}||f(y)|dy \\
&\leq C\sum_{N=1}^{\infty}\int_{|y-x_0|\leq 2^{N+1}d} |x-y|^{-2n}|b_1(y)-(b_1)_{2Q}||f(y)|dy \\
&\leq C\sum_{N=1}^{\infty}\left(2^{N+1}d\right)^{-2n+n}\left(\frac{1}{(2^{N+1}d)^n}\int_{|y-x_0|\leq 2^{N+1}d}|b_1(y)-(b_1)_{2Q}|^{r'}dy\right)^{1/r'} \\
&\quad \times\left(\frac{1}{(2^{N+1}d)^n}\int_{|y-x_0|\leq 2^{N+1}d}|f(y)|^r dy\right)^{1/r} \\
&\leq C\sum_{N=1}^{\infty}(2^{N+1}d)^{-n}N||b_1||_{BMO}M_r(f)(\tilde{x}) \\
&\leq C||b_1||_{BMO}M_r(f)(\tilde{x}),
\end{aligned}$$

thus

$$C_2 \leq C||b_1||_{BMO}M_r(f)(\tilde{x}).$$

Combining all the estimates, we finish the case $m = 1$ and $d > 1$.

Now, we consider the case $m \geq 2$ and $d \leq 1$.

Let $f(x) = f_1(x) + f_2(x)$, with $f_1(x) = f(x)\chi_J(x)$ and $f_2(x) = f(x)\chi_{J^c}(x)$, where J is a cube concentric with Q of side-length d^{1-a}.

$$\begin{aligned}
T_{\vec{b}}(f)(x) &= \int_{R^n} K(x, x-y) \prod_{j=1}^{m}((b_j(x) - (b_j)_J) - (b_j(y) - (b_j)_J))f(y)dy \\
&= \sum_{j=0}^{m} \sum_{\gamma \in C_j^m} (-1)^{m-j}(b(x) - b_J)_\gamma \int_{R^n} K(x, x-y)(b(y) - b_J)_{\gamma^c} f(y)dy \\
&= \prod_{j=1}^{m}(b_j(x) - (b_j)_J) \int_{R^n} K(x, x-y)f(y)dy \\
&\quad + \sum_{j=1}^{m-1} \sum_{\gamma \in C_j^m} (-1)^{m-j}(b(x) - b_J)_\gamma \int_{R^n} K(x, x-y)(b(y) - b_J)_{\gamma^c} f(y)dy \\
&\quad + (-1)^m \int_{R^n} K(x, x-y) \prod_{j=1}^{m}(b_j(y) - (b_j)_J)f(y)dy \\
&= (2^m - 1) \prod_{j=1}^{m}(b_j(x) - (b_j)_J) \int_{R^n} K(x, x-y)f(y)dy \\
&\quad + \sum_{j=1}^{m-1} \sum_{\gamma \in C_j^m} (b(x) - b_J)_\gamma \int_{R^n} K(x, x-y)(b(x) - b(y))_{\gamma^c} f(y)dy \\
&\quad + (-1)^m \int_{R^n} K(x, x-y) \prod_{j=1}^{m}(b_j(y) - (b_j)_J)f(y)dy \\
&= (2^m - 1) \prod_{j=1}^{m}(b_j(x) - (b_j)_J) \int_{R^n} K(x, x-y)f(y)dy \\
&\quad + \sum_{j=1}^{m-1} \sum_{\gamma \in C_j^m} (b(x) - b_J)_\gamma \int_{R^n} K(x, x-y)(b(x) - b(y))_{\gamma^c} f(y)dy \\
&\quad + (-1)^m \int_{R^n} K(x, x-y) \prod_{j=1}^{m}(b_j(y) - (b_j)_J)f_1(y)dy \\
&\quad + (-1)^m \int_{R^n} K(x, x-y) \prod_{j=1}^{m}(b_j(y) - (b_j)_J)f_2(y)dy \\
&= (2^m - 1) \prod_{j=1}^{m}(b_j(x) - (b_j)_J)T(f)(x)
\end{aligned}$$

$$+\sum_{j=1}^{m-1}\sum_{\gamma\in C_j^m}(b(x)-b_J)_\gamma T_{\vec{b_{\gamma^c}}}(f)(x)$$
$$+(-1)^m T(\prod_{j=1}^m(b_j-(b_j)_J)f_1)(x)$$
$$+(-1)^m T(\prod_{j=1}^m(b_j-(b_j)_J)f_2)(x),$$

thus

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q |T_{\vec{b}}(f)(x)-T(\prod_{j=1}^m((b_j)_J-b_j)f_2)(x_0)|dx\\
\leq\ & \frac{2^m-1}{|Q|}\int_Q\prod_{j=1}^m|b_j(x)-(b_j)_J||T(f)(x)|dx\\
&+\sum_{j=1}^{m-1}\sum_{\gamma\in C_j^m}\frac{1}{|Q|}\int_Q|(b(x)-b_J)_\gamma||T_{\vec{b_{\gamma^c}}}(f)(x)|dx\\
&+\frac{1}{|Q|}\int_Q|T(\prod_{j=1}^m(b_j-(b_j)_J)f_1)(x)|dx\\
&+\frac{1}{|Q|}\int_Q|T(\prod_{j=1}^m(b_j-(b_j)_J)f_2)(x)-T(\prod_{j=1}^m(b_j-(b_j)_J)f_2)(x_0)|dx\\
=\ & D_1+E_1+F_1+G_1.
\end{aligned}$$

For D_1, choose $p_j, 1\leq j\leq m$, such that $1/p_1+\cdots+p_m+1/r=1$. By Hölder's inequality and lemma 1, we get

$$\begin{aligned}
D_1 &\leq C\prod_{j=1}^m\left(\frac{1}{|J|}\int_J|b_j(x)-(b_j)_J|^{p_j}|dx\right)^{1/p_j}\left(\frac{1}{|Q|}\int_Q|T(f)(x)|^r dx\right)^{1/r}\\
&\leq C\prod_{j=1}^m||b_j||_{BMO}M_r(T(f))(x_0)\\
&\leq C||b||_{BMO}M_r(T(f))(\tilde{x}).
\end{aligned}$$

For E_1, by Hölder's inequality with exponent $1/r+1/r'=1$ and lemma 1, we get

$$E_1 \leq C\sum_{j=1}^{m-1}\sum_{\gamma\in C_j^m}\left(\frac{1}{|J|}\int_J|(b(x)-b_J)_\gamma|^{r'}|dx\right)^{1/r'}\left(\frac{1}{|Q|}\int_Q|T_{\vec{b_{\gamma^c}}}(f)(x)|^r dx\right)^{1/r}$$

$$
\begin{aligned}
&\leq C\sum_{j=1}^{m-1}\sum_{\gamma\in C_j^m}||b_\gamma||_{BMO}M_r(T_{\vec{b_{\gamma^c}}}(f))(\tilde{x})\\
&\leq C\sum_{j=1}^{m-1}\sum_{\gamma\in C_j^m}||b||_{BMO}M_r(T_{\vec{b_{\gamma^c}}}(f))(\tilde{x}).
\end{aligned}
$$

For F_1, choose $q, s, s_j, 1 \leq j \leq m$ such that $qs = r$ and $1/s_1 + \cdots + 1/s_m + 1/s = 1$. By Hölder's inequality and boundedness of T on the $L^p(R^n)$ and lemma 1, we get

$$
\begin{aligned}
F_1 &\leq C\left(\frac{1}{|Q|}\int_Q |T(\prod_{j=1}^m (b_j-(b_j)_J)f_1)(x)|^q dx\right)^{1/q}\\
&\leq C\left(\frac{1}{|J|}\int_{R^n} |T(\prod_{j=1}^m (b_j-(b_j)_J)f_1)(x)|^q dx\right)^{1/q}\\
&\leq C\left(\frac{1}{|J|}\int_{R^n} \prod_{j=1}^m |b_j(x)-(b_j)_J|^q|f_1(x)|^q dx\right)^{1/q}\\
&\leq C\prod_{j=1}^m\left(\frac{1}{|J|}\int_J |b_j(x)-(b_j)_J|^{qs_j}dx\right)^{1/qs_j}\left(\frac{1}{|J|}\int_J |f(x)|^{qs}dx\right)^{1/qs}\\
&\leq C\prod_{j=1}^m ||b_j||_{BMO}M_r(f)(\tilde{x})\\
&\leq C||b||_{BMO}M_r(f)(\tilde{x}).
\end{aligned}
$$

For G_1, choose $v, 1 < v < \infty, q_j, 1 \leq j \leq m$ such that $1/v + 1/r + 1/2 = 1$ and $1/q_1 + \cdots + 1/q_m = 1$. By Hölder's inequality and lemma 2,3 and $n/2 < m < n/2 + 1/(1-a)$, we get

$$
\begin{aligned}
G_1(x) &= \left|\int_{R^n}(K(x,x-y)-K(x_0,x_0-y))\prod_{j=1}^m(b_j(y)-(b_j)_J)f_2(y)dy\right|\\
&\leq \sum_{N=0}^\infty \int_{(2^Nd)^{1-a}\leq|y-x_0|\leq(2^{N+1}d)^{1-a}} |K(x,x-y)-K(x_0,x_0-y)|\\
&\times \prod_{j=1}^m |b_j(y)-(b_j)_J||f(y)|dy\\
&\leq C\sum_{N=0}^\infty\left(\int_{(2^Nd)^{1-a}\leq|y-x_0|\leq(2^{N+1}d)^{1-a}} |K(x,x-y)-K(x_0,x_0-y)|^2dy\right)^{1/2}
\end{aligned}
$$

$$\begin{aligned}
&\times \left(\int_{|y-x_0|\le(2^{N+1}d)^{1-a}} \prod_{j=1}^m |b_j(y)-(b_j)_J|^v dy\right)^{1/v} \left(\int_{|y-x_0|\le(2^{N+1}d)^{1-a}} |f(y)|^r dy\right)^{1/r}\\
&\le C\sum_{N=0}^\infty |x-x_0|^{(1-a)(m-n/2)}(2^{N+1}d)^{-m(1-a)+n(1-a)/2}\\
&\times \prod_{j=1}^m \left(\frac{1}{(2^{N+1}d)^{n(1-a)}}\int_{|y-x_0|\le(2^{N+1}d)^{1-a}} |b_j(y)-(b_j)_J|^{vq_j}dy\right)^{1/vq_j}\\
&\times \left(\frac{1}{(2^{N+1}d)^{n(1-a)}}\int_{|y-x_0|\le(2^{N+1}d)^{1-a}} |f(y)|^r dy\right)^{1/r}\\
&\le C\sum_{N=0}^\infty d^{(1-a)(m-n/2)}(2^{N+1}d)^{(1-a)(n/2-m)}N^m\prod_{j=1}^m ||b_j||_{BMO}M_r(f)(\tilde{x})\\
&\le C\sum_{N=0}^\infty 2^{(N+1)(1-a)(n/2-m)}N^m||b||_{BMO}M_r(f)(\tilde{x})\\
&\le C||b||_{BMO}M_r(f)(\tilde{x}),
\end{aligned}$$

thus

$$G_1 \le C||b||_{BMO}M_r(f)(\tilde{x}).$$

Combining all the estimates, we finish the case $m\ge 2$ and $d\le 1$.

In case $m\ge 2$ and $d>1$, we proceeds the case as follows.

Let $f(x)=f_1(x)+f_2(x)$, with $f_1(x)=f(x)\chi_{2Q}(x)$ and $f_2(x)=f(x)\chi_{(2Q)^c}(x)$. We have

$$\begin{aligned}
T_{\vec{b}}(f)(x) &= (2^m-1)\prod_{j=1}^m (b_j(x)-(b_j)_{2I})T(f)(x)\\
&\quad+\sum_{j=1}^{m-1}\sum_{\gamma\in C_j^m}(b(x)-b_{2Q})_\gamma T_{\vec{b}_{\gamma^c}}(f)(x)\\
&\quad+(-1)^m T(\prod_{j=1}^m (b_j-(b_j)_{2Q})f_1)(x)\\
&\quad+(-1)^m T(\prod_{j=1}^m (b_j-(b_j)_{2Q})f_2)(x),
\end{aligned}$$

thus

$$\frac{1}{|Q|}\int_{|}T_{\vec{b}}(f)(x)|dx$$

$$
\begin{aligned}
&\le \frac{2^m-1}{|Q|}\int_Q \prod_{j=1}^m |b_j(x)-(b_j)_{2Q}||T(f)(x)|dx \\
&\quad + \sum_{j=1}^{m-1}\sum_{\gamma\in C_j^m}\frac{1}{|Q|}\int_Q |(b(x)-b_{2I})_\gamma||T_{\vec{b}_{\gamma^c}}(f)(x)|dx \\
&\quad + \frac{1}{|Q|}\int_Q |T(\prod_{j=1}^m (b_j-(b_j)_{2Q})f_1)(x)|dx \\
&\quad + \frac{1}{|Q|}\int_Q |T(\prod_{j=1}^m (b_j-(b_j)_{2Q})f_2)(x)|dx \\
&= D_2+E_2+F_2+G_2.
\end{aligned}
$$

Similar to D_1, $D_2 \le C||b||_{BMO}M_r(T(f))(\tilde{x})$.
Similar to E_1, $E_2 \le C\sum_{j=1}^{m-1}\sum_{\gamma\in C_j^m}||b||_{BMO}M_r(T_{\vec{b}_{\gamma^c}}(f))(\tilde{x})$.
Similar to F_1, $F_2 \le C||b||_{BMO}M_r(f)(\tilde{x})$.

For G_2, choose $p_j, 1\le j\le m$ such that $1/p_1+\cdots+1/p_m+1/r=1$. By Hölder's inequality and lemma 2, 4, we get

$$
\begin{aligned}
G_2(x) &= \left|T(\prod_{j=1}^m (b_j-(b_j)_{2I})f_2)(x)\right| \\
&\le C\sum_{N=1}^\infty \int_{2^N d\le|y-x_0|\le 2^{N+1}d} |x-y|^{-2n}\prod_{j=1}^m |b_j(y)-(b_j)_{2I}||f(y)|dy \\
&\le C\sum_{N=1}^\infty \int_{|y-x_0|\le 2^{N+1}d} |x-y|^{-2n}\prod_{j=1}^m |b_j(y)-(b_j)_{2I}||f(y)|dy \\
&\le C\sum_{N=1}^\infty (2^{N+1}d)^{-2n+n}\prod_{j=1}^m\left(\frac{1}{(2^{N+1}d)^n}\int_{|y-x_0|\le 2^{N+1}d}|b_j(y)-(b_j)_{2I}|^{p_j}dy\right)^{1/p_j} \\
&\quad \times\left(\frac{1}{(2^{N+1}d)^n}\int_{|y-x_0|\le 2^{N+1}d}|f(y)|^r dy\right)^{1/r} \\
&\le C\sum_{N=1}^\infty (2^{N+1}d)^{-n}N^m\prod_{j=1}^m ||b_j||_{BMO}M_r(f)(x_0) \\
&\le C||b||_{BMO}M_r(f)(\tilde{x}),
\end{aligned}
$$

thus

$$G_2 \le C||b||_{BMO}M_r(f)(\tilde{x}).$$

Combining all the estimates, we finish the case $m\ge 2$ and $d>1$. This completes the proof of theorem 1.

Proof of Theorem 2. Choose $2 < r < p$ in Theorem 1. We first consider the case $m = 1$. By Theorem 1 and boundedness of T and M_r on $L^p(R^n)$,

$$\begin{aligned} ||T_{\vec{b_1}}(f)(x)||_{L^p} &\leq ||M(T_{\vec{b_1}}(f))(x)||_{L^p} \leq C||T^{\#}_{\vec{b_1}}(f)(x)||_{L^p} \\ &\leq C||M_r(T(f))(x)||_{L^p} + C||M_r(f)(x)||_{L^p} \\ &\leq C||T(f)(x)||_{L^p} + C||f(x)||_{L^p} \\ &\leq C||f(x)||_{L^p}. \end{aligned}$$

When $m \geq 2$, we may get the conclusion of Theorem 2 by induction. This finishes the proof of Theorem 2.

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