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ON THE UNIVALENCE OF A CERTAIN INTEGRAL OPERATOR

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Abstract. In view of an integral operator $H_{\gamma_1,\gamma_2,\ldots,\gamma_{[\mathrm{Re}\eta]},\beta,\eta}$ for analytic functions f in the open unit disk $\mathcal{U}$, sufficient conditions for univalence of this integral operator are discussed.

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1. Introduction

Let $\mathcal{A}$ be the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let S denote the subclass of $\mathcal{A}$ consisting of all univalent functions f in $\mathcal{U}$.

For $f \in \mathcal{A}$, the integral operator G_α is defined by

$$G_\alpha(z) = \int_0^z \left(\frac{f(u)}{u}\right)^{\frac{1}{\alpha}} du \tag{1.1}$$

for some complex numbers $\alpha (\alpha \neq 0)$.

In [1] Kim-Merkes prove that the integral operator G_α is in the class $\mathcal{S}$ for $\frac{1}{|\alpha|} \leq \frac{1}{4}$ and $f \in \mathcal{S}$.

Also, the integral operator J_γ for $f \in \mathcal{A}$ is given by

$$M_\gamma(z) = \left\{\frac{1}{\gamma}\int_0^z u^{-1}\left(f(u)\right)^{\frac{1}{\gamma}} du\right\}^{\gamma} \tag{1.2}$$

γ be a complex number, $\gamma \neq 0$.

Miller and Mocanu in [3] have studied that the integral operator M_γ is in the class S for $f \in \mathcal{S}^*, \gamma > 0, \mathcal{S}^*$ is the subclass of $\mathcal{S}$ consisting of all starlike functions f in $\mathcal{U}$.

We introduce the general integral operator

$$H_{\gamma_1,\gamma_2,\ldots,\gamma_{[\mathrm{Re}\eta]},\beta,\eta}(z) = \left\{ \eta\beta \int_0^z u^{\eta\beta-1} \left(\frac{f_1(u)}{u}\right)^{\frac{1}{\gamma_1}} \cdots \left(\frac{f_{[\mathrm{Re}\eta]}(u)}{u}\right)^{\frac{1}{\gamma_{[\mathrm{Re}\eta]}}} du \right\}^{\frac{1}{\eta\beta}} \quad (1.3)$$

for $f_j \in \mathcal{A}$, γ_j, η, β complex numbers, $[\mathrm{Re}\eta] \geq 1$, $\gamma_j \neq 0$, $j = \overline{1, [\mathrm{Re}\eta]}$, $\beta \neq 0$, and $[\mathrm{Re}\eta]$ is the integer part of η.

If in (1.3) we take $\eta = 1$, $\gamma_1 = \gamma$, $\beta = \frac{1}{\gamma}$ and $f_1 = f$ we obtain the integral operator M_γ.

From (1.3) we take $\eta \cdot \beta = 1$, $[\mathrm{Re}\eta] = 1$, $\gamma_1 = \alpha$, $f_1 = f$, we obtain the integral operator G_γ, given by (1.1).

2.Preliminary results

We need the following lemmas.

Lemma 2.1.[5] *Let α be a complex number,* $\mathrm{Re}\,\alpha > 0$ *and $f \in \mathcal{A}$. If*

$$\frac{1-|z|^{2\mathrm{Re}\,\alpha}}{\mathrm{Re}\,\alpha} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1 \quad (2.1)$$

for all $z \in \mathcal{U}$, then for any complex number β, $\mathrm{Re}\,\beta \geq \mathrm{Re}\,\alpha$ *the function*

$$F_\beta(z) = \left[\beta \int_0^z u^{\beta-1} f'(u) du \right]^{\frac{1}{\beta}} \quad (2.2)$$

is in the class S.

Lemma 2.2.(Schwarz [2]) *Let f the function regular in the disk $\mathcal{U}_R = \{z \in \mathbb{C} : |z| < R\}$ with $|f(z)| < M$, M fixed. If $f(z)$ has in $z = 0$ one zero with multiply $\geq m$, then*

$$|f(z)| \leq \frac{M}{R^m}|z|^m, \; z \in \mathcal{U}_R \quad (2.3)$$

the equality (in the inequality (2.3) for $z \neq 0$) can hold only if

$$f(z) = e^{i\theta}\frac{M}{R^m}z^m,$$

where θ is constant.

3.Main results

Theorem 3.1. *Let γ_j, η complex numbers,* $\mathrm{Re}\eta \geq 1$ $j = \overline{1, [\mathrm{Re}\eta]}$, $a = \sum_{j=1}^{[\mathrm{Re}\eta]} Re\, \frac{1}{\gamma_j} > 0$ *and* $f_j \in \mathcal{A}$, $f_j(z) = z + b_{2j}z^2 + b_{3j}z^3 + \ldots$, $j = \overline{1, [\mathrm{Re}\eta]}$.
If

$$\left| \frac{zf_j'(z)}{f_j(z)} - 1 \right| \leq \frac{(2a+1)^{\frac{2a+1}{2a}}}{2\,[\mathrm{Re}\eta]} |\gamma_j|, \quad j = \overline{1, [\mathrm{Re}\eta]}, \tag{3.1}$$

for all $z \in \mathcal{U}$*, then for any complex number* β*,* $Re\, \eta\beta \geq a$*, the function*

$$H_{\gamma_1,\gamma_2,\ldots,\gamma_{[\mathrm{Re}\eta]},\beta,\eta}(z) = \left\{ \eta\beta \int_0^z u^{\eta\beta-1} \left(\frac{f_1(u)}{u} \right)^{\frac{1}{\gamma_1}} \ldots \left(\frac{f_{[\mathrm{Re}\eta]}(u)}{u} \right)^{\frac{1}{\gamma_{[\mathrm{Re}\eta]}}} du \right\}^{\frac{1}{\eta\beta}} \tag{3.2}$$

is in the class $\mathcal{S}$.

Proof. We consider the function

$$g(z) = \int_0^z \left(\frac{f_1(u)}{u} \right)^{\frac{1}{\gamma_1}} \ldots \left(\frac{f_{[\mathrm{Re}\eta]}(u)}{u} \right)^{\frac{1}{\gamma_{[\mathrm{Re}\eta]}}} du \tag{3.3}$$

The function g is regular in $\mathcal{U}$. We define the function $p(z) = \frac{zg''(z)}{g'(z)}$, $z \in \mathcal{U}$ and we obtain

$$p(z) = \frac{zg''(z)}{g'(z)} = \sum_{j=1}^{[\mathrm{Re}\eta]} \left[\frac{1}{\gamma_j} \left(\frac{zf_j'(z)}{f_j(z)} - 1 \right) \right], \quad z \in \mathcal{U} \tag{3.4}$$

From (3.1) and (3.4) we have

$$|p(z)| \leq \frac{(2a+1)^{\frac{2a+1}{2a}}}{2} \tag{3.5}$$

for all $z \in \mathcal{U}$ and applying Lemma 2.2 we get

$$|p(z)| \leq \frac{(2a+1)^{\frac{2a+1}{2a}}}{2} |z|, \quad z \in \mathcal{U} \tag{3.6}$$

From (3.4) and (3.6) we have

$$\frac{1-|z|^{2a}}{a} \left| \frac{zg''(z)}{g'(z)} \right| \leq \frac{(1-|z|^{2a})|z|}{a} \cdot \frac{(2a+1)^{\frac{2a+1}{2a}}}{2} \tag{3.7}$$

for all $z \in \mathcal{U}$.

Because $\max\limits_{|z|\leq 1} \frac{(1-|z|^{2a})|z|}{a} = \frac{2}{(2a+1)^{\frac{2a+1}{2a}}}$, from (3.7) we have

$$\frac{1-|z|^{2a}}{a}\left|\frac{zg''(z)}{g'(z)}\right| \leq 1 \tag{3.8}$$

for all $z \in \mathcal{U}$. So, by the Lemma 2.1, the integral operator $H_{\gamma_1,\gamma_2,\ldots,\gamma_{[\mathrm{Re}\eta]},\beta,\eta}$ belongs to class $\mathcal{S}$.

Corollary 3.2. *Let γ be a complex number, $a = Re\ \frac{1}{\gamma} > 0$ and $f \in \mathcal{A}, f(z) = z + b_{21}z^2 + b_{31}z^3 + \ldots$.*

If

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{(2a+1)^{\frac{2a+1}{2a}}}{2}|\gamma| \tag{3.9}$$

for all $z \in \mathcal{U}$, then the integral operator M_γ define by (3.9) belongs to the class $\mathcal{S}$.

Proof. We take $\eta = 1, \beta = \frac{1}{\gamma}, \gamma_1 = \gamma, f_1 = f$ in Theorem 3.1.

Corollary 3.3. *Let α, η complex numbers $a = Re\ \frac{1}{\alpha} \in (0,1]$, $\mathrm{Re}\eta \geq 1$ and $f_j \in \mathcal{A},\ \ f_j(z) = z + b_{2j}z^2 + \ldots,\ \ j = \overline{1,[\mathrm{Re}\eta]}$.*

If

$$\left|\frac{zf_j'(z)}{f_j(z)} - 1\right| \leq \frac{(2a+1)^{\frac{2a+1}{2a}}}{2\,[\mathrm{Re}\eta]}|\alpha|,\ \ j = \overline{1,[\mathrm{Re}\eta]} \tag{3.10}$$

for all $z \in \mathcal{U}$, then the function

$$L_{\alpha,\eta}(z) = \int_0^z \left(\frac{f_1(u)}{u}\right)^{\frac{1}{\alpha}} \cdots \left(\frac{f_{[\mathrm{Re}\eta]}(u)}{u}\right)^{\frac{1}{\alpha}} du \tag{3.11}$$

is in the class $\mathcal{S}$.

Proof. For $\gamma_1 = \gamma_2 = ... = \gamma_{[\mathrm{Re}\eta]} = \alpha$ and $\eta\beta = 1$ in Theorem 3.1. we have the Corollary 3.3.

Remark. For $[\mathrm{Re}\eta] = 1$, $f_1 = f$, $f \in \mathcal{A}$, $a = \mathrm{Re}\frac{1}{\alpha} \in (0,1]$ from Corollary 3.3, we obtain that the function $G_\alpha(z)$ is in the class $\mathcal{S}$.

Theorem 3.5. *Let γ_j, η complex numbers, $[\mathrm{Re}\eta] \geq 1$. $j = \overline{1,[\mathrm{Re}\eta]}$, $a = \sum\limits_{j=1}^{[\mathrm{Re}\eta]} Re\ \frac{1}{\gamma_j} \geq \frac{1}{2}$ and $f_j \in \mathcal{S},\ f_j(z) = z + \sum\limits_{k=2}^{\infty} b_{kj}z^k,\ j = \overline{1,[\mathrm{Re}\eta]}$.*

If

$$\sum_{j=1}^{[\mathrm{Re}\eta]} \frac{1}{|\gamma_j|} \leq \frac{1}{4},\ \ for\ \ a \geq \frac{1}{2} \tag{3.12}$$

then for any complex number β, *Re* $\eta\beta \geq a$, *the integral operator* $H_{\gamma_1,\gamma_2,\ldots,\gamma_{[\mathrm{Re}\eta]},\beta,\eta}$ *given by (1.3) is in the class* $\mathcal{S}$.

Proof. We consider the function

$$g(z) = \int_0^z \left(\frac{f_1(u)}{u}\right)^{\frac{1}{\gamma_1}} \ldots \left(\frac{f_{[\mathrm{Re}\eta]}(u)}{u}\right)^{\frac{1}{\gamma_{[\mathrm{Re}\eta]}}} du \tag{3.13}$$

The function g is regular in $\mathcal{U}$. We have

$$\frac{1-|z|^{2a}}{a}\left|\frac{zg''(z)}{g'(z)}\right| \leq \frac{1-|z|^{2a}}{a}\sum_{j=1}^{[\mathrm{Re}\eta]}\left[\frac{1}{|\gamma_j|}\left|\frac{zf_j'(z)}{f_j(z)}-1\right|\right]. \tag{3.14}$$

Because $f_j \in \mathcal{S},\ j=\overline{1,[\mathrm{Re}\eta]}$ we have

$$\left|\frac{zf_j'(z)}{f_j(z)}\right| \leq \frac{1+|z|}{1-|z|},\quad z\in\mathcal{U},\quad j=\overline{1,[\mathrm{Re}\eta]} \tag{3.15}$$

From (3.14) and (3.15) we obtain

$$\frac{1-|z|^{2a}}{a}\left|\frac{zg''(z)}{g'(z)}\right| \leq \frac{1-|z|^{2a}}{a}\frac{2}{1-|z|}\sum_{j=1}^{[\mathrm{Re}\eta]}\frac{1}{|\gamma_j|} \tag{3.16}$$

for all $z\in\mathcal{U}$.

For $a \geq \frac{1}{2}$ we have $\max\limits_{|z|\leq 1}\frac{1-|z|}{1-|z|}^{2a} = 2a$ and from (3.12), (3.16) we obtain

$$\frac{1-|z|^{2a}}{a}\left|\frac{zg''(z)}{g'(z)}\right| \leq\ 1,\quad z\in\mathcal{U} \tag{3.17}$$

From (3.17) and Lemma 2.1 it results that the integral operator $H_{\gamma_1,\gamma_2,\ldots,\gamma_{[\mathrm{Re}\eta]},\beta,\eta}$ belongs to class $\mathcal{S}$.

Corollary 3.6. *Let* α, η *complex numbers,* $[\mathrm{Re}\eta] \geq 1$, $a = Re\ \frac{1}{\alpha} \in \left[\frac{1}{2},1\right]$ *and* $f_j \in \mathcal{S},\quad f_j(z) = z+\sum\limits_{k=2}^{\infty} b_{kj}z^k,\ j=\overline{1,[\mathrm{Re}\eta]}$.

If

$$\frac{1}{|\alpha|} \leq \frac{1}{4}, \tag{3.18}$$

then the integral operator $L_{\alpha,\eta}$ *given by (3.11) is in the class* $\mathcal{S}$.

Proof. We take $\eta\beta = 1, \gamma_1 = \gamma_2 = \ldots = \gamma_{[\mathrm{Re}\eta]} = \alpha$ in Theorem 3.5.

Corollary 3.7. *Let γ be a complex number, $a = Re\,\frac{1}{\gamma} \geq \frac{1}{2}$ and $f \in \mathcal{S}$ $\quad f(z) = z + b_{21}z^2 + b_{31}z^3 + ...$*

If

$$\frac{1}{|\gamma|} \leq \frac{1}{4}, \tag{3.19}$$

then the integral operator M_γ define by (1.2) belongs to class $\mathcal{S}$.

Proof. For $\eta = 1,\ \beta = \frac{1}{\gamma},\ \gamma_1 = \gamma,\ f_1 = f$ in Theorem 3.5 we have the Corollary 3.7.

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