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ON (h,k) -GROWTH OF EVOLUTION OPERATORS IN BANACH SPACES

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Abstract. The paper considers some concept of uniform and nonuniform asymptotical growth and polynomial growth as particular cases of (h,k)-stability of evolution operators in Banach spaces. Some illustrating examples clarify the relations between these properties.

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1. Introduction

Let T be the set defined by $T = \{(t,s) \in \mathbf{R}_+^2 : t \geq s \geq 0\}$ We recall that an operator-valued function $\Phi : T \to B(X)$ is called an *evolution operator* on the Banach spaces X iff:
e_1) $\Phi(t,t)$ =I (the identity operator on X) for every $t \geq 0$;
e_2) $\Phi(t,s)\,\Phi(s,t_0) = \Phi(t,t_0)$ for all (t,s) and $(s,t_0) \in T$.
Furthermore, if
e_3) there are $M \geq 1$ and a nondecreasing function $\varphi : \mathbf{R}_+ \to [1,\infty)$ such that:

$$\|\Phi(t,s)\| \leq M\varphi(t-s) \text{ for all } (t,s) \in T$$

then Φ is called with *uniform growth*.
If $h,k : \mathbf{R}_+ \to [1,\infty)$ then we introduce the concept of (h,k)-stability by
Definition 1.1. *If $h,k : \boldsymbol{R}_+ \to (0,\infty)$ are nondecreasing functions then the evolution operator $\Phi : T \to \mathcal{B}(X)$ is said to be (h,k)-stable (and we denote (h,k)-s) iff there are $N \geq 1$ and $t_0 \geq 0$ such that:*

$$\frac{h(t)}{h(s)}\,\|\Phi(t,t_0)x_0\| \leq Nk(s)\,\|\Phi(s,t_0)x_0\| \tag{1}$$

for all $t \geq s \geq t_0 \geq 0$ and all $x_0 \in X$.
Remark 1.1. If $h, k : \mathbf{R}_+ \to (0, \infty)$ are nondecreasing functions, then an evolution operator $\Phi : T \to \mathcal{B}(X)$ is *(h,k)-stable* iff there are $N \geq 1$ and $t_0 \geq 0$ such that:

$$\frac{h(t)}{h(s)} \|\Phi(t,s)x\| \leq Nk(s) \|x\| \tag{2}$$

for all $(t, s, x) \in T \times X$ with $s \geq t_0 \geq 0$.
Another concepts of stability are given by
Definition 1.2. *The evolution operator $\Phi : T \to B(X)$ is called:*
The evolution operator $\Phi : T \to \mathcal{B}(X)$ is called:
(i) uniformly exponentially stable (and denote u.e.s) iff there are $N \geq 1$, $t_0 \geq 0$ and $\alpha > 0$ such that :

$$e^{\alpha(t-s)} \|\Phi(t,s)x\| \leq N\|x\| \tag{3}$$

for all $(t, s, x) \in T \times X$ with $s \geq t_0 \geq 0$;
(ii) exponentially stable in the sense Barreira-Valls (and denote B.V.e.s) iff there are $N \geq 1$, $t_0 \geq 0$, $\alpha > 0$ and $\beta \geq 0$ such that:

$$e^{\alpha(t-s)} \|\Phi(t,s)x\| \leq Ne^{\beta s}\|x\| \tag{4}$$

for all $(t, s, x) \in T \times X$ with $s \geq t_0 \geq 0$;
(iii)(nonuniformly) exponentially stable (and denote e.s) iff there are $N \geq 1$, $t_0 \geq 0$, $\alpha > 0$ and a nondecreasing function $k : \boldsymbol{R}_+ \to [1, \infty)$ such that:

$$e^{\alpha(t-s)} \|\Phi(t,s)x\| \leq Nk(s)\|x\| \tag{5}$$

for all $(t, s, x) \in T \times X$ with $s \geq t_0 \geq 0$.
The particular cases of (h,k)-stability considered in this paper are : the uniform exponential growth, the exponential growth in the sense Barreira-Valls, the non-uniform exponential growth,uniform polynomial growth, polynomial growth in the sense Barreira-Valls and non-uniform polinomial growth. In what follows we present some relations between these concepts (implications and counterexamples).

2.Exponential growth

Let $\Phi : T \to B(X)$ be an evolution operator on Banach space X.
Definition 2.1. *The evolution operator $\Phi : T \to \mathcal{B}(X)$ is with :*
(i) uniform exponential growth iff there are $N \geq 1$, $t_0 \geq 0$ and $\alpha > 0$ such that :

$$\|\Phi(t,s)x\| \leq Ne^{\alpha(t-s)} \|x\| \tag{6}$$

for all $(t,s,x) \in T \times X$ *with* $s \geq t_0 \geq 0$*;*
(ii) exponential growth in the sense Barreira-Valls iff there are $N \geq 1$, $t_0 \geq 0$, $\alpha > 0$ *and* $\beta \geq 0$ *such that:*

$$\|\Phi(t,s)x\| \leq Ne^{\alpha(t-s)}e^{\beta s}\|x\| \tag{7}$$

for all $(t,s,x) \in T \times X$ *with* $s \geq t_0 \geq 0$*;*
(iii)(nonuniform) exponential growth iff there are $N \geq 1$, $\alpha > 0$, $t_0 \geq 0$ *and a nondecreasing function* $k : \boldsymbol{R}_+ \to [1,+\infty)$ *such that:*

$$\|\Phi(t,s)x\| \leq Ne^{\alpha(t-s)}k(s)\|x\| \tag{8}$$

for all $(t,s,x) \in T \times X$ *with* $s \geq t_0 \geq 0$.
Remark 2.1. The evolution operator $\Phi : T \to \mathcal{B}(X)$ is with *exponential growth in the sense Barreira-Valls* iff there are $N \geq 1$, $t_0 \geq 0$, $a > 0$ and $0 < b \leq a$ such that:

$$\|\Phi(t,s)x\| \leq Ne^{at}e^{-bs}\|x\| \tag{9}$$

for all $(t,s,x) \in T \times X$ with $s \geq t_0 \geq 0$.
Let $\mathcal{E}^-$ the set of all functions $f : \mathbf{R}_+ \to [1,\infty)$ with the property that there is $\alpha > 0$ such that $f(t) = e^{-\alpha t}$ for every $t \geq 0$.
Remark 2.2. We have that:
i) Φ is u.e.g. iff there is $h \in \mathcal{E}^-$ such that Φ is (h,h)-stable;
ii)Φ is B.V.e.g. iff there are $h,k \in \mathcal{E}^-$ such that Φ is (h,k)-stable;
iii) Φ is e.g. iff there exist $h \in \mathcal{E}^-$ such that Φ is (h,k)-stable.
Remark 2.3. It is obvious that: *u.e.g* $\Rightarrow$ *B.V.e.g* $\Rightarrow$ *e.g*
Example 2.1.(*Evolution operator with B.V.e.g and without u.e.g*)
Let $u : \mathbf{R}_+ \to (0,\infty)$ be the function defined by $u(t) = exp(3t - tcost)$. Then $\Phi : T \to \mathcal{B}(\mathbf{R}), \Phi(t,s)x = \frac{u(t)}{u(s)}x$ is an evolution operator on $X = \mathbf{R}$ with:

$$|\Phi(t,s)x| = |x|\, exp(3t - tcost - 3s + scoss) \leq |x|\, exp(4t - 2s)$$

for all $(t,s,x) \in T \times X$ with $s \geq t_0 = 1$. This shows, by Remark 2.1. with $a = 4$ and $b = 2$ that Φ is with B.V.e.g. If we suppose that Φ is with u.e.g then there exist $N \geq 1$ and $\alpha > 0$ such that: $exp(3t - tcost - 3s + scoss) \leq Nexp\alpha(t-s)$ for all $t \geq s \geq t_0 \geq 0$. For $t = 2n\pi + \frac{\pi}{2}$ and $s = 2n\pi$ we obtain a contradiction.
Proposition 2.1. *If the evolution operator* Φ *is u.e.s then it is with u.e.g.*

Proof. It is immediate from Definition 1.2 and Definition 2.1.

Proposition 2.2. *If the evolution operator* Φ *is B.V.e.s then it is with B.V.e.g.*

Proof. If the evolution operator Φ is B.V.e.s then there are $N \geq 1$, $\beta > 0$, $\alpha > 0$ and $t_0 > 0$ such that:

$$\|\Phi(t,s)x\| \leq Ne^{-\alpha t}e^{\beta s}\|x\| \leq Ne^{(\alpha+2\beta)t}e^{-\beta s}\|x\|$$

for all $t \geq s \geq t_0$ and all $x \in X$.

Proposition 2.3 *If the evolution operator* Φ *is e.s then it is with e.g.*

Proof. It is trivial.

3.Polynomial growth

Definition 3.1. *The evolution operator* $\Phi : T \to \mathcal{B}(X)$ *is said to be with:*
(i) uniform polynomial growth (and denote u.p.g) iff there are $N \geq 1$, $\alpha > 0$ *and* $t_0 \geq 1$ *such that :*

$$t^{-\alpha} s^{\alpha} \|\Phi(t,s)x\| \leq N\|x\| \tag{10}$$

for all $(t, s, x) \in T \times X$ *with* $s \geq t_0 \geq 1$*;*
(ii) polynomial growth in the sense Barreira-Valls (and denote B.V.p.g) iff there are $N \geq 1$, $\alpha > 0$, $\beta \geq 0$ *and* $t_0 \geq 1$ *such that:*

$$t^{-\alpha} s^{\alpha} \|\Phi(t,s)x\| \leq N s^{\beta}\|x\| \tag{11}$$

for all $t \geq s \geq t_0 \geq 1$ *and all* $x \in X$*;*
(iii)(nonuniform) polynomial growth (and denote p.g) iff there are $N \geq 1$, $\alpha > 0$, $t_0 \geq 1$ *and a nondecreasing function* $k : \boldsymbol{R}_+ \to (0, \infty)$ *such that:*

$$t^{-\alpha} s^{\alpha} \|\Phi(t,s)x\| \leq N k(s)\|x\| \tag{12}$$

for all $(t, s, x) \in T \times X$ *with* $s \geq t_0 \geq 1$.

Remark 3.1. The evolution operator $\Phi : T \to \mathcal{B}(X)$ is with *polynomial growth in the sense Barreira-Valls* iff there are $N \geq 1$, $t_0 \geq 1$, $a > 0$ and $0 < b \leq a$ such that:

$$\|\Phi(t,s)x\| \leq N t^{a} s^{-b} \|x\| \tag{13}$$

for all $(t, s, x) \in T \times X$ with $s \geq t_0 \geq 1$.

Let $\mathcal{P}^-$ the set of all functions $f : \mathbf{R}_+ \to [1, \infty)$ with the property that there is $\alpha > 0$ such that $f(t) = t^{-\alpha}$ for every $t \geq 0$.

Remark 3.2. The preceding definition shows that:
i) Φ is u.p.g. iff there are $h \in \mathcal{P}^-$ and $k = const. \geq 1$ such that Φ is (h, k)-stable;
ii)Φ is B.V.p.g. iff there are $h \in \mathcal{P}^-$ and $k \in \mathcal{P}^+$ such that Φ is (h, k)-stable;
iii) Φ is p.g. iff there are $h \in \mathcal{P}^-$ and a nondecreasing function $k : \mathbf{R}_+ \to (0, \infty)$

such that Φ is (h,k)-stable.

Remark 3.3. It is obvious that: $u.p.g \Rightarrow B.V.p.g \Rightarrow p.g$
The following examples show that the converse implications are not valid.
Example 3.1. (*Evolution operator with B.V.p.g and without u.p.g.*)
The evolution operator (on $\mathbf{R}$)$\Phi : \Delta \to \mathcal{B}(\mathbf{R}), \Phi(t,s)x = \frac{(t+1)^2(s+1)^{cosln(s+1)}}{(s+1)^2(t+1)^{cosln(t+1)}}x$ satisfies the inequality $s\,|\Phi(t,s)x| \leq \frac{s}{(s+1)}(t+1)^3\,|x| \leq 8t^3\,|x|\,, t^{-3}s^3\,|\Phi(t,s)x| \leq 8s^2\,|x|$ for all $(t,s,x) \in \Delta \times \mathbf{R}$ with $s \geq t_0 = 1$. It follows that Φ is with B.V.p.g. If we suppose that Φ is with u.p.g. then there are $N \geq 1$, $\alpha \geq 1$ and $t_0 \geq 1$ such that :

$$s^{\alpha}(t+1)^2(s+1)^{cosln(s+1)} \leq Nt^{\alpha}(s+1)^2(t+1)^{cosln(t+1)}$$

for all $t \geq s \geq t_0 \geq 1$.
From here, for $t = exp(2n\pi + \frac{\pi}{2}) - 1$ and $s = exp(2n\pi) - 1$ taking $n \to \infty$ we obtain a contradiction.

Example 3.2. (*Evolution operator with p.g. and without B.V.p.g.*)
Let $u : \mathbf{R}_+ \to [1,\infty)$ be a function with $u(n) = e^n$ and $u\left(n + \frac{1}{n}\right) = e^2$ for every $n \in \mathbf{N}^*$.
Then

$$\Phi : \Delta \to \mathcal{B}(\mathbf{R}), \Phi(t,s)x = \frac{t^2u(s)}{s^2u(t)}x$$

is an evolution operator on $\mathbf{R}$ with the property

$$|\Phi(t,s)x| = \frac{t^2u(s)\,|x|}{s^2u(t)} \leq N(s)t^2s^{-2}\,|x|$$

for all $(t,s,x) \in T \times \mathbf{R}$, where

$$N(s) = 1 + u(s)$$

This shows that Φ is with p.g.

If we suppose that Φ is B.V.p.s then there are $N \geq 1$, $\alpha > 0$, $\beta \geq 0$ and $t_0 > 0$ such that

$$s^{\alpha-\beta-2}u(s) \leq Nt^{\alpha-2}u(t)$$

for all $t \geq s \geq t_0$.

Then for $s = n$ and $t = n + \frac{1}{n}$ we obtain a contradiction and hence Φ does not have B.V.p.g.

Proposition 3.1. *If the evolution operator Φ is with u.p.g then it has u.e.g.*

Proof. Using the fact that the function $\varphi(t) = \frac{e^t}{t}$ is nondecreasing on $[1, +\infty)$ we obtain that if Φ is with u.p.g then by Definition 2.1 we have:

$$\|\Phi(t, t_0)x\| \leq N t^{\alpha} s^{-\alpha} \|x\| \leq N e^{\alpha t} e^{-\alpha s} \|x\| \leq N e^{\alpha(t-s)} \|x\|$$

for all $(t, s, x) \in T \times X$ with $s \geq t_0 \geq 1$.

Proposition 3.2. *If the evolution operator Φ has p.g then it is with e.g.*

Proof. It is immediate from Definition 2.1 and Definition 3.1, using the inequality $t \leq e^t$ for all $t \geq 1$.

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