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ON CERTAIN SUBCLASS OF P-VALENT FUNCTIONS INVOLVING THE DZIOK-SRIVASTAVA OPERATOR

Ali Muhammad

Abstract. In this paper, we introduce a class $T_k(\lambda, \alpha_1, p, q, s, \rho)$. We investigate a number of inclusion relationships, radius problem and some other interesting properties of p-valent functions which are defined here by means of a certain linear integral operator $H_{p,q,s}(\alpha_1)$.

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1. Introduction

Let $A(p)$ denote the class of functions $f(z)$ normalized by

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \ (p \in \mathbb{N} = \{1, 2, 3 \cdots\}, \tag{1}$$

which are analytic and p-valent in the unit disk $E = \{|z| : z \in C, |z| < 1\}$.

For functions $f_j(z) \in A(p)$, given by (1) we define the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

$$f_j(z) = z^p + \sum_{k=1}^{\infty} a_{p+k,j} z^{p+k} \ (j = \{1, 2, 3 \cdots\}, \tag{2}$$

we define the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

$$(f_1 * f_2)(z) = z^p + \sum_{k=1}^{\infty} a_{p+k,1} a_{p+k,2} z^{p+k} = (f_2 * f_1)(z), \ (z \in E). \tag{3}$$

Let $P_k(\rho)$ be the class of functions $p(z)$ analytic in E satisfying the properties $p(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{\mathrm{Re} p(z) - \rho}{1-\rho} \right| d\theta \leq k\pi,, \tag{4}$$

where $z = re^{i\theta}, k \geq 2$ and $0 \leq \rho < 1$. This class has been introduced in [10]. We note, for $\rho = 0$, we obtain the class P_k defined and studied in [11], and for $\rho = 0, k = 2$, we have the well-known class P of functions with positive real part. The case $k = 2$ gives the class $P(\rho)$ of functions with positive real part greater than ρ. From (4) we can easily deduce that $p \in P_k(\rho)$ if and only if, there exists $p_1, p_2 \in P(\rho)$ such that for $z \in E$,

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z). \tag{5}$$

Making use of the Hadamard product (or convolution) given by (3), we now define the Dziok-Srivastava operator,

$$H_p(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots \beta_q) : A(p) \to A(p).$$

which was introduced and studied in a series of recent papers by Dziok and Srivastava [1], [2], see also [5], [6]. Indeed, for complex parameters
$\alpha_1, \cdots \alpha_q$ and $\beta_1, \cdots, \beta_s$, $(\beta_j \notin Z_0^- = \{0, -1, -2, -3, \cdots\}; j = 1, \cdots s)$,
the generalized hypergeometric function

$${}_qF_s(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s; z)$$

is given by

$${}_qF_s(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_q)_n}{(\beta_1, \cdots, \beta_s)_n n!} z^n \tag{6}$$

$(q \leq s+1; q, s \in N_0 = \mathbb{N} \cup \{0\}; \mathbb{N} = \{1, 2, \cdots\};$ $z \in E$, where $(v)_k$ is the Pochhammer symbol (or the shifted factorial) defined in (terms of the Gamma function) by

$$(v)_k = \frac{\Gamma(v+k)}{\Gamma(v)} = \left\{ \begin{array}{cc} 1 & \text{if } k = 0,\ v \in C\backslash\{0\} \\ v(v+1), \cdots (v+k-1) & \text{if } k \in N, \mathrm{v} \in C. \end{array} \right\}$$

Corresponding to a function

$$\mathcal{F}_p(\alpha_1, \cdots \alpha_q; \beta_1, \cdots, \beta_s; z)$$

defined by

$$\mathcal{F}_p(\alpha_1, \cdots \alpha_q; \beta_1, \cdots, \beta_s; z) = z^p {}_qF_s(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s; z).$$

Dziok and Srivastava [1] considered a linear operator defined by the following Hadamard product (or convolution):

$$H_p(\alpha_1, \cdots \alpha_q; \beta_1, \cdots, \beta_s) f(z) = \mathcal{F}_p(\alpha_1, \cdots \alpha_q; \beta_1, \cdots, \beta_s; z) * f(z). \tag{7}$$

For convenience, we write

$$H_{p,q,s}(\alpha_1) = H_p(\alpha_1, \cdots \alpha_q; \beta_1, \cdots, \beta_s). \tag{8}$$

Thus after some calculations, we have

$$z(H_{p,q,s}(\alpha_1) f(z))' = \alpha_1 H_{\lambda,q,s}(\alpha_1 + 1) f(z) - (\alpha_1 - p) H_{p,q,s}(\alpha_1) f(z). \tag{9}$$

Many interesting subclasses of analytic functions, associated with the Dziok-Srivastava operator $H_{p,q,s}(\alpha_1)$ and its many special cases, were investigated recently by Dziok and Srivastava [1], [2], Gangadharan et. al [3], Liu and Srivastava [5], [6], see also [5], [9], [13].

Definition 1.1. *Let $f \in A(p)$. Then $f \in T_k(\lambda, \alpha_1, p, q, s, \rho)$, if and only if*

$$\left\{(1-\lambda)\frac{H_{p,q,s}(\alpha_1) f(z)}{z^p} + \lambda \frac{H_{p,q,s}(\alpha_1 + 1) f(z)}{z^p}\right\} \in P_k(\rho),\ z \in E,$$

where $\lambda > 0, k \geq 2$ and $0 \leq \rho < p$.

2. Preliminary Results

Lemma 2.1.[12] *If $p(z)$ is analytic in E with $p(0) = 1$, and if λ_1 is a convex number satisfying $Re(\lambda_1) \geq 0, (\lambda_1 \neq 0)$, then*

$$Re\left\{p(z) + \lambda_1 z p'(z)\right\} > \beta \ (0 \leq \beta < 1)$$

implies

$$Rep(z) > \beta + (1-\beta)(2\gamma - 1),$$

where γ is given by

$$\gamma = \gamma(Re\lambda_1 = \int_0^1 (1 + t^{Re\lambda_1)^{-1}} dt,$$

which is an increasing function of $Re(\lambda_1)$ and $\frac{1}{2} \leq \gamma < 1$. The estimate is sharp in the sense that the bound cannot be improved.

Lemma 2.2.[14] *If $p(z)$ is analytic in $E, p(0) = 1$ and $Rep(z) > \frac{1}{2}, z \in E$, then for any function F analytic in E, the function $p * F$ takes the value in the convex hull of the image of E under F.*

3. Main Results

Theorem 3.1. *Let* $Re\alpha_1 > 0$. *Then* $T_k(\lambda, \alpha_1, p, q, s, \rho) \subset T_k(0, \alpha_1, p, q, s, \rho_1)$, *where* ρ_1*is given by*

$$\rho_1 = \rho + (1-\rho)(2\gamma - 1), \tag{10}$$

and

$$\int_0^1 \left(1 + t^{\mathrm{Re}\left(\frac{\lambda}{\alpha_1}\right)}\right)^{-1} dt.$$

Proof. Let $f \in T_k(\lambda, \alpha_1, p, q, s, \rho)$ and set

$$\frac{H_{p,q,s}(\alpha_1)f(z)}{z^p} = h(z) = \left(\frac{k}{4} + \frac{1}{2}\right) h_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) h_2(z) \tag{11}$$

Then $h(z)$ is analytic in E with $h(0) = 1$. By a simple computation, we have

$$\left\{(1-\lambda)\frac{H_{p,q,s}(\alpha_1)f(z)}{z^p} + \lambda\frac{H_{p,q,s}(\alpha_1+1)f(z)}{z^p}\right\} = \left\{h(z) + \frac{\lambda z h'(z)}{\alpha_1}\right\} \in P_k(\rho)$$

for $z \in E$.

This implies that $\mathrm{Re}\left\{h_i(z) + \frac{\lambda z h_i'(z)}{\alpha_1}\right\} > \rho, i = 1, 2.$

Using Lemma 2.1, we see that $\mathrm{Re}h_i(z) > \rho_1$, where ρ_1 is given by (10). Consequently $h \in P_k(\rho_1)$, where ρ_1 is given by (10) for $z \in E$ and proof is complete.

Theorem 3.2. *Let* $f \in T_k(0, \alpha_1, p, q, s, \rho)$ *for* $z \in E$. *Then* $f \in T_k(\lambda, \alpha_1, p, q, s, \rho)$ *for* $|z| < R(\alpha_1, \lambda)$, *where*

$$R(\alpha_1, \lambda) = \frac{|\alpha_1|}{\lambda + \sqrt{(\lambda^2 + |\alpha_1|^2)}}. \tag{12}$$

Proof. Set

$$\frac{H_{p,q,s}(\alpha_1)f(z)}{z^p} = (p-\rho)h(z) + \rho, \qquad h \in P_k.$$

Now proceeding as in Theorem 3.1, we have

$$\left\{(1-\lambda)\frac{H_{p,q,s}(\alpha_1)f(z)}{z^p} + \lambda\frac{H_{p,q,s}(\alpha_1+1)f(z)}{z^p} - \rho\right\} = (p-\rho)\left\{h(z) + \frac{\lambda z h'(z)}{\alpha_1}\right\}$$

$$= (p-\rho)\left[(\frac{k}{4} + \frac{1}{2})\left\{h_1(z) + \frac{\lambda z h_1'(z)}{\alpha_1}\right\} - (\frac{k}{4} - \frac{1}{2})\left\{h_2(z) + \frac{\lambda z h_2'(z)}{\alpha_1}\right\}\right] \tag{13}$$

where we have used (5) and $h_1, h_2 \in P, z \in E$. Using the following well-known estimates, see [7]

$$|zh_i'(z)| \leq \frac{2r}{1-r^2}\text{Re}h_i(z),\ (|z|=r<1), i=1,2,$$

we have

$$\begin{aligned}\text{Re}\left\{h_i(z)+\frac{\lambda z h_i'(z)}{|\alpha_1|}\}\right\} &\geq \text{Re}\left\{h_i(z)+\frac{\lambda z h_i'(z)}{|\alpha_1|}\}\right\} \\ &\geq \text{Re}h_i(z)\left\{1-\frac{2\lambda r}{|\alpha_1|(1-r^2)}\right\}.\end{aligned}$$

The right hand side of this inequality is positive if $r < R(\alpha_1, \lambda)$, where $R(\alpha_1, \lambda)$ is given by (12). Consequently it follows from (13) that $f \in T_k(\lambda, \alpha, p, q, s, \rho)$ for $|z| < R(\alpha_1, \lambda)$. Sharpness of this result follows by taking $h_i(z) = \frac{1+z}{1-z}$ in (13), $i = 1, 2$.

Theorem 3.3. $T_k(\lambda_1, \alpha_1, p, q, s, \rho) \subset T_k(\lambda_2, \alpha_1, p, q, s, \rho)$ for $0 \leq \lambda_2 < \lambda_1$.
Proof. For $\lambda_2 = 0$ the proof is immediate. Let $\lambda_2 > 0$ and let $f \in T_k(\lambda_1, \alpha_1, p, q, s, \rho)$. Then there exist two functions $H_1, H_2 \in P_k(\rho)$ such that, from Definition 1.1 and Theorem 3.1,

$$(1-\lambda_1)\frac{H_{p,q,s}(\alpha_1)f(z)}{z^p}+\lambda_1\frac{H_{p,q,s}(\alpha_1+1)}{z^p}=H_1(z),$$

and

$$\frac{H_{p,q,s}(\alpha_1)}{z^p}=H_2(z).$$

Hence

$$(1-\lambda_2)\frac{H_{p,q,s}(\alpha_2)f(z)}{z^p}+\lambda_2\frac{H_{p,q,s}(\alpha_1+1)}{z^p}=\frac{\lambda_2}{\lambda_1}H_1(z)+(1-\frac{\lambda_2}{\lambda 1})H_2(z). \qquad (14)$$

Since the class $P_k(\rho)$ is a convex set, see [8], it follows that the right hand side of (14) belongs to $P_k(\rho)$ and this proves the result.

Theorem 3.4. *Let $f \in T_k(\lambda, \alpha_1, p, q, s, \rho)$ and let $\phi \in C(\rho)$ is the class of p-valent convex functions. Then $\phi * f \in T_k(\lambda, \alpha_1, p, q, s, \rho)$.*

Proof. Let $F = \phi * f$. Then we have

$$\left\{(1-\lambda)\frac{H_{p,q,s}(\alpha_1)F(z)}{z^p}+\lambda\frac{H_{p,q,s}(\alpha_1+1)F(z)}{z^p}\right\}=\frac{\phi(z)}{z^p}*G(z),$$

where

$$G(z) = \left\{(1-\lambda)\frac{H_{p,q,s}(\alpha_1)f(z)}{z^p} + \lambda\frac{H_{p,q,s}(\alpha_1+1)f(z)}{z^p}\right\} \in P_k(\rho).$$

Therefore,we have

$$\frac{\phi(z)}{z^p}*G(z) = (\frac{k}{4}+\frac{1}{2})\left\{(p-\rho)\left(\frac{\phi(z)}{z^p}*g_1(z)\right)+\rho\right\}-(\frac{k}{4}-\frac{1}{2})\left\{(p-\rho)\left(\frac{\phi(z)}{z^p*g_2(z)}\right)+\rho\right\}$$

where $g_1, g_2 \in P$.

Since $\phi \in C(p), \text{Re}\left\{\frac{\phi(z)}{z^p}\right\} > \frac{1}{2}$, $z \in E$, and so using Lemma 2.2, we conclude that $F = \phi * f \in T_k(\lambda, \alpha_1, p, q, s, \rho)$.

Theorem 3.5. *Let $f(z) \in A(p)$ and define the one-parameter integral operator $J_c(c > -p)$ by*

$$J_c f(z) = \frac{c+p}{z^c}\int_0^z t^{c-1}f(t)dt \;\; (f \in A(p); c > p). \tag{15}$$

If

$$\left\{(1-\lambda)\frac{H_{p,q,s}(\alpha_1)J_cf(z)}{z^p} + \lambda\frac{H_{p,q,s}(\alpha_1)f(z)}{z^p}\right\} \in P_k(\rho). \tag{16}$$

then

$$\frac{H_{p,q,s}(\alpha_1)J_cf(z)}{z^p} \in P_k(\rho_2),$$

where ρ_2 is given by

$$\rho_2 = \rho + (1-\rho)(2\gamma_1 - 1), \tag{17}$$

and

$$\gamma_1 = \int_0^1 \left(1 + t^{\text{Re}(\frac{\lambda}{(c+p)}}\right)^{-1} dt.$$

Proof. First of all it follows from the Definition 3.6, that

$$z(H_{p,q,s}(\alpha_1)J_cf(z))' = (c+p)H_{\lambda,q,s}(\alpha_1)f(z) - cH_{p,q,s}(\alpha_1)J_cf(z). \tag{18}$$

Let

$$\frac{H_{p,q,s}(\alpha_1 J_cf(z)}{z^p} = h(z) = (\frac{k}{4}+\frac{1}{2})h_1(z) - (\frac{k}{4}-\frac{1}{2})h_2(z), \tag{19}$$

then the hypothesis (16) in conjection with (18) would yield

$$\left\{(1-\lambda)\frac{H_{p,q,s}(\alpha_1)J_c f(z)}{z^p}+\lambda\frac{H_{p,q,s}(\alpha_1)f(z)}{z^p}\right\}=\left\{h(z)+\frac{\lambda z h'(z)}{c+p}\right\}\in P_k(\rho)\ \text{for} z\in E.$$

Consequently

$$\left\{h_i(z)+\frac{\lambda z h_i'(z)}{c+p}\right\}\in P(\rho),\ \ i=1,2,\ \ 0\leq\rho<p, \text{and} z\in E.$$

Using Lemma 2.1 with $\lambda_1=\frac{\lambda}{(c+p)}$, we have $\mathrm{Re}h_i(z)>\rho_2$, where ρ_2 is given by (17), and the proof is complete.

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Ali Muhammad,
Department of Basic Sciences
University of Engineering and Technology Peshawar Pakistan.
email: *ali7887@gmail.com*