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ON THE FEKETE-SZEGÖ INEQUALITY FOR A CLASS OF ANALYTIC FUNCTIONS DEFINED BY USING GENERALIZED DIFFERENTIAL OPERATOR

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Abstract. In this present investigation, the Fekete- Szegö inequality for certain normalized analytic functions f defined on the open unit disk for which $\frac{z\left(D^k_{\alpha,\beta,\lambda,\delta}f(z)\right)'}{D^k_{\alpha,\beta,\lambda,\delta}f(z)}$ $(k \in N_0, \alpha, \beta, \lambda, \delta \geq 0)$ lies in a region starlike with respect to 1 and is symmetric with respect to the real axis will be obtained. In addition, certain applications of the main result for a class of functions defined by convolution are given. As a special case of this result, Fekete- Szegö inequality for a class of functions defined by fractional derivative is obtained. The motivation of this paper is due to the work given by Srivastava and Mishra in [6].

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1. Introduction

Let A be the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disk $U = \{z : z \in C \text{ and } |z| < 1\}$. Further, let S denote the class of functions which are univalent in U. For a function $f \in A$, we define

$$D^0 f(z) = f(z)$$

$$D^{1}_{\alpha,\beta,\lambda,\delta}f(z) = [1-(\lambda-\delta)(\beta-\alpha)]\, f(z) + (\lambda-\delta)(\beta-\alpha)\, zf'(z)$$

$$= z + \sum_{n=2}^{\infty} [(\lambda-\delta)(\beta-\alpha)(n-1)+1]\, a_n z^n$$

$$\vdots$$

$$D^{k}_{\alpha,\beta,\lambda,\delta}f(z) = D^{1}_{\alpha,\beta,\lambda,\delta}\left(D^{k-1}_{\alpha,\beta,\lambda,\delta}f(z)\right)$$

$$D^{k}_{\alpha,\beta,\lambda,\delta}f(z) = z + \sum_{n=2}^{\infty} [(\lambda-\delta)(\beta-\alpha)(n-1)+1]^{k}\, a_n z^n \tag{2}$$

$$for \ \ (\alpha \geq 0,\ \beta \geq 0,\ \lambda \geq 0,\ \delta \geq 0,\ \lambda > \delta,\ \beta > \alpha) \ \ and \ \ k \in \{0,1,2,....\}.$$

Remark 1.

(i) *When* $\alpha = 0,\ \delta = 0,\ \lambda = 1,\ \beta = 1$, *it reduces to Sălăgean differential operator[9].*

(ii) *When* $\alpha = 0$, *reduces to Darus and Ibrahim differential operator [2].*

(iii) *And when* $\alpha = 0,\ \delta = 0,\ \beta = 1$, *reduces to Al- Oboudi differential operator [1].*

Let $\phi(z)$ be an analytic function with positive real part on U with $\phi(z) = 1,\ \ \phi'(z) > 0$ which maps the unit disk U onto a region starlike with respect to 1 which is symmetric with respect to the real axis. Let $S^{*}(\phi)$ be the class of functions in $f(z) \in S$ for which

$$\frac{zf'(z)}{f(z)} \prec \phi(z), \ \ (z \in U)$$

and $C(\phi)$ be the class of functions in $f(z) \in S$ for which

$$1 + \frac{zf''(z)}{f'(z)} \prec \phi(z), \ \ (z \in U)$$

where $\prec$ denotes the subordination between analytic functions. These classes were investigated and studied by Ma and Minda [4]. They have obtain the Fekete- Szegö inequality for the functions in the class $C(\phi)$. Since $f \in C(\phi)$ if and only if $zf'(z) \in S^{*}(\phi)$, we get the Fekete- Szegö inequality for functions in the class $S^{*}(\phi)$. For a brief history of the Fekete- Szegö problem for class of starlike, convex, and close to convex functions, see the recent paper by Srivastava et.al [7]. In the present paper, we obtain the Fekete- Szegö inequality for the class $M^{k}_{\alpha,\beta,\lambda,\delta}(\phi)$ as defined below. Also we give applications of our result to certain functions defined through convolution (or Hadamard product) and in particular we consider a class $M^{k}_{\alpha,\beta,\lambda,\delta}(\phi)$

defined by fractional derivatives. The object of this paper is to generalize the Fekete-Szegö inequality of that given by Srivastava and Mishra [6].

Definition 1. *Let $\phi(z)$ be a univalent starlike function with respect to 1 which maps the unit disk U onto a region in the right half plane which is symmetric with respect to the real axis, $\phi(0) = 1$ and $\phi'(0) > 0$. A function $f \in A$ is in the class $M^k_{\alpha,\beta,\lambda,\delta}(\phi)$ if*

$$\frac{z\left(D^k_{\alpha,\beta,\lambda,\delta}f(z)\right)'}{D^k_{\alpha,\beta,\lambda,\delta}f(z)} \prec \phi(z).$$

For fixed $g \in A$, we define the class $M^{k,g}_{\alpha,\beta,\lambda,\delta}(\phi)$ to be the class of functions $f \in A$ for which $(f * g) \in M^k_{\alpha,\beta,\lambda,\delta}(\phi)$. In order to derive our main results, we have to recall here the following lemma [4]:

Lemma 1. *If $p_1(z) = 1 + c_1 z + c_2 z^2 + ...$ is an analytic function with positive real part in U, then*

$$\left|c_2 - \nu c_1^2\right| \leq \begin{cases} -4\nu + 2 & if \quad \nu \leq 0; \\ 2 & if \quad 0 \leq \nu \leq 1; \\ 4\nu - 2 & if \quad \nu \geq 1. \end{cases}$$

When $\nu < 0$ or $\nu > 1$, the equality holds if and only if $p_1(z)$ is $(1+z)/(1-z)$ or one of its rotations. If $0 < \nu < 1$, then the equality holds if and only if $p_1(z)$ is $\left(1+z^2\right)/\left(1-z^2\right)$ or one of its rotations. If $\nu = 0$, the equality holds if and only if

$$p_1(z) = \left(\frac{1+\gamma}{2}\right)\frac{1+z}{1-z} + \left(\frac{1-\gamma}{2}\right)\frac{1-z}{1+z}, \quad (0 \leq \gamma \leq 1)$$

or one of its rotations. If $\nu = 1$, the equality holds if and only if $p_1(z)$ is the reciprocal of one of the functions such that the equality holds in the case of $\nu = 0$. Also the above upper bound is sharp, and it can be improved as follows when $0 < \nu < 1$:

$$\left|c_2 - \nu c_1^2\right| + \nu\left|c_1\right|^2 \leq 2 \qquad \left(0 < \nu \leq \frac{1}{2}\right)$$

and

$$\left|c_2 - \nu c_1^2\right| + (1-\nu)\left|c_1\right|^2 \leq 2 \qquad \left(\frac{1}{2} < \nu \leq 1\right).$$

2.Fekete-Szegö problem

Our main result is the following:

Theorem 1. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + ...$ If $f(z)$ given by (1) belongs to*

$M^k_{\alpha,\beta,\lambda,\delta}(\phi)$, *then*

$|a_3 - \mu a_2^2| \leq$

$$\begin{cases} \frac{B_2}{[2(\lambda-\delta)(\beta-\alpha)+1]^k} - \frac{2\mu B_1^2}{[(\lambda-\delta)(\beta-\alpha)+1]^{2k}} + \frac{B_1^2}{[2(\lambda-\delta)(\beta-\alpha)+1]^k} & if\ \mu \leq \sigma_1; \\ \frac{B_1}{2[2(\lambda-\delta)(\beta-\alpha)+1]^k} & if \sigma_1 \leq \mu \leq \sigma_2; \\ -\frac{B_2}{[2(\lambda-\delta)(\beta-\alpha)+1]^k} + \frac{2\mu B_1^2}{[(\lambda-\delta)(\beta-\alpha)+1]^{2k}} - \frac{B_1^2}{[2(\lambda-\delta)(\beta-\alpha)+1]^k} & if\ \mu \geq \sigma_2, \end{cases} \tag{3}$$

where

$$\sigma_1 := \frac{[(\lambda-\delta)(\beta-\alpha)+1]^{2k}\left\{(B_2 - B_1) + B_1^2\right\}}{2\left[2(\lambda-\delta)(\beta-\alpha)+1\right]^k B_1^2}$$

$$\sigma_2 := \frac{[(\lambda-\delta)(\beta-\alpha)+1]^{2k}\left\{(B_2 + B_1) + B_1^2\right\}}{2\left[2(\lambda-\delta)(\beta-\alpha)+1\right]^k B_1^2}.$$

The result is sharp.

Proof. For $f \in M^k_{\alpha,\beta,\lambda,\delta}(\phi)$, let

$$p(z) = \frac{z\left(D^k_{\alpha,\beta,\lambda,\delta} f(z)\right)'}{D^k_{\alpha,\beta,\lambda,\delta} f(z)} = 1 + b_1 z + b_2 z^2 + \dots. \tag{4}$$

From (4), we obtain

$$\begin{gathered} [(\lambda-\delta)(\beta-\alpha)+1]^k a_2 = b_1, \\ 2\left[2(\lambda-\delta)(\beta-\alpha)+1\right]^k a_3 = [(\lambda-\delta)(\beta-\alpha)+1]^{2k} a_2^2 + b_2. \end{gathered} \tag{5}$$

Since $\phi(z)$ is univalent and $p \prec \phi$, the function

$$p_1(z) = \frac{1+\phi^{-1}(p(z))}{1-\phi^{-1}(p(z))} = 1 + c_1 z + c_2 z^2 + \dots.$$

is analytic and positive real in U.

Also we have

$$p(z) = \phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right), \tag{6}$$

and from this equality and (4), $1+b_1z+b_2z^2+....=\phi\left(\frac{c_1z+c_2z^2+....}{2+c_1z+c_2z^2+....}\right)$
$=\phi\left(\frac{1}{2}\ c_1z+\frac{1}{2}\left(c_2-\frac{1}{2}\ c_1^2\right)z^2+....\right)=1+B_1\frac{1}{2}\ c_1z+B_1\frac{1}{2}\left(c_2-\frac{1}{2}\ c_1^2\right)z^2+....+B_2\frac{1}{4}\ c_1^2z^2+$
.... we obtain $b_1=\frac{1}{2}B_1c_1$ *and* $b_2=\frac{1}{2}B_1\left(c_2-\frac{1}{2}\ c_1^2\right)+\frac{1}{4}B_2c_1^2$. Therefore, we have

$$a_3-\mu a_2^2=\frac{B_1}{4\left[2\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^k}\left[c_2-c_1^2\left\{\frac{1}{2}\left(1-\frac{B_2}{B_1}\right.\right.\right.$$

$$\left.\left.-\frac{\left[\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^{2k}-2\mu\left[2\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^k}{\left[\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^{2k}}B_1\right)\right\},$$

$$a_3-\mu a_2^2=\frac{B_1}{4\left[2\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^k}\ \left\{c_2-\nu c_1^2\right\}.$$

Where

$$\nu=\frac{1}{2}\left(1-\ \frac{B_2}{B_1}\ -\frac{\left[\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^{2k}-2\mu\left[2\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^k}{\left[\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^{2k}}B_1\right).$$

If $\mu\leq\sigma_1$, then by applying Lemma 1, we get

$$\left|a_3-\mu a_2^2\right|\leq\frac{B_2}{\left[2\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^k}-\frac{2\mu B_1^2}{\left[\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^{2k}}+\frac{B_1^2}{\left[2\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^k}$$

which is the first part of assertion (3).
Next, if $\mu\geq\sigma_2$, by applying Lemma 1, we get

$$\left|a_3-\mu a_2^2\right|\leq-\frac{B_2}{\left[2\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^k}+\frac{2\mu B_1^2}{\left[\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^{2k}}-\frac{B_1^2}{\left[2\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^k}.$$

if $\mu=\sigma_1$, then equality holds if and only if

$$p_1\left(z\right)=\left(\frac{1+\gamma}{2}\right)\frac{1+z}{1-z}+\left(\frac{1-\gamma}{2}\right)\frac{1-z}{1+z},\quad\left(0\leq\gamma\leq1,\ z\in U\right)$$

or one of its rotations. if $\mu=\sigma_2$, then

$$\frac{1}{2}\left(1-\ \frac{B_2}{B_1}\ -\frac{\left[\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^{2k}-2\mu\left[2\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^k}{\left[\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^{2k}}B_1\right)=0.$$

Therefore,

$$\frac{1}{p_1\left(z\right)}=\left(\frac{1+\gamma}{2}\right)\frac{1+z}{1-z}+\left(\frac{1-\gamma}{2}\right)\frac{1-z}{1+z},\quad\left(0<\gamma<1,\ z\in U\right).$$

Finally, we see that

$$\left|a_3-\mu a_2^2\right| = \frac{B_1}{4\left[2\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^k}\left|c_2-c_1^2\left\{\frac{1}{2}\left(1-\frac{B_2}{B_1}\right.\right.\right.$$

$$\left.\left.\left.-\frac{\left[\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^{2k}-2\mu\left[2\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^k}{\left[\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^{2k}}B_1\right|\right\}$$

and

$$\max\left|\frac{1}{2}\left(1-\ \frac{B_2}{B_1}\ -\frac{\left[\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^{2k}-2\mu\left[2\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^k}{\left[\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^{2k}}B_1\right)\right| \leq 1,$$

$$\left(\sigma_1\leq\mu\leq\sigma_2\right).$$

Therefore using Lemma 1, we get

$$\left|a_3-\mu a_2^2\right| = \frac{B_1\left|c_1\right|}{4\left[2\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^k} \leq \frac{B_1}{2\left[2\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^{2k}},\ \ \left(\sigma_1\leq\mu\leq\sigma_2\right).$$

If $\sigma_1<\mu<\sigma_2$, then we have

$$p_1\left(z\right)=\frac{1+\nu z^2}{1-\nu z^2},\ \ \left(0\leq\nu\leq1\right).$$

Our result now follows by an application of Lemma 1. To show that the bounds are sharp, we define the function K_s^{ϕ} $(s=2,3,...)$ by

$$p\left(z\right)=\frac{z\left(D^k_{\alpha,\beta,\lambda,\delta}K_s^{\phi}\left(z\right)\right)'}{D^k_{\alpha,\beta,\lambda,\delta}K_s^{\phi}\left(z\right)}=\phi\left(z^{s-1}\right),\ \ K_s^{\phi}\left(0\right)=0=\left[K_s^{\phi}\left(0\right)\right]'-1$$

and the function F_γ and G_γ, $(0\leq\gamma\leq1)$ by

$$\frac{z\left(D^k_{\alpha,\beta,\lambda,\delta}F_\gamma\left(z\right)\right)'}{D^k_{\alpha,\beta,\lambda,\delta}F_\gamma\left(z\right)}=\phi\left(\frac{z\left(z+\nu\right)}{1+\nu z}\right)\ ,\ \ F_\gamma\left(0\right)=0=\left[F_\gamma\left(0\right)\right]'-1$$

and

$$\frac{z\left(D^k_{\alpha,\beta,\lambda,\delta}G_\gamma\left(z\right)\right)'}{D^k_{\alpha,\beta,\lambda,\delta}G_\gamma\left(z\right)}=\phi\left(-\frac{z\left(z+\nu\right)}{1+\nu z}\right)\ ,\ \ G_\gamma\left(0\right)=0=\left[G_\gamma\left(0\right)\right]'-1.$$

Clearly the functions $K_s^{\phi},\ F_{\gamma}\ ,\ G_{\gamma} \in M^{k}_{\alpha,\beta,\lambda,\delta}(\phi)$. Also we write $K^{\phi} = K_2^{\phi}$.

If $\mu < \sigma_1$ or $\mu > \sigma_2$, then the equality holds if and only if f is K_s^{ϕ} or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, the equality holds if and only if f is K_3^{ϕ} or one of its rotations. If $\mu = \sigma_1$, then the equality holds if and only if f is F_{γ} or one of its rotations. If $\mu = \sigma_2$, then the equality holds if and only if f is G_{γ} or one of its rotations.

Remark 2. *If $\sigma_1 \le \mu \le \sigma_2$, then in view of Lemma 1, Theorem 1 can be improved.* Let σ_3 be given by

$$\sigma_3 := \frac{[(\lambda-\delta)(\beta-\alpha)+1]^{2k}\left(B_2+B_1^2\right)}{2\left[2(\lambda-\delta)(\beta-\alpha)+1\right]^{k} B_1^2}.$$

If $\sigma_1 \le \mu \le \sigma_3$, then

$$\left|a_3-\mu a_2^2\right| + \frac{[(\lambda-\delta)(\beta-\alpha)+1]^{2k}}{2\left[2(\lambda-\delta)(\beta-\alpha)+1\right]^{k} B_1^2}\Bigg[B_1 - B_2 + \frac{2\mu\left[2(\lambda-\delta)(\beta-\alpha)+1\right]^{k} - [(\lambda-\delta)(\beta-\alpha)+1]^{2k}}{[(\lambda-\delta)(\beta-\alpha)+1]^{2k}} B_1^2\Bigg] |a_2|^2 \le \frac{B_1}{2\left[2(\lambda-\delta)(\beta-\alpha)+1\right]^{k}}.$$

If $\sigma_3 \le \mu \le \sigma_2$, then

$$\left|a_3-\mu a_2^2\right| + \frac{[(\lambda-\delta)(\beta-\alpha)+1]^{2k}}{2\left[2(\lambda-\delta)(\beta-\alpha)+1\right]^{k} B_1^2}\Bigg[B_1 + B_2 - \frac{2\mu\left[2(\lambda-\delta)(\beta-\alpha)+1\right]^{k} - [(\lambda-\delta)(\beta-\alpha)+1]^{2k}}{[(\lambda-\delta)(\beta-\alpha)+1]^{2k}} B_1^2\Bigg] |a_2|^2 \le \frac{B_1}{2\left[2(\lambda-\delta)(\beta-\alpha)+1\right]^{k}}.$$

Proof. For the values of $\sigma_1 \le \mu \le \sigma_3$, we have

$$\left|a_3-\mu a_2^2\right| + (\mu-\sigma_1)|a_2|^2 =$$

$$\frac{B_1}{4\left[2(\lambda-\delta)(\beta-\alpha)+1\right]^{k}}\left|c_2-\nu c_1^2\right| + (\mu-\sigma_1)\frac{B_1^2}{4\left[(\lambda-\delta)(\beta-\alpha)+1\right]^{2k}}|c_1|^2$$

$$= \frac{B_1}{4\left[2\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^k}\left|c_2-\nu c_1^2\right|+$$

$$+\left(\mu-\frac{\left[\left(\lambda-\delta\right)\left(\beta-\alpha\right)+\right]^{2k}\left\{\left(B_2-B_1\right)+B_1^2\right\}}{2\left[2\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^k B_1^2}\right)\frac{B_1^2}{4\left[\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^{2k}}\left|c_1\right|^2=$$

$$=\frac{B_1}{2\left[2\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^k}\left\{\frac{1}{2}\left[\left|c_2-\nu c_1^2\right|+\nu\left|c_1\right|^2\right]\right\}\leq\frac{B_1}{2\left[2\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^k}.$$

Similarly, for the value of $\sigma_3 \leq \mu \leq \sigma_2$, we write

$$\left|a_3-\mu a_2^2\right|+\left(\sigma_2-\mu\right)\left|a_2\right|^2=$$

$$\frac{B_1}{4\left[2\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^k}\left|c_2-\nu c_1^2\right|+\left(\sigma_2-\mu\right)\frac{B_1^2}{4\left[\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^{2k}}\left|c_1\right|^2$$

$$= \frac{B_1}{4\left[2\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^k}\left|c_2-\nu c_1^2\right|+$$

$$+\left(\frac{\left[\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^{2k}\left\{\left(B_2+B_1\right)+B_1^2\right\}}{2\left[2\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^k B_1^2}-\mu\right)\frac{B_1^2}{4\left[\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^{2k}}\left|c_1\right|^2=$$

$$=\frac{B_1}{2\left[2\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^k}\left\{\frac{1}{2}\left[\left|c_2-\nu c_1^2\right|+\left(1-\nu\right)\left|c_1\right|^2\right]\right\}\leq\frac{B_1}{2\left[2\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^k}.$$

Thus, Remark 2 holds.

3. Applications to Functions Defined by Fractional Derivatives

For two analytic functions $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n$, their convolution (or Hadamard product) is defined to be the function $(f * g)(z)$ given by

$$(f*g)(z) = f(z) * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

Definition 2.[8] *Let f be analytic in a simply-connected region of the z-plane containing the region. The fractional derivative of f of order γ is defined by*

$$D_z^{\gamma} f(z) \frac{1}{\Gamma(1-\gamma)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\gamma}} d\zeta, \quad (0 \leq \gamma < 1),$$

where the multiplicity of $(z-\zeta)^{\gamma}$ is removed by requiring that $\log(z-\zeta)$ is real for $z-\zeta > 0$. Using the above definition and its known extensions involving fractional derivatives and fractional integrals, Owa and Srivastava [5] introduced the operator $\Omega^{\gamma} : A \to A$ defined by

$$(\Omega^{\gamma} f)(z) = \Gamma(2-\gamma) z^{\gamma} D_z^{\gamma} f(z), \quad (\gamma \neq 2, 3, 4, ...).$$

The class $M_{\alpha,\beta,\lambda,\delta}^{k,\gamma}(\phi)$ consists of functions $f \in A$ for which $\Omega^{\gamma} f \in M_{\alpha,\beta,\lambda,\delta}^{k}(\phi)$. Note that $M_{\alpha,\beta,\lambda,\delta}^{k,\gamma}(\phi)$ is the special case of the class $M_{\alpha,\beta,\lambda,\delta}^{k,g}(\phi)$ when

$$g(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\gamma)}{\Gamma(n+1-\gamma)} z^n. \tag{7}$$

Let

$$g(z) = z + \sum_{n=2}^{\infty} g_n z^n, \quad (g_n > 0).$$

Since

$$D_{\alpha,\beta,\lambda,\delta}^{k} f(z) = z + \sum_{n=2}^{\infty} \left[(\lambda-\delta)(\beta-\alpha)(n-1)+1\right]^k a_n z^n \in M_{\alpha,\beta,\lambda,\delta}^{g}(\phi)$$

if and only if

$$\left(D_{\alpha,\beta,\lambda,\delta}^{k} f * g\right)(z) = z + \sum_{n=2}^{\infty} \left[(\lambda-\delta)(\beta-\alpha)(n-1)+1\right]^k a_n g_n z^n \in M_{\alpha,\beta,\lambda,\delta}^{k}(\phi) \tag{8}$$

we obtain the coefficient estimate for functions in the class $M_{\alpha,\beta,\lambda,\delta}^{k,g}(\phi)$, from the corresponding estimate for functions in the class $M_{\alpha,\beta,\lambda,\delta}^{k}(\phi)$. Applying Theorem 1 for the operator (8), we get the following Theorem 2 after an obvious change of the parameter μ :

Theorem 2. *Let $g(z) = z + \sum_{k=2}^{\infty} g_k z^k$, $(g_k > 0)$ and let the function $\phi(z)$ be given by $\phi(z) = 1 + \sum_{n=1}^{\infty} B_n z^n$. If operator (2) belongs to $M_{\alpha,\beta,\lambda,\delta}^{k,g}(\phi)$, then*

$|a_3 - \mu a_2| \leq$

$$\begin{cases} \frac{1}{g_3}\left[\frac{B_2}{[2(\lambda-\delta)(\beta-\alpha)+1]^k} - \frac{2\mu g_3 B_1^2}{[(\lambda-\delta)(\beta-\alpha)+1]^{2k}} + \frac{B_1^2}{[2(\lambda-\delta)(\beta-\alpha)+1]^k}\right] & if\, \mu \leq \sigma_1; \\ \frac{1}{g_3}\left[\frac{B_1}{2[2(\lambda-\delta)(\beta-\alpha)+1]^k}\right] & if \sigma_1 \leq \mu \leq \sigma_2; \\ \frac{1}{g_3}\left[-\frac{B_2}{[2(\lambda-\delta)(\beta-\alpha)+1]^k} + \frac{2\mu g_3 B_1^2}{[(\lambda-\delta)(\beta-\alpha)+1]^{2k}} - \frac{B_1^2}{[2(\lambda-\delta)(\beta-\alpha)+1]^k}\right] & if\ \mu \geq \sigma_2, \end{cases}$$

where

$$\sigma_1 := \frac{g_2^2\left[(\lambda-\delta)(\beta-\alpha)+1\right]^{2k}\left\{(B_2-B_1)+B_1^2\right\}}{2g_3\left[2(\lambda-\delta)(\beta-\alpha)+1\right]^k B_1^2}$$

$$\sigma_2 := \frac{g_2^2\left[(\lambda-\delta)(\beta-\alpha)+1\right]^{2k}\left\{(B_2+B_1)+B_1^2\right\}}{2g_3\left[2(\lambda-\delta)(\beta-\alpha)+1\right]^k B_1^2}.$$

The result is sharp.

Since

$$\left(\Omega^\gamma D^k_{\alpha,\beta,\lambda,\delta} f\right)(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\gamma)}{\Gamma(n+1-\gamma)}\left[(\lambda-\delta)(\beta-\alpha)(n-1)+1\right]^k a_n z^n$$

we have

$$g_2 := \frac{\Gamma(3)\Gamma(2-\gamma)}{\Gamma(3-\gamma)} = \frac{2}{2-\gamma} \tag{9}$$

and

$$g_3 : \frac{\Gamma(4)\Gamma(3-\gamma)}{\Gamma(4-\gamma)} = \frac{6}{(2-\gamma)(3-\gamma)}. \tag{10}$$

For g_2 and g_3 given by (9) and (10), Theorem 2 reduces to the following:

Theorem 3. *Let* $g(z) = z + \sum_{k=2}^{\infty} g_k z^k$, $(g_k > 0)$ *and let the function* $\phi(z)$ *be given by* $\phi(z) = 1 + \sum_{n=1}^{\infty} B_n z^n$. *If* $D^k_{\alpha,\beta,\lambda,\delta} f$ *given by (2) belongs to* $M^{k,\gamma}_{\alpha,\beta,\lambda,\delta}(\phi)$, *then*

$|a_3 - \mu a_2| \leq$

$$\begin{cases} \frac{(2-\gamma)(3-\gamma)}{6}\left[\frac{B_2}{[2(\lambda-\delta)(\beta-\alpha)+1]^k}-\frac{3(2-\gamma)\mu B_1^2}{(3-\gamma)[(\lambda-\delta)(\beta-\alpha)+1]^{2k}}+\frac{B_1^2}{[2(\lambda-\delta)(\beta-\alpha)+1]^k}\right] if\ \mu\leq\sigma_1; \\ \frac{(2-\gamma)(3-\gamma)}{6}\left[\frac{B_1}{2[2(\lambda-\delta)(\beta-\alpha)+1]^k}\right] \qquad if \sigma_1\leq\mu\leq\sigma_2; \\ \frac{(2-\gamma)(3-\gamma)}{6}\left[-\frac{B_2}{[2(\lambda-\delta)(\beta-\alpha)+1]^k}+\frac{3(2-\gamma)\mu B_1^2}{(3-\gamma)[(\lambda-\delta)(\beta-\alpha)+1]^{2k}}-\frac{B_1^2}{[2(\lambda-\delta)(\beta-\alpha)+1]^k}\right] \ if\ \mu\geq\sigma_2, \end{cases}$$

where

$$\sigma_1 := \frac{(3-\gamma)\left[(\lambda-\delta)(\beta-\alpha)+1\right]^{2k}\left\{(B_2-B_1)+B_1^2\right\}}{3(2-\gamma)\left[2(\lambda-\delta)(\beta-\alpha)+1\right]^k B_1^2}$$

$$\sigma_2 := \frac{(3-\gamma)\left[(\lambda-\delta)(\beta-\alpha)+1\right]^{2k}\left\{(B_2+B_1)+B_1^2\right\}}{3(2-\gamma)\left[2(\lambda-\delta)(\beta-\alpha)+1\right]^k B_1^2}.$$

The result is sharp.

Remark 3. When $k = 0$, $\lambda = 1$, $\beta = 1$, $\alpha = 0$, $\delta = 0$, $B_1 = \frac{8}{\pi^2}$ *and* $B_2 = \frac{16}{3\pi^2}$ the above Theorem 3 reduces to a recent result of Srivastava and Mishra ([6], Theorem 8, P.64) for a class of functions for which $\Omega^\gamma f(z)$ is a parabolic starlike functions (see [3],[5]). Note also, other work related to the upper bounds of the Fekete-Szegó theorem can be found in [10].

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