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CONVERGENCE AND STABILITY RESULTS FOR SOME ITERATIVE SCHEMES

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Abstract. We obtain strong convergence results for quasi-contractive operators in arbitrary Banach space via some recently introduced iterative schemes and also establish stablity theorems for Kirk's iterative process. Our results generalize and extend some well-known results in the literature.

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1. Introduction

Our purpose in this paper is to obtain some strong convergence results for quasi-contractive operators in arbitrary Banach space via the iterative schemes introduced in [13, 14]. We also establish stablity theorems for Kirk's iterative process. The convergence results obtained are generalizations and extensions of those of [3, 8, 9, 18, 19] while our stability results generalize some of the results of Rhoades [20, 21] and the results of the author [12].

Let $(E, ||.||)$ be a normed linear space and $T : E \to E$ a selfmap of E. Suppose that $F_T = \{ p \in E \mid Tp = p \}$ is the set of fixed points of T.

In a normed linear space or a Banach space setting, we have several iterative processes that have been defined by many researchers to approximate the fixed points of different operators. Some of them are the following:
For $x_0 \in E$, define the sequence $\{x_n\}_{n=0}^{\infty}$ by

$$x_{n+1} = \sum_{i=0}^{k} \alpha_i T^i x_n,\ x_0 \in E,\ n = 0, 1, 2, \cdots ,\ \sum_{i=0}^{k} \alpha_i = 1, \tag{1}$$

$\alpha_i \geq 0,\ \alpha_0 \neq 0,\ \alpha_i \in [0,1]$, where k is a fixed integer. See Kirk [11] for this iterative process.

Another iterative process of interest is the Ishikawa scheme defined as follows: For $x_0 \in E$, define the sequence $\{x_n\}_{n=0}^{\infty}$ by

$$\left.\begin{array}{l} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T z_n, \\ z_n = (1-\beta_n)x_n + \beta_n T x_n \end{array}\right\} \ n = 0, 1, \cdots, \tag{2}$$

For the iterative process in (2), see Ishikawa [7].

The following two iterative processes have been recently introduced in [13, 14]:

(I) For $x_0 \in E$, define the sequence $\{x_n\}_{n=0}^{\infty}$ by

$$\left.\begin{array}{l} x_{n+1} = \alpha_{n,0}x_n + \sum_{i=1}^{k} \alpha_{n,i} T^i z_n,\ \ \sum_{i=0}^{k} \alpha_{n,i} = 1,\ n = 0, 1, 2, \cdots, \\ z_n = \sum_{j=0}^{s} \beta_{n,j} T^j x_n,\ \ \sum_{j=0}^{s} \beta_{n,j} = 1, \end{array}\right\} \tag{3}$$

$k \geq s,\ \alpha_{n,i} \geq 0,\ \alpha_{n,0} \neq 0,\ \beta_{n,j} \geq 0,\ \beta_{n,0} \neq 0,\ \alpha_{n,i},\ \beta_{n,j} \in [0,1]$, where k and s are fixed integers.

(II) For $x_0 \in E$, define the sequence $\{x_n\}_{n=0}^{\infty}$ by

$$\left.\begin{array}{l} x_{n+1} = \alpha_{n,0}\ x_n + \sum_{i=1}^{k} \alpha_{n,i} T_i z_n,\ \ \sum_{i=0}^{k} \alpha_{n,i} = 1,\ n = 0, 1, 2, \cdots, \\ z_n = \sum_{j=0}^{s} \beta_{n,j} T_j x_n,\ \ \sum_{j=0}^{s} \beta_{n,j} = 1, \end{array}\right\} \tag{4}$$

$k \geq s,\ \alpha_{n,i} \geq 0,\ \alpha_{n,0} \neq 0,\ \beta_{n,j} \geq 0,\ \beta_{n,0} \neq 0,\ \alpha_{n,i},\ \beta_{n,j} \in [0,1]$, where k and s are fixed integers and S_0 is an identity operator.

Remark 1.1 It has been shown in [13, 14] that the iterative algorithms defined in (3) and (4) generalize many well-known schemes in the literature.

Definition 1.1 [6]. *Let (E, d) be a complete metric space, $T : E \to E$ a selfmap of E. Suppose that $F_T \neq \phi$ (i.e. nonempty) is the set of fixed points of T. Let $\{x_n\}_{n=0}^{\infty} \subset E$ be the sequence generated by an iterative procedure involving T which is defined by*

$$x_{n+1} = f(T, x_n),\ n = 0, 1, \cdots, \tag{$\star$}$$

where $x_0 \in E$ is the initial approximation and f is some function. Suppose $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point p of T. Let $\{y_n\}_{n=0}^{\infty} \subset E$ and set $\epsilon_n = d(y_{n+1}, f(T, y_n)),\ (n = 0, 1, \cdots)$. Then, the iterative procedure $(\star)$ is said to be T-stable or stable with respect to T if and only if $\lim_{n\to\infty} \epsilon_n = 0$ implies $\lim_{n\to\infty} y_n = p$.

Remark 1.2. (i) Since the metric is induced by the norm, we have $\epsilon_n = ||y_{n+1} - f(T, y_n)||,\ n = 0, 1, \cdots$, in place of $\epsilon_n = d(y_{n+1}, f(T, y_n)),\ n = 0, 1, \cdots$, in the definitions of stability stated above whenever we are working in normed linear space or Banach space.

(ii) we obtain the Kirk's iterative process from $(\star)$ if $f(T, x_n) = \sum_{i=0}^{k} \alpha_i T^i x_n,\ n = 0, 1, 2, \cdots,\ \sum_{i=0}^{k} \alpha_i = 1$, where $\alpha_i \geq 0,\ \alpha_0 \neq 0,\ \alpha_i \in [0, 1]$ and k is a fixed integer. Any other iterative process can be obtained in a similar manner from $(\star)$.

Several stability results established in metric spaces and normed linear spaces are available in the literature. Some of the various authors whose contributions are of colossal value in the study of stability of the fixed point iterative procedures are Ostrowski [17], Harder and Hicks [6], Rhoades [18, 20], Osilike and Udomene [16], and Berinde [3, 4]. The first stability result on $T-$ stable mappings was due to Ostrowski [17].

Definition 1.2 [4, 22] (a) *A function* $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *is called a comparison function if it satisfies the following conditions:*
(i) ψ *is monotone increasing;*
(ii) $\lim\limits_{n\to\infty} \psi^n(t) = 0,\ \forall\ t \geq 0.$

Remark 1.3. (i) Every comparison function satisfies $\psi(0) = 0$.
(ii) $\psi^n(t)$ is the $n-$th iterate of $\psi(t)$.
We shall employ the following contractive definitions: Let E be an arbitrary Banach space,
(i) for an operator $T : E \to E$, there exist $a \in [0, 1)$ and a monotone increasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, with $\varphi(0) = 0$, such that

$$||Tx - Ty|| \leq \varphi(||x - Tx||) + a||x - y||,\ \forall\ x,\ y \in E. \tag{5}$$

(ii) also, for operators $T_i : E \to E,\ i = 0, 1, 2, \cdots, k$ there exist $a_i \in [0, 1),\ i = 0, 1, 2, \cdots, k$, and a monotone increasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$, with $\varphi(0) = 0$, such that

$$||T_i x - T_i y|| \leq \varphi(||x - T_i x||) + a_i||x - y||,\ \forall\ x,\ y \in E, \tag{6}$$

where T_0 =identity operator.

Other forms of the contractive conditions which shall be used are given in two of our Lemmas in the sequel:
Lemma 1.1 [12]. *Let* $(E,\ ||\cdot||)$ *be a normed linear space and let* $T : E \to E$ *be a selfmap of* E *satisfying (5), where* $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *is a sublinear, monotone increasing function such that* $\varphi(0) = 0,\ \varphi(Lu) \leq L\varphi(u),\ L \geq 0$. *Then,* $\forall\ i \in \mathbb{N}$, *and* $\forall\ x,\ y \in E$, $||T^i x - T^i y|| \leq \sum_{j=1}^{i} \binom{i}{j} b^{i-j} \varphi^j(||x - Tx||) + b^i||x - y||,\ \forall\ x,\ y \in E.$
Lemma 1.2 [15]. *Let* $\{\psi^k(t)\}_{k=0}^{n}$ *be a sequence of comparison functions. Then, any convex linear combination* $\sum_{j=0}^{n} c_j \psi^j(t)$ *of the comparison functions is also a comparison function, where* $\sum_{j=0}^{n} c_j = 1$ *and* $c_o, c_1, \cdots, c_n$ *are positive constants.*
Lemma 1.3 [15]. *If* $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *is a subadditive comparison function and* $\{\epsilon_n\}_{n=0}^{\infty}$

is a sequence of positive numbers such that $\lim_{n\to\infty} \epsilon_n = 0$, *then for any sequence of positive numbers* $\{u_n\}_{n=0}^{\infty}$ *satisfying*

$$u_{n+1} \leq \sum_{k=0}^{m} \delta_k \psi^k(u_n) + \epsilon_n, \; n = 0, 1, \cdots, \tag{7}$$

where $\delta_0, \delta_1, \cdots, \delta_{\mathrm{m}} \in [0,1]$ *with* $0 \leq \sum_{k=0}^{m} \delta_k \leq 1$, *we have* $\lim_{n\to\infty} u_n = 0$.

Lemma 1.4. *Let* $(E, \; ||\cdot||)$ *be a normed linear space and let* $T : E \to E$ *be a selfmap of* E *satisfying*

$$||Tx - Ty|| \leq \varphi(||x - Tx||) + \psi(||x - y||), \; \forall \; x, \; y \in E, \tag{8}$$

where $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *is a sublinear comparison function and* $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$, *a sublinear monotone increasing function such that* $\varphi(0) = 0$ *and* $\psi^s(\varphi^r(x)) \leq \varphi^r(\psi^s(x))$, $\forall \; x \in \mathbb{R}_+, \; r, \; s \in \mathbb{N}$. *Then,* $\forall \; i \in \mathbb{N}$, *we have*

$$||T^i x - T^i y|| \leq \sum_{j=1}^{i} \binom{i}{j} \varphi^j(\psi^{i-j}(||x - Tx||)) + \psi^i(||x - y||), \; \forall \; x, \; y \in E. \tag{9}$$

Proof: We first proof the sublinearity of both φ and ψ as follows: In order to show that ψ^i (i.e. iterate of ψ) is sublinear, we have to show that ψ^i is both subadditive and positively homogeneous. We first establish that ψ subadditive implies that each iterate ψ^i of ψ is also subadditive: Since ψ is subadditive, we have $\psi(x + y) \leq \psi(x) + \psi(y), \; \forall \; x, \; y \in \mathbb{R}^+$. Therefore,using subadditivity of ψ in ψ^2 yields

$$\psi^2(x+y) = \psi(\psi(x+y)) \leq \psi(\psi(x)+\psi(y)) \leq \psi(\psi(x)) + \psi(\psi(y)) = \psi^2(x) + \psi^2(y),$$

which implies that ψ^2 is subadditive. Similarly, applying subadditivity of ψ^2 in ψ^3, we get

$$\psi^3(x+y) = \psi(\psi^2(x+y)) \leq \psi(\psi^2(x) + \psi^2(y)) \leq \psi(\psi^2(x))+\psi(\psi^2(y)) = \psi^3(x) + \psi^3(y),$$

which implies that ψ^3 is also subadditive. Hence, in general, each $\psi^n, \; n = 1, 2, \cdots$, is subadditive. We now prove that ψ positively homogeneous implies that each iterate ψ^i of ψ is also positively homogeneous: Therefore, we have that $\psi(\alpha x) = \alpha\psi(x), \; \forall \; x \in \mathbb{R}^+, \; \alpha > 0$. Using positive homogeneity of ψ in ψ^2, we have

$$\psi^2(\alpha x) = \psi(\psi(\alpha x)) = \psi(\alpha\psi(x)) = \alpha\psi(\psi(x)) = \alpha\psi^2(x), \; \forall \; x \in \mathbb{R}^+, \; \alpha > 0,$$

which implies that ψ^2 is positively homogeneous. Hence, in general, each $\psi^n, \; n = 1, 2, \cdots$, is positively homogeneous. Thus, we have that $\psi^n, \; n = 1, 2, \cdots$, is sublinear. In a similar manner, we can prove that $\varphi^j, \; j = 1, 2, \cdots$, is sublinear.

The second part of the proof of this lemma is by induction on i as follows: If $i = 1$, then (9) becomes
$||Tx-Ty|| \leq \sum_{j=1}^{1} \binom{1}{j}\varphi^j(\psi^{1-j}(||x-Tx||))+\psi(||x-y||) = \varphi(||x-Tx||)+\psi(||x-y||)$, that is, (9) reduces to (8) when $i = 1$ and hence the result holds. Assume as an inductive hypothesis that (9) holds for $i = m,\ m \in \mathbb{N}$, i.e.

$$||T^m x - T^m y|| \leq \sum_{j=1}^{m} \binom{m}{j} \varphi^j(\psi^{m-j}(||x - Tx||)) + \psi^m(||x - y||),\ \forall\ x,\ y \in E.$$

We then show that the statement is true for $i = m + 1$;

$$\begin{aligned}
||T^{m+1}x - T^{m+1}y|| &= ||T^m(Tx) - T^m(Ty)|| \\
&\leq \textstyle\sum_{j=1}^{m} \binom{m}{j}\varphi^j(\psi^{m-j}(||Tx - T^2x||)) + \psi^m(||Tx - Ty||) \\
&\leq \textstyle\sum_{j=1}^{m} \binom{m}{j}\varphi^j(\psi^{m-j}(\varphi(||x - Tx||) + \psi(||x - Tx||))) \\
&\quad + \psi^m(\varphi(||x - Tx||) + \psi(||x - y||)) \\
&\leq \textstyle\sum_{j=1}^{m} \binom{m}{j}\varphi^j[\psi^{m-j}(\varphi(||x - Tx||)) + \psi^{m+1-j}(||x - Tx||)] \\
&\quad + \psi^m(\varphi(||x - Tx||)) + \psi^{m+1}(||x - y||) \\
&\leq \textstyle\sum_{j=1}^{m} \binom{m}{j}\varphi^j[\varphi(\psi^{m-j}(||x - Tx||)) + \psi^{m+1-j}(||x - Tx||)] \\
&\quad + \varphi(\psi^m(||x - Tx||)) + \psi^{m+1}(||x - y||) \\
&\leq \textstyle\sum_{j=1}^{m} \binom{m}{j}\varphi^{j+1}(\psi^{m-j}(||x - Tx||)) + \sum_{j=1}^{m} \binom{m}{j}\varphi^j(\psi^{m+1-j}(||x \\
&\quad - Tx||)) + \varphi(\psi^m(||x - Tx||)) + \psi^{m+1}(||x - y||) \\
&= \tbinom{m}{m}\varphi^{m+1}(||x - Tx||) + [\tbinom{m}{m-1} + \tbinom{m}{m}]\varphi^m(\psi(||x - Tx||)) \\
&\quad + [\tbinom{m}{m-2} + \tbinom{m}{m-1}]\varphi^{m-1}(\psi^2(||x - Tx||)) \\
&\quad + \cdots + [\tbinom{m}{1} + \tbinom{m}{0}]\varphi(\psi^m(||x - Tx||)) + \psi^{m+1}(||x - y||) \\
&= \tbinom{m+1}{m+1}\varphi^{m+1}(||x - Tx||) + \tbinom{m+1}{m}\varphi^m(\psi(||x - Tx||)) \\
&\quad + \tbinom{m+1}{m-1}\varphi^{m-1}(\psi^2(||x - Tx||)) + \cdots \\
&\quad + \tbinom{m+1}{2}\varphi^2(\psi^{m-1}(||x - Tx||)) + \tbinom{m+1}{1}\varphi(\psi^m(||x - Tx||)) \\
&\quad + \psi^{m+1}(||x - y||) \\
&= \textstyle\sum_{j=1}^{m+1} \binom{m+1}{j}\varphi^j(\psi^{m+1-j}(||x - Tx||)) + \psi^{m+1}(||x - y||).
\end{aligned}$$

Lemma 1.5. *Let* $(E,\ ||\cdot||)$ *be a normed linear space and let* $T : E \to E$ *be a selfmap of* E *satisfying*

$$||Tx - Ty|| \leq L||x - Tx|| + \psi(||x - y||),\ \forall\ x,\ y \in E,\ L \geq 0, \tag{10}$$

where $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *is a sublinear comparison function. Then,* $\forall\ i \in \mathbb{N}$, *we have*

$$||T^i x - T^i y|| \leq \sum_{j=1}^{i} \binom{i}{j} L^j \psi^{i-j}(||x - Tx||) + \psi^i(||x - y||),\ \forall\ x,\ y \in E. \tag{11}$$

Proof: The proof of sublinearity of ψ^i (i.e. iterate of ψ) for each $i = 1, 2, \cdots$, is the same as that of Lemma 1.4. The second part of the proof of this lemma is by induction on i: If $i = 1$, then (11) becomes
$||Tx - Ty|| \leq \sum_{j=1}^{1} \binom{1}{j} L^j \psi^{1-j}(||x - Tx||) + \psi(||x - y||) = L||x - Tx|| + \psi(||x - y||)$, that is, (11) reduces to (10) when $i = 1$ and hence the result holds. Assume as an inductive hypothesis that (11) holds for $i = m,\ m \in \mathbb{N}$, i.e.

$$||T^m x - T^m y|| \leq \sum_{j=1}^{m} \binom{m}{j} L^j \psi^{m-j}(||x - Tx||) + \psi^m(||x - y||),\ \forall\ x,\ y \in E.$$

We then show that the statement is true for $i = m + 1$;

$$\begin{aligned}
||T^{m+1}x - T^{m+1}y|| &= ||T^m(Tx) - T^m(Ty)|| \\
&\leq \textstyle\sum_{j=1}^{m} \binom{m}{j} L^j \psi^{m-j}(||Tx - T^2x||) + \psi^m(||Tx - Ty||) \\
&\leq \textstyle\sum_{j=1}^{m} \binom{m}{j} L^j \psi^{m-j}(L||x - Tx|| + \psi(||x - Tx||)) \\
&\quad + \psi^m(L||x - Tx|| + \psi(||x - y||)) \\
&\leq \textstyle\sum_{j=1}^{m} \binom{m}{j} L^{j+1} \psi^{m-j}(||x - Tx||) + \sum_{j=1}^{m} \binom{m}{j} L^j \psi^{m+1-j}(||x \\
&\quad - Tx||) + L\psi^m(||x - Tx||) + \psi^{m+1}(||x - y||) \\
&= \binom{m}{m} L^{m+1}||x - Tx|| + [\binom{m}{m-1} + \binom{m}{m}] L^m \psi(||x - Tx||) \\
&\quad + [\binom{m}{m-2} + \binom{m}{m-1}] L^{m-1} \psi^2(||x - Tx||) + \cdots \\
&\quad + [\binom{m}{1} + \binom{m}{2}] L^2 \psi^{m-1}(||x - Tx||) + [\binom{m}{1} + \binom{m}{0}] L\psi^m(||x \\
&\quad - Tx||) + \psi^{m+1}(||x - y||) \\
&= \binom{m+1}{m+1} L^{m+1}||x - Tx|| + \binom{m+1}{m} L^m \psi(||x - Tx||) \\
&\quad + \binom{m+1}{m-1} L^{m-1} \psi^2(||x - Tx||) + \cdots \\
&\quad + \binom{m+1}{2} L^2 \psi^{m-1}(||x - Tx||) + \binom{m+1}{1} L\psi^m(||x - Tx||) \\
&\quad + \psi^{m+1}(||x - y||) \\
&= \textstyle\sum_{j=1}^{m+1} \binom{m+1}{j} L^j \psi^{m+1-j}(||x - Tx||) + \psi^{m+1}(||x - y||).
\end{aligned}$$

Remark 1.4. In addition to both (5) and (6), the contractive conditions (8) and (10) shall also be used to prove some of our results.

2. Main results

We now establish some convergence results:

Theorem 2.1. *Let E be an arbitrary Banach space, K a closed convex subset of E and $T : K \to K$ an operator satisfying (5), where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a subadditive monotone increasing function such that $\varphi(0) = 0,\ \varphi(Lu) \leq L\varphi(u),\ u \in \mathbb{R}_+$. Let $x_0 \in K,\ \{x_n\}_{n=0}^{\infty}$ be the iterative process defined by (3). Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to the fixed point p of T.*

Proof: We shall employ Lemma 1.1 and the triangle inequality to establish that

$\lim_{n\to\infty} x_n = p$. However, we shall first establish that T satisfying the given contractive condition has a unique fixed point. Suppose that there exist $p_1,\ p_2 \in F_T,\ p_1 \neq p_2$, with $||p_1 - p_2|| > 0$. Therefore, we have

$$\begin{aligned} 0 < ||p_1 - p_2|| = ||T^i p_1 - T^i p_2|| &\leq \textstyle\sum_{j=1}^{i} \binom{i}{j} a^{i-j} \varphi^j(||p_1 - Tp_1||) + a^i ||p_1 - p_2|| \\ &= \textstyle\sum_{j=1}^{i} \binom{i}{j} a^{i-j} \varphi^j(0) + a^i ||p_1 - p_2||, \end{aligned}$$

from which we have that $(1 - a^i)||p_1 - p_2|| \leq 0$. Since $a \in [0, 1)$, then, $1 - a^i > 0$ and $||p_1 - p_2|| \leq 0$.

Also, since norm is nonnegative we have $||p_1 - p_2|| = 0$. That is, $p_1 = p_2$.

Thus, T has a unique fixed point.

We now prove that $\{x_n\}_{n=0}^{\infty}$ converges strongly to a fixed point of T : Then, we have

$$\begin{aligned} ||x_{n+1} - p|| &\leq \textstyle\sum_{i=1}^{k} \alpha_{n,i} ||T^i p - T^i z_n|| + \alpha_{n,0} ||x_n - p|| \\ &\leq \textstyle\sum_{i=1}^{k} \alpha_{n,i} \left\{ \sum_{j=1}^{i} \binom{i}{j} a^{i-j} \varphi^j(||p - Tp||) + a^i ||p - z_n|| \right\} + \alpha_{n,0} ||x_n - p|| \\ &= \left(\textstyle\sum_{i=1}^{k} \alpha_{n,i} a^i \right) ||p - z_n|| + \alpha_{n,0} ||x_n - p|| \\ &= \left(\textstyle\sum_{i=1}^{k} \alpha_{n,i} a^i \right) || \sum_{r=0}^{s} \beta_{n,r}\ T^r p - \sum_{r=0}^{s} \beta_{n,r}\ T^r x_n|| + \alpha_{n,0} ||x_n - p|| \\ &\leq \left(\textstyle\sum_{i=1}^{k} \alpha_{n,i} a^i \right) \{ \sum_{r=1}^{s} \beta_{n,r} ||T^r p - T^r x_n|| + \beta_{n,0} ||x_n - p|| \} \\ &\quad + \alpha_{n,0} ||x_n - p|| \\ &= \left(\textstyle\sum_{i=1}^{k} \alpha_{n,i} a^i \right) \sum_{r=1}^{s} \beta_{n,r} ||T^r p - T^r x_n|| + \left(\sum_{i=1}^{k} \alpha_{n,i} a^i \right) \beta_{n,0} ||x_n - p|| \\ &\quad + \alpha_{n,0} ||x_n - p|| \\ &\leq \left(\textstyle\sum_{i=1}^{k} \alpha_{n,i} a^i \right) \sum_{r=1}^{s} \beta_{n,r} [\ \sum_{j=1}^{r} \binom{r}{j} a^{r-j} \varphi^j(||p - Tp||) + a^r ||p - x_n||\ ] \\ &\quad + \left(\textstyle\sum_{i=1}^{k} \alpha_{n,i} a^i \right) \beta_{n,0} ||x_n - p|| + \alpha_{n,0} ||x_n - p|| \\ &= [(\textstyle\sum_{i=1}^{k} \alpha_{n,i} a^i)(\sum_{r=0}^{s} \beta_{n,r} a^r) + \alpha_{n,0}\ ]||x_n - p|| \\ &\leq \Pi_{\nu=0}^{n} [\ (\textstyle\sum_{i=1}^{k} \alpha_{\nu,i} a^i)(\sum_{r=0}^{s} \beta_{\nu,r} a^r) + \alpha_{\nu,0}\ ]||x_0 - p|| \to 0 \text{ as n} \to \infty. \quad (12) \end{aligned}$$

We claim that $0 \leq (\sum_{i=1}^{k} \alpha_{\nu,i} a^i)(\sum_{r=0}^{s} \beta_{\nu,r} a^r) + \alpha_{\nu,0} < 1$ as follows:

$$\begin{aligned} \textstyle\sum_{i=1}^{k} \alpha_{\nu,i} a^i &\leq \textstyle\sum_{i=1}^{k} |\alpha_{\nu,i} a^i| = |\alpha_{\nu,1} a| + |\alpha_{\nu,2} a^2| + \cdots + |\alpha_{\nu,k} a^k| \\ &= |\alpha_{\nu,1}||a| + |\alpha_{\nu,2}||a|^2 + \cdots + |\alpha_{\nu,k}||a|^k \\ &< |\alpha_{\nu,1}| + |\alpha_{\nu,2}| + \cdots + |\alpha_{\nu,k}| = \alpha_{\nu,1} + \alpha_{\nu,2} + \cdots + \alpha_{\nu,k} \\ &= 1 - \alpha_{\nu,0},\ \ \alpha_{\nu,0} \in (0, 1). \end{aligned}$$

Similarly, we have that

$$\begin{aligned} \textstyle\sum_{r=0}^{s} \beta_{\nu,r} a^r &\leq \textstyle\sum_{r=0}^{s} |\beta_{\nu,r} a^r| = |\beta_{\nu,0}| + |\beta_{\nu,1} a| + |\beta_{\nu,2} a^2| + \cdots + |\beta_{\nu,k} a^s| \\ &= |\beta_{\nu,0}| + |\beta_{\nu,1}||a| + |\beta_{\nu,2}||a|^2 + \cdots + |\beta_{\nu,s}||a|^s \\ &< |\beta_{\nu,0}| + |\beta_{\nu,1}| + \cdots + |\beta_{\nu,s}| = \beta_{\nu,0} + \beta_{\nu,1} + \cdots + \beta_{\nu,s} = 1. \end{aligned}$$

Therefore, $(\sum_{i=1}^k \alpha_{\nu,i} a^i)(\sum_{r=0}^s \beta_{\nu,r} a^r) + \alpha_{\nu,0} < 1 - \alpha_{\nu,0} + \alpha_{\nu,0} = 1$.

Hence, we obtain from (12) that $||x_{n+1} - p|| \to 0$ as n $\to \infty$,
i.e. $\{x_n\}_{n=0}^{\infty}$ converges strongly to p.

Theorem 2.2. *Let E be an arbitrary Banach space, K a closed convex subset of E and $T_i : K \to K,\ (i = 0, 1, \cdots, k)$, selfoperators satisfying (6), where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a monotone increasing function such that $\varphi(0) = 0$. Let $x_0 \in K,\ \{x_n\}_{n=0}^{\infty}$ be the iterative process defined by (4). Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique common fixed point p of T_i* (for each i).

Proof. Let F_{T_i} be the set of common fixed points of T_i $(i = 0, 1, \cdots, k)$. Suppose that there exist $p_1,\ p_2 \in F_{T_i},\ p_1 \neq p_2$, with $||p_1 - p_2|| > 0$. Therefore, we have

$$\begin{aligned} 0 < ||p_1 - p_2|| = ||T_i p_1 - T_i p_2|| &\leq \varphi(||p_1 - T_i p_1||) + a_i ||p_1 - p_2|| \\ &= a_i ||p_1 - p_2||, \end{aligned}$$

from which it follows that we have that $p_1 = p_2$. That is, T_i $(i = 0, 1, \cdots, k)$ have a unique common fixed point.

We now prove that $\{x_n\}_{n=0}^{\infty}$ converges strongly to a fixed point of T_i.

Then, we have

$$\begin{aligned} ||x_{n+1} - p|| &\leq \sum_{i=1}^k \alpha_{n,i} ||T_i p - T_i z_n|| + \alpha_{n,0} ||x_n - p|| \\ &\leq \sum_{i=1}^k \alpha_{n,i} \{\varphi(||p - T_i p||) + a_i ||p - z_n||\} + \alpha_{n,0} ||x_n - p|| \\ &= \left(\sum_{i=1}^k \alpha_{n,i} a_i\right) ||p - z_n|| + \alpha_{n,0} ||x_n - p|| \\ &= \left(\sum_{i=1}^k \alpha_{n,i} a_i\right) ||\sum_{r=0}^s \beta_{n,r} T_r p - \sum_{r=0}^s \beta_{n,r} T_r x_n|| + \alpha_{n,0} ||x_n - p|| \\ &\leq \left(\sum_{i=1}^k \alpha_{n,i} a_i\right) \{\sum_{r=0}^s \beta_{n,r} ||T_r p - T_r x_n||\} + \alpha_{n,0} ||x_n - p|| \\ &\leq \left(\sum_{i=1}^k \alpha_{n,i} a_i\right) \sum_{r=0}^s \beta_{n,r} [\varphi(||p - T_r p||) + a_r ||p - x_n||\ ] + \alpha_{n,0} ||x_n - p|| \\ &= [\ (\sum_{i=1}^k \alpha_{n,i} a_i)(\sum_{r=0}^s \beta_{n,r} a_r) + \alpha_{n,0}\ ]||x_n - p|| \\ &\leq \Pi_{\nu=0}^n [\ (\sum_{i=1}^k \alpha_{\nu,i} a_i)(\sum_{r=0}^s \beta_{\nu,r} a_r) + \alpha_{\nu,0}\ ]||x_0 - p|| \to 0 \text{ as n} \to \infty, \quad (13) \end{aligned}$$

where as in Theorem 2.1, we can show that $0 \leq (\sum_{i=1}^k \alpha_{\nu,i} a_i)(\sum_{r=0}^s \beta_{\nu,r} a_r) + \alpha_{\nu,0} < 1$.

Hence, we obtain from (13) that $||x_{n+1} - p|| \to 0$ as n $\to \infty$,
i.e. $\{x_n\}_{n=0}^{\infty}$ converges strongly to p.

Theorem 2.3. *Let E be an arbitrary Banach space, K a closed convex subset of E, and $T : K \to K$ an operator satisfying*

$$||Tx - Ty|| \leq \frac{\varphi(||x - Tx||) + a||x - y||}{1 + M||x - Tx||},\ \forall\ x,\ y \in E,\ a \in [0, 1),\ M \geq 0, \qquad (14)$$

where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *is a monotone increasing function such that* $\varphi(0) = 0$. *Let* $x_0 \in K$, $\{x_n\}_{n=0}^{\infty}$ *defined by (2) be the Ishikawa iterative process with* α_n, $\beta_n \in [0,1]$ and $\sum_{\mathrm{k}=0}^{\infty} \alpha_{\mathrm{k}} = \infty$. *Then, the Ishikawa iterative process converges strongly to the fixed point of* T.

Proof. We shall first establish that T has a unique fixed point by using condition (14): Suppose not. Then, there exist x^*, $y^* \in F_T$, $x^* \neq y^*$ and $||\mathrm{x}^* - \mathrm{y}^*|| > 0$. Therefore, we have

$$\begin{aligned} 0 < ||x^* - y^*|| = ||Tx^* - Ty^*|| &\leq \tfrac{\varphi(||x^*-Tx^*||)+a||x^*-y^*||}{1+M||x^*-Tx^*||} \\ &= a||x^* - y^*|| \end{aligned}$$

from which it follows that $(1-a)||x^* - y^*|| \leq 0$, which leads to $1 - a > 0$ (since $\mathrm{a} \in [0,1)$), but $||x^* - y^*|| \leq 0$ (which is a contradiction).

Therefore, since norm is nonnegative, $||x^* - y^*|| = 0$ i.e. $x^* = y^* = p$, thus proving the uniqueness of the fixed point for T. Hence, $F_T = \{p\}$.

We now prove that $\{x_n\}_{n=0}^{\infty}$ converges strongly to the fixed point p using condition (14). Therefore, we have

$$\begin{aligned} ||x_{n+1} - p|| &\leq ||(1-\alpha_n)x_n + \alpha_n T z_n - (1 - \alpha_n + \alpha_n)p|| \\ &\leq (1-\alpha_n)||x_n - p|| + a\alpha_n||p - z_n|| \\ &\leq [1 - \alpha_n(1-a) - a\alpha_n\beta_n(1-a)]||x_n - p|| \\ &\leq [1 - (1-a)\alpha_n]||x_n - p|| \\ &\leq \Pi_{k=0}^{n}[1 - (1-a)\alpha_k]||x_0 - p|| \\ &\leq \Pi_{k=0}^{n} e^{-(1-a)\alpha_k}||x_0 - p|| \\ &= e^{-[(1-a)\sum_{k=0}^{n}\alpha_k]}||x_0 - p|| \to 0 \text{ as n} \to \infty, \end{aligned} \tag{15}$$

since $\sum_{k=0}^{\infty} \alpha_k = \infty$ and $a \in [0,1)$. Hence, we obtain from (15) that $||x_{n+1} - p|| \to 0$ as $\mathrm{n} \to \infty$, i.e. $\{x_n\}_{n=0}^{\infty}$ converges strongly to p.

Remark 2.1. If in each of Theorem 2.1 and Theorem 2.2, the iterative processes defined by

$$x_{n+1} = \sum_{i=0}^{k} \alpha_{n,i} T^i x_n, \ \ \sum_{i=0}^{k} \alpha_{n,i} = 1, \ \ n = 0, 1, 2, \cdots, \tag{16}$$

$\alpha_{n,i} \geq 0$, $\alpha_{n,0} \neq 0$, $\alpha_{n,i} \in [0,1]$, where k is a fixed integer;
and

$$x_{n+1} = \sum_{i=0}^{k} \alpha_{n,i} T_i x_n, \ \ \sum_{i=0}^{k} \alpha_{n,i} = 1, \ \ n = 0, 1, 2, \cdots, \tag{17}$$

$\alpha_{n,i} \geq 0$, $\alpha_{n,0} \neq 0$, $\alpha_{n,i} \in [0,1]$, where k is a fixed integer and T_0 =identity operator; are employed, then we obtain corresponding results for the one-step processes defined in (16) and (17). Again, we refer to [13, 14] for these iterative processes too.

Remark 2.2. Theorem 2.1, Theorem 2.2 and Theorem 2.3 are generalizations and extensions of both Theorem 1 and Theorem 2 of Berinde [3], Theorem 2 and Theorem 3 of Kannan [8], Theorem 3 of Kannan [9], Theorem 4 of Rhoades [18] as well as Theorem 8 of Rhoades [19]. See also Berinde [4] for the results of Rhoades [18, 19].

We prove the following stability results:

Theorem 2.4. *Let $(E, ||.||)$ is a normed linear space and $T : E \to E$ a selfmap of E satisfying (8). Let $x_0 \in E$ and $\{x_n\}_{n=0}^{\infty}$ defined by (1) be the Kirk's iterative process. Suppose that T has a fixed point p. Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous sublinear comparison function and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ a sublinear monotone increasing function such that $\varphi(0) = 0$ and $\psi^s(\varphi^r(x)) \leq \varphi^r(\psi^s(x))$, $\forall\, x \in \mathbb{R}^+$, $r,\ s \in \mathbb{N}$. Then, the Kirk iteration process is T-stable.*

Proof. Let $\{y_n\}_{n=0}^{\infty} \subset E$ and $\epsilon_n = ||y_{n+1} - \sum_{i=0}^{k} \alpha_i T^i y_n||$. Let $\lim\limits_{n\to\infty} \epsilon_n = 0$. Then, we shall prove that $\lim\limits_{n\to\infty} y_n = p$. By using both Lemma 1.4 and the triangle inequality we have that:

$$\begin{aligned}
||y_{n+1} - p|| &\leq ||y_{n+1} - \textstyle\sum_{i=0}^{k} \alpha_i T^i y_n|| + ||\textstyle\sum_{i=0}^{k} \alpha_i T^i y_n - p|| \\
&= \epsilon_n + ||\textstyle\sum_{i=0}^{k} \alpha_i T^i y_n - \textstyle\sum_{i=0}^{k} \alpha_i T^i p|| \\
&= \epsilon_n + ||\textstyle\sum_{i=0}^{k} \alpha_i (T^i y_n - T^i p)|| \\
&\leq \textstyle\sum_{i=0}^{k} \alpha_i ||T^i p - T^i y_n|| + \epsilon_n \\
&= \alpha_0 ||p - y_n|| + \textstyle\sum_{i=1}^{k} \alpha_i ||T^i p - T^i y_n|| + \epsilon_n \\
&\leq \textstyle\sum_{i=0}^{k} \alpha_i \psi^i(||y_n - p||) + \epsilon_n, \qquad (18)
\end{aligned}$$

where $\psi^{i-j}(0) = \psi(0) = 0$ and $\varphi^{j}(0) = \varphi(0) = 0$.

We have by Lemma 1.2 that $\sum_{i=0}^{k} \alpha_i \psi^i(||y_n - p||)$ is a comparison function. Therefore, using Lemma 1.3 in (18) yields $\lim\limits_{n\to\infty} ||y_n - p|| = 0$, that is, $\lim\limits_{n\to\infty} y_n = p$.

Conversely, let $\lim\limits_{n\to\infty} y_n = p$. Then, by Lemma 1.4 and the triangle inequality, we have

$$\begin{aligned}
\epsilon_n &= ||y_{n+1} - \textstyle\sum_{i=0}^{k} \alpha_i T^i y_n|| \\
&\leq ||y_{n+1} - p|| + ||p - \textstyle\sum_{i=0}^{k} \alpha_i T^i y_n|| \\
&= ||y_{n+1} - p|| + ||\textstyle\sum_{i=0}^{k} \alpha_i T^i p - \textstyle\sum_{i=0}^{k} \alpha_i T^i y_n|| \\
&= ||y_{n+1} - p|| + ||\textstyle\sum_{i=0}^{k} \alpha_i (T^i p - T^i y_n)|| \\
&\leq ||y_{n+1} - p|| + \textstyle\sum_{i=0}^{k} \alpha_i ||T^i p - T^i y_n|| \\
&= ||y_{n+1} - p|| + \alpha_0 ||p - y_n|| + \textstyle\sum_{i=1}^{k} \alpha_i ||T^i p - T^i y_n|| \\
&\leq ||y_{n+1} - p|| + \textstyle\sum_{i=0}^{k} \alpha_i \psi^i(||y_n - p||) \to 0 \text{ as n} \to \infty.
\end{aligned}$$

Theorem 2.5. *Let $(E, ||.||)$ is a normed linear space and $T : E \to E$ a selfmap of E satisfying (10). Let $x_0 \in E$ and $\{x_n\}_{n=0}^{\infty}$ defined by (1) be the Kirk's iterative* *process. Suppose that T has a fixed point p. Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous sublinear comparison function and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ a sublinear monotone increasing function such that $\varphi(0) = 0$. Then, the Kirk iteration process is T-stable.*

Proof. The proof of this result is similar to that of Theorem 2.4 except for the application of Lemma 1.2, Lemma 1.3 and Lemma 1.5.

Remark 2.3. Our stability results generalize some of the results of Rhoades [20, 21]. In particular, both Theorem 2.4 and Theorem 2.5 are generalizations of the results of Olatinwo [12].

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