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ON LACUNARY ALMOST CONVERGENT SEQUENCES

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Abstract. The purpose of this paper is to define and study the spaces $[\hat{c}, M, p]^{\theta}(\Delta_u^m, q, s)$, $[\hat{c}, M, p,]_0^{\theta}(\Delta_u^m, q, s)$ and $[\hat{c}, M, p,]_\infty^{\theta}(\Delta_u^m, q, s)$ of lacunary convergent sequences. We also study some inclusion relations between these spaces and some properties and theorems.

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1.Introduction and definitions

Let w denote the set of all complex sequences $x = (x_i)$, and l_∞, c, and c_0 be the linear spaces of bounded, convergent, and null sequences with complex terms, respectively, normed by $\| x \|= \sup_i | x_i |$, where $i \in \mathbb{N}$, the set of positive integers.

A bounded sequence $x = (x_i)$ is said to be almost convergent (see [8]) if all Banach limits of x coincide. In [8], it was shown that :

$$\hat{c} = \{x = (x_i) : \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} x_{i+t} \text{ exists, uniformly in } t\}.$$

Maddox ([9], [10]) defined a sequence $x = (x_i)$ to be strongly almost convergent to a number L if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} | x_{i+t} - L |= 0, \text{ uniformly in } t.$$

The spaces of lacunary strong convergence have been introduced by Freedman et al. [3] A sequence of positive integers $\theta = (k_r)$ is called lacunary if $k_0 = 0$, $0 < k_r < k_{r+1}$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ are denoted by $I_r = (k_{r-1}, k_r]$ and the ratio k_r/k_{r-1} will be denoted by q_r. The space of lacunary strongly convergent sequences N_θ was defined as follows :

$$N_\theta = \{x = (x_i) : \lim_{r\to\infty} h_r^{-1} \sum_{i\in I_r} | x_i - s |= 0, \text{ for some } s\}.$$

For any sequence $x = (x_i)$, the difference sequence Δx is defined by $\Delta x = (\Delta x_i)_{i=1}^{\infty} = (x_i - x_{i+1})_{i=1}^{\infty}$. Kizmaz [5] defined the sequence spaces :

$$l_\infty(\Delta) = \{x \in w : \Delta x \in l_\infty\},$$

$$c\ (\Delta) = \{x \in w : \Delta x \in c\},$$

and

$$c_0(\Delta) = \{x \in w : \Delta x \in c_0\}.$$

Et and Colak [2] generalized the notion of difference sequence spaces and defined the spaces

$$l_\infty(\Delta^m) = \{x \in w : \Delta^m x \in l_\infty\},$$

$$c\ (\Delta^m) = \{x \in w : \Delta^m x \in c\},$$

and

$$c_0(\Delta^m) = \{x \in w : \Delta^m x \in c_0\},$$

where $\Delta^m x_i = \Delta^{m-1} x_i - \Delta^{m-1} x_{i+1}$ and $\Delta^0 x_i = x_i$, for all $i \in \mathbb{N}$. We recall that an

Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, nondecreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$, as $x \to \infty$. The Orlicz function M can always be represented in the following integral form (see Krasnoselskii and Rutickii [6])

$$M(x) = \int_0^x \varphi(t)dt,$$

where φ, known as the kernel of M, is right-differentiable for $t \geq 0, \varphi(0) = 0, \varphi(t) > 0$ for $t > 0, \varphi$ is nondecreasing and $\varphi(t) \to \infty$, as $t \to \infty$.

If convexity of M is replaced by $M(x + y) \leq M(x) + M(y)$, then it is called a modulus function, defined and discussed by Ruckle [12] and Maddox [11].

An Orlicz function M is said to satisfy the Δ_2-condition for all values of v, if there exist a constant $L > 0$ such that :

$$M(2v) \leq LM(v)\ (v \geq 0).$$

The Δ_2-condition is equivalent to the satisfaction of the inequality :

$$M(Tv) \leq LTvM(v)$$

for all values of v and for all $T > 1$ (see Krasnoselskii and Rutitsky [6]).

Remark 1. An Orlicz function satisfies the inequality $M(\lambda u) \leq \lambda M(u)$ for all λ with $0 < \lambda < 1$.

Lindenstrauss and Tzafriri [7] used the idea of Orlicz function to construct the Orlicz sequence space :

$$l_M = \{x = (x_i) : \sum_{i=1}^{\infty} M(\frac{| x_i |}{\rho}) < \infty, \text{ for some } \rho > 0\},$$

which is a Banach space with the norm :

$$\| x \|_M = \inf\{\rho > 0 : \sum_{i=1}^{\infty} M(\frac{| x_i |}{\rho}) \leq 1\}.$$

If $M(x) = x^p, 1 \leq p < \infty$, the space l_M coincide with the classical sequence space l_p.

Let M be an Orlicz function, $p = (p_i)$ be any bounded sequence of strictly positive real numbers, then Güngör and Et [4] defined the sequence spaces :

$$[\hat{c}, M, p](\Delta^m) = \{x = (x_i) : \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} [M(\frac{| \Delta^m x_{i+t} - L |}{\rho})]^{p_i} = 0, \text{uniformly in } t, \text{ for some } \rho > 0 \text{ and } L > 0\},$$

$$[\hat{c}, M, p]_0(\Delta^m) = \{x = (x_i) : \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} [M(\frac{| \Delta^m x_{i+t} |}{\rho})]^{p_i} = 0, \text{uniformly in } t, \text{ for some } \rho > 0 \},$$

and

$$[\hat{c}, M, p]_\infty(\Delta^m) = \{x = (x_i) : \sup_{n,t} \frac{1}{n} \sum_{i=0}^{n} [M(\frac{| \Delta^m x_{i+t} |}{\rho})]^{p_i} < \infty, \text{ for some } \rho > 0 \}.$$

Esi [1] defined the sequence spaces :

$$[\hat{c}, M, p]^{\theta}(\Delta^m) = \{x = (x_i) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} [M(\frac{|\ \Delta^m x_{i+t} - L\ |}{\rho})]^{p_i} = 0, \text{uniformly in } t, \\ \text{for some } \rho > 0 \text{ and } L > 0\},$$

$$[\hat{c}, M, p]_0^{\theta}(\Delta^m) = \{x = (x_i) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} [M(\frac{|\ \Delta^m x_{i+t}\ |}{\rho})]^{p_i} = 0, \text{uniformly in } t, \\ \text{for some } \rho > 0\ \},$$

and

$$[\hat{c}, M, p]_{\infty}^{\theta}(\Delta^m) = \{x = (x_i) : \sup_{r,t} \frac{1}{h_r} \sum_{i\in I_r} [M(\frac{|\ \Delta^m x_{i+t}\ |}{\rho})]^{p_i} < \infty,\ \text{for some } \rho > 0\ \}.$$

Now, if $u = (u_i)$ is any sequence such that $u_i \neq 0$ for each $i, w(X)$ denotes the space of all sequences with elements in X, where (X, q) denotes a seminormed space, seminormed by q, and s is any real number such that $s \geq 0$, then we define the following sequence spaces :

$$[\hat{c}, M, p]^{\theta}(\Delta_u^m, q, s) = \{x = (x_i) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} i^{-s}[M(q(\frac{|\ \Delta_u^m x_{i+t} - L\ |}{\rho}))]^{p_i} = 0, \\ \text{uniformly in } t,\ \text{for some } \rho > 0 \text{ and } L > 0\},$$

$$[\hat{c}, M, p]_0^{\theta}(\Delta_u^m, q, s) = \{x = (x_i) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} i^{-s}[M(q(\frac{|\ \Delta_u^m x_{i+t}\ |}{\rho}))]^{p_i} = 0, \\ \text{uniformly in } t,\ \text{for} \rho > 0\ \},$$

and

$$[\hat{c}, M, p]_{\infty}^{\theta}(\Delta_u^m, q, s) = \{x = (x_i) : \sup_{r,t} \frac{1}{h_r} \sum_{i\in I_r} i^{-s}[M(q(\frac{|\ \Delta_u^m x_{i+t}\ |}{\rho}))]^{p_i} < \infty, for \rho > 0\ \}.$$

where

$$\Delta_u^0 x = u_i x_i,$$

$$\Delta_u^1 x = u_i x_i - u_{i+1} x_{i+1},$$

$$\Delta_u^2 x = \Delta(\Delta_u^1 x),$$

$$\vdots$$

$$\Delta_u^m x = \Delta(\Delta_u^{m-1} x),$$

so that

$$\Delta_u^m x = \Delta_{u_i}^m x_i = \sum_{r=0}^{m} (-1)^r \binom{m}{r} u_{i+r} x_{i+r}.$$

If $u = e = (1,1,1,\cdots)$, $q(x_i) = x_i$ for each i, and $s = 0$, then the above spaces reduce to those defined and studied by Esi [1].

If $x = (x_i) \in [\hat{c}, M, p]^\theta(\Delta_u^m, q, s)$, we say that $x = (x_i)$ is lacunary almost $(\Delta_u^m, q, s)-$convergent to L with respect to Orlicz function M.

When $M(x) = x$, then we write $[\hat{c}, p]^\theta(\Delta_u^m, q, s), [\hat{c}, p]_0^\theta(\Delta_u^m, q, s)$, and $[\hat{c}, p]_\infty^\theta(\Delta_u^m, q, s)$ for the spaces $[\hat{c}, M, p]^\theta(\Delta_u^m, q, s), [\hat{c}, M, p]_0^\theta(\Delta_u^m, q, s)$, and $[\hat{c}, M, p]_\infty^\theta(\Delta_u^m, q, s)$, respectively.

If $p_i = 1$ for each i, then $[\hat{c}, M, p]^\theta(\Delta_u^m, q, s), [\hat{c}, M, p]_0^\theta(\Delta_u^m, q, s)$, and $[\hat{c}, M, p]_\infty^\theta(\Delta_u^m, q, s)$ reduce to $[\hat{c}, M]^\theta(\Delta_u^m, q, s), [\hat{c}, M]_0^\theta(\Delta_u^m, q, s)$, and $[\hat{c}, M]_\infty^\theta(\Delta_u^m, q, s)$, respectively.

The following inequality will be used throughout this paper :

$$| x_i + y_i |^{p_i} \leq K(| x_i |^{p_i} + | y_i |^{p_i}), \tag{1}$$

where x_i and y_i are complex numbers, $K = \max(1, 2^{H-1})$ and $H = \sup_i p_i < \infty$.

2.Main results

In this section we prove the following theorems :

Theorem 1. Let M be an Orlicz function and $p = (p_i)$ be a bounded sequence of strictly positive real numbers. Then $[\hat{c}, M, p]^\theta(\Delta_u^m, q, s), [\hat{c}, M, p]_0^\theta(\Delta_u^m, q, s)$, and $[\hat{c}, M, p]_\infty^\theta(\Delta_u^m, q, s)$ are linear spaces over the set of complex numbers $\mathbb{C}$.

Proof. We will prove it for $[\hat{c}, M, p]_0^\theta(\Delta_u^m, q, s)$ and the others are similar. Let $x = (x_i)$, $y = (y_i) \in [\hat{c}, M, p]_0^\theta(\Delta_u^m, q, s)$ and $\alpha, \beta \in \mathbb{C}$. then there exists positive numbers ρ_1, ρ_2 such that

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} i^{-s}[M(q(\frac{\mid \Delta_u^m x_{i+t} \mid}{\rho_1}))]^{p_i} = 0, \text{uniformly in } t,$$

and

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} i^{-s}[M(q(\frac{\mid \Delta_u^m x_{i+t} \mid}{\rho_2}))]^{p_i} = 0, \text{uniformly in } t.$$

Let $\rho_3 = \max(2 \mid \alpha \mid \rho_1, 2 \mid \beta \mid \rho_2)$. Since M is nondecreasing and convex and using inequality (1), we see that

$$\begin{aligned}
& \frac{1}{h_r} \sum_{i\in I_r} i^{-s}[M(q(\frac{\mid \Delta_u^m(\alpha x_{i+t} + \beta y_{i+t}) \mid}{\rho_3}))]^{p_i} \\
= & \frac{1}{h_r} \sum_{i\in I_r} i^{-s}[M(q(\frac{\mid \alpha\Delta_u^m(x_{i+t}) + \beta\Delta_u^m(y_{i+t}) \mid}{\rho_3}))]^{p_i} \\
\leq & K\frac{1}{h_r} \sum_{i\in I_r} \frac{1}{2^{p_i}} i^{-s}[M(q(\frac{\mid \Delta_u^m(x_{i+t}) \mid}{\rho_1}))]^{p_i} + K\frac{1}{h_r} \sum_{i\in I_r} \frac{1}{2^{p_i}} i^{-s}[M(q(\frac{\mid \Delta_u^m(y_{i+t}) \mid}{\rho_2}))]^{p_i} \\
\leq & K\frac{1}{h_r} \sum_{i\in I_r} i^{-s}[M(q(\frac{\mid \Delta_u^m(x_{i+t}) \mid}{\rho_1}))]^{p_i} + K\frac{1}{h_r} \sum_{i\in I_r} i^{-s}[M(q(\frac{\mid \Delta_u^m(y_{i+t}) \mid}{\rho_2}))]^{p_i} \\
\to & 0 \text{ as } r \to \infty, \text{uniformly in } t.
\end{aligned}$$

This shows that $\alpha x + \beta y \in [\hat{c}, M, p]_0^\theta(\Delta_u^m, q, s)$. Hence $[\hat{c}, M, p]_0^\theta(\Delta_u^m, q, s)$ is linear.

Theorem 2. Let M be an Orlicz function and $p = (p_i)$ be a bounded sequence of strictly positive real numbers. Then $[\hat{c}, M, p]_0^\theta(\Delta_u^m, q, s)$ is a topological linear space paranormed by

$$h(x) = \inf\{\rho^{\frac{p_r}{h}} : (\frac{1}{h_r} \sum_{i\in I_r} i^{-s}[M(q(\frac{\mid \Delta_u^m(x_{i+t}) \mid}{\rho}))]^{p_i})^{\frac{1}{H}} \leq 1, r = 1, 2, \cdots, \ t = 1, 2, \cdots\},$$

where $H = \max(1, \sup_i p_i)$.

Proof. Clearly $h(x) \geq 0$ for all $x = (x_i) \in [\hat{c}, M, p]_0^\theta(\Delta_u^m, q, s)$. Since$M(0) = 0$, we see that $h(0) = 0$. Conversely, suppose that $h(x) = 0$, then

$$\inf\{\rho^{\frac{p_r}{h}} : (\frac{1}{h_r} \sum_{i\in I_r} i^{-s}[M(q(\frac{\mid \Delta_u^m(x_{i+t}) \mid}{\rho}))]^{p_i})^{\frac{1}{H}} \leq 1, r = 1, 2, \cdots, \ t = 1, 2, \cdots\} = 0.$$

This implies that for a given $\varepsilon > 0$, there exists some $\rho_\varepsilon(0 < \rho_\varepsilon < \varepsilon)$ such that

$$(\frac{1}{h_r}\sum_{i\in I_r} i^{-s}[M(q(\frac{\mid \Delta_u^m(x_{i+t})\mid}{\rho_\varepsilon}))]^{p_i})^{\frac{1}{H}} \leq 1.$$

Thus

$$(\frac{1}{h_r}\sum_{i\in I_r} i^{-s}[M(q(\frac{\mid \Delta_u^m(x_{i+t})\mid}{\varepsilon}))]^{p_i})^{\frac{1}{H}} \leq (\frac{1}{h_r}\sum_{i\in I_r} i^{-s}[M(q(\frac{\mid \Delta_u^m(x_{i+t})\mid}{\rho_\varepsilon}))]^{p_i})^{\frac{1}{H}} \leq 1$$

for each r and t. Suppose that $x_i \neq 0$ for each $i \in \mathbb{N}$. Then $\Delta_u^m(x_{i+t}) \neq 0$,for each $i, t \in \mathbb{N}$. Let $\varepsilon \to 0$, then $\frac{\Delta_u^m(x_{i+t})}{\varepsilon} \to \infty$.
This implies that $(\frac{1}{h_r}\sum_{i\in I_r} i^{-s}[M(q(\frac{|\Delta_u^m(x_{i+t})|}{\varepsilon}))]^{p_i})^{\frac{1}{H}} \to \infty$ which is contradiction. Therefore $\Delta_u^m(x_{i+t}) = 0$, for each $i, t \in \mathbb{N}$, and thus $x_i = 0$, for each $i \in \mathbb{N}$. Let $\rho_1 > 0$ and $\rho_2 > 0$ be such that :

$$(\frac{1}{h_r}\sum_{i\in I_r} i^{-s}[M(q(\frac{\mid \Delta_u^m(x_{i+t})\mid}{\rho_1}))]^{p_i})^{\frac{1}{H}} \leq 1$$

and

$$(\frac{1}{h_r}\sum_{i\in I_r} i^{-s}[M(q(\frac{\mid \Delta_u^m(x_{i+t})\mid}{\rho_2}))]^{p_i})^{\frac{1}{H}} \leq 1,$$

for each r and t. Let $\rho = \rho_1 + \rho_2$. Then we have :

$$\begin{aligned}
& (\frac{1}{h_r}\sum_{i\in I_r} i^{-s}[M(q(\frac{\mid \Delta_u^m(x_{i+t}+y_{i+t})\mid}{\rho}))]^{p_i})^{\frac{1}{H}} \\
\leq\ & (\frac{1}{h_r}\sum_{i\in I_r} i^{-s}[M(q(\frac{\mid \Delta_u^m(x_{i+t})\mid + \Delta_u^m(y_{i+t})\mid}{\rho_1+\rho_2}))]^{p_i})^{\frac{1}{H}} \\
\leq\ & (\frac{1}{h_r}\sum_{i\in I_r} i^{-s}[\frac{\rho_1}{\rho_1+\rho_2}[M(q(\frac{\mid \Delta_u^m(x_{i+t})\mid}{\rho_1})) + \frac{\rho_2}{\rho_1+\rho_2}M(q(\frac{\mid \Delta_u^m(y_{i+t})\mid}{\rho_1}))]]^{p_i})^{\frac{1}{H}} \\
\leq\ & \frac{\rho_1}{\rho_1+\rho_2}(\frac{1}{h_r}\sum_{i\in I_r} i^{-s}[M(q(\frac{\mid \Delta_u^m(x_{i+t})\mid}{\rho_1}))]^{p_i})^{\frac{1}{H}} \\
& + \frac{\rho_2}{\rho_1+\rho_2}(\frac{1}{h_r}\sum_{i\in I_r} i^{-s}[M(q(\frac{\mid \Delta_u^m(y_{i+t})\mid}{\rho_1}))]^{p_i})^{\frac{1}{H}}, \text{ using Minkowski's inequality} \\
\leq\ & 1.
\end{aligned}$$

Since the ρ's are nonnegative, so we get that :

$$\begin{aligned} h(x+y) &= \inf\{\rho^{\frac{p_r}{h}} : (\frac{1}{h_r}\sum_{i\in I_r} i^{-s}[M(q(\frac{\mid \Delta_u^m(x_{i+t}+y_{i+t})\mid}{\rho}))]^{p_i})^{\frac{1}{H}} \leq 1, \\ r &= 1,2,\cdots,\ t=1,2,\cdots\}, \\ &\leq \inf\{\rho_1^{\frac{p_r}{h}} : (\frac{1}{h_r}\sum_{i\in I_r} i^{-s}[M(q(\frac{\mid \Delta_u^m(x_{i+t})\mid}{\rho_1}))]^{p_i})^{\frac{1}{H}} \leq 1, \\ r &= 1,2,\cdots,\ t=1,2,\cdots\} \\ +\inf\{\rho_2^{\frac{p_r}{h}} &: (\frac{1}{h_r}\sum_{i\in I_r} i^{-s}[M(q(\frac{\mid \Delta_u^m(y_{i+t})\mid}{\rho_2}))]^{p_i})^{\frac{1}{H}} \leq 1, r=1,2,\cdots,\ t=1,2,\cdots\}. \end{aligned}$$

Therefore $h(x+y) \leq h(x)+h(y)$. Finally, we prove that the scalar multiplication is continuous. let λ be any complex number. Then by definition :

$$h(\lambda x) = \inf\{\rho^{\frac{p_r}{h}} : (\frac{1}{h_r}\sum_{i\in I_r} i^{-s}[M(q(\frac{\mid \Delta_u^m(x_{i+t}+y_{i+t})\mid}{\rho}))]^{p_i})^{\frac{1}{H}} \leq 1, r,t=1,2,\cdots\}.$$

Then

$$h(\lambda x) = \inf\{(\mid \lambda \mid k)^{\frac{p_r}{h}} : (\frac{1}{h_r}\sum_{i\in I_r} i^{-s}[M(q(\frac{\mid \Delta_u^m(x_{i+t})\mid}{k}))]^{p_i})^{\frac{1}{H}} \leq 1, r,t=1,2,\cdots\},$$

where $k = \frac{\rho}{|\lambda|}$. Since $\mid \lambda \mid^{p_r} \leq \max(1, \mid \lambda \mid^{\sup p_r})$, we have :

$$\begin{aligned} h(\lambda x) &\leq \max(1,\mid \lambda \mid^{\sup p_r}) \inf\{(k)^{\frac{p_r}{h}} : (\frac{1}{h_r}\sum_{i\in I_r} i^{-s}[M(q(\frac{\mid \Delta_u^m(x_{i+t})\mid}{k}))]^{p_i})^{\frac{1}{H}} \leq 1, \\ r &= 1,2,\cdots,\ t=1,2,\cdots\}. \end{aligned}$$

Then the scalar multiplication is continuous follows from the above inequality.

Theorem 3. Let M be an Orlicz function. If $\sup_i [M(x)]^{p_i} < \infty$, for all fixed $x > 0$, then

$$[\hat{c}, M, p]_0^{\theta}(\Delta_u^m, q, s) \subset [\hat{c}, M, p]_{\infty}^{\theta}(\Delta_u^m, q, s).$$

Proof. Let $x = (x_i) \in [\hat{c}, M, p]_0^{\theta}(\Delta_u^m, q, s)$. The there exists some ρ_1 such that :

$$\lim_{r\to\infty} \frac{1}{h_r}\sum_{i\in I_r} i^{-s}[M(q(\frac{\mid \Delta_u^m(x_{i+t})\mid}{\rho_1}))]^{p_i} = 0, \text{ uniformly in } t.$$

Define $\rho = 2\rho_1$. Since M is nondecreasing and convex and using Inequality (1), we get that :

$$\begin{aligned}
& \sup_{r,t} \frac{1}{h_r} \sum_{i\in I_r} i^{-s}[M(q(\frac{\mid \Delta_u^m(x_{i+t}) \mid}{\rho}))]^{p_i} \\
= & \sup_{r,t} \frac{1}{h_r} \sum_{i\in I_r} i^{-s}[M(q(\frac{\mid \Delta_u^m(x_{i+t}) - L + L \mid}{\rho}))]^{p_i} \\
\leq & K \sup_{r,t} \frac{1}{h_r} \sum_{i\in I_r} i^{-s}\frac{1}{2^{p_i}}[M(q(\frac{\mid \Delta_u^m(x_{i+t}) - L \mid}{\rho_1}))]^{p_i} \\
& + K \sup_{r,t} \frac{1}{h_r} \sum_{i\in I_r} i^{-s}\frac{1}{2^{p_i}}[M(q(\frac{\mid L \mid}{\rho_1}))]^{p_i} \\
\leq & K \sup_{r,t} \frac{1}{h_r} \sum_{i\in I_r} i^{-s}[M(q(\frac{\mid \Delta_u^m(x_{i+t}) - L \mid}{\rho_1}))]^{p_i} \\
& + K \sup_{r,t} \frac{1}{h_r} \sum_{i\in I_r} i^{-s}[M(q(\frac{\mid L \mid}{\rho_1}))]^{p_i} \\
< & \infty.
\end{aligned}$$

Hence $x = (x_i) \in [\hat{c}, M, p]^{\theta}_{\infty}(\Delta_u^m, q, s)$.

Theorem 4. Let $0 < \inf p_i = h \leq \sup p_i = H < \infty$ and M, M_1 be Orlicz functions satisfying Δ_2-condition, then $[\hat{c}, M_1, p]^{\theta}_0(\Delta_u^m, q, s) \subset [\hat{c}, M \circ M_1, p]^{\theta}_0(\Delta_u^m, q, s)$, $[\hat{c}, M_1, p]^{\theta}(\Delta_u^m, q, s) \subset [\hat{c}, M \circ M_1, p]^{\theta}(\Delta_u^m, q, s)$ and $[\hat{c}, M_1, p]^{\theta}_{\infty}(\Delta_u^m, q, s) \subset [\hat{c}, M \circ M_1, p]^{\theta}_{\infty}(\Delta_u^m, q, s)$.

Proof. Let $x = (x_i) \in [\hat{c}, M_1, p]^{\theta}_0(\Delta_u^m, q, s)$. The we have :

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} i^{-s}[M_1(q(\frac{\mid \Delta_u^m(x_{i+t}) - L \mid}{\rho}))]^{p_i} = 0, \text{ uniformly in } t, \text{ for some } L.$$

Let $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M(t) < \varepsilon$ for $0 \leq t < \delta$. Let $y_{i,t} = M_1(q(\frac{|\Delta_u^m(x_{i+t}) - L|}{\rho}))$ for all $i, t \in \mathbb{N}$. We can write :

$$\frac{1}{h_r} \sum_{i\in I_r} i^{-s}[M(y_{i,t})]^{p_i} = \frac{1}{h_r} \sum_{\substack{i\in I_r \\ y_{i,t} \leq \delta}} i^{-s}[M(y_{i,t})]^{p_i} + \frac{1}{h_r} \sum_{\substack{i\in I_r \\ y_{i,t} > \delta}} i^{-s}[M(y_{i,t})]^{p_i}$$

By Remark 1, we get that :

$$\frac{1}{h_r} \sum_{\substack{i\in I_r \\ y_{i,t} \leq \delta}} i^{-s}[M(y_{i,t})]^{p_i} \leq [M(1)]^H \frac{1}{h_r} \sum_{\substack{i\in I_r \\ y_{i,t} \leq \delta}} i^{-s}[M(y_{i,t})]^{p_i}$$

$$\leq [M(2)]^H \frac{1}{h_r} \sum_{\substack{i\in I_r \\ y_{i,t}\leq\delta}} i^{-s}[M(y_{i,t})]^{p_i}. \tag{2}$$

For $y_{i,t} > \delta$,

$$y_{i,t} < \frac{y_{i,t}}{\delta} < 1 + \frac{y_{i,t}}{\delta}.$$

Since M is nondecreasing and convex, it follows that :

$$M(y_{i,t}) < M(1 + \frac{y_{i,t}}{\delta}) < \frac{1}{2}M(2) + \frac{1}{2}M(\frac{2y_{i,t}}{\delta}).$$

Since M sastisfies Δ_2-condition, we can write

$$M(y_{i,t}) < \frac{1}{2}T\frac{y_{i,t}}{\delta}M(2) + \frac{1}{2}T\frac{y_{i,t}}{\delta}M(2) = T\frac{y_{i,t}}{\delta}M(2).$$

Hence

$$\frac{1}{h_r}\sum_{\substack{i\in I_r \\ y_{i,t}>\delta}} i^{-s}[M(y_{i,t})]^{p_i} \leq \max(1, \frac{TM(2)}{\delta})\frac{1}{h_r}\sum_{\substack{i\in I_r \\ y_{i,t}>\delta}} i^{-s}[M(y_{i,t})]^{p_i}. \tag{3}$$

By (2) and (3), we see that $x \in [\hat{c}, M \circ M_1, p]_0^\theta(\Delta_u^m, q, s)$.

Following similar arguments we can prove the others.

Theorem 5. Let M be an Orlicz function, then the following are equivalent :

(a) $[\hat{c}, p]_\infty^\theta(\Delta_u^m, q, s) \subset [\hat{c}, M, p]_\infty^\theta(\Delta_u^m, q, s)$,

(b) $[\hat{c}, p]_0^\theta(\Delta_u^m, q, s) \subset [\hat{c}, M, p]_\infty^\theta(\Delta_u^m, q, s)$,

(c) $\sup_r \frac{1}{h_r}\sum_{i\in I_r} i^{-s}[M(q(\frac{l}{\rho}))]^{p_i} < \infty$ $(l, \rho > 0)$.

Proof. (a)$\Rightarrow$(b) : It is obvious since $[\hat{c}, p]_0^\theta(\Delta_u^m, q, s) \subset [\hat{c}, p]_\infty^\theta(\Delta_u^m, q, s)$.

(b)$\Rightarrow$(c) : Let $[\hat{c}, p]_0^\theta(\Delta_u^m, q, s) \subset [\hat{c}, M, p]_0^\theta(\Delta_u^m, q, s)$. Suppose that (c) does not hold. Then for some $l, \rho > 0$,

$$\sup_r \frac{1}{h_r}\sum_{i\in I_r} i^{-s}[M(q(\frac{l}{\rho}))]^{p_i} = \infty$$

and therefore we can find a subinterval $I_{r(j)}$ of the set of intervals I_r such that

$$\frac{1}{h_{r(j)}}\sum_{i\in I_{r(j)}} i^{-s}[M(q(\frac{j^{-1}}{\rho}))]^{p_i} > j, \;\; j = 1, 2, \cdots \tag{4}$$

Define the sequence $x = (x_i)$ by

$$\Delta_u^m(x_{i+t}) = \begin{cases} j^{-1}, i \in I_{r(j)} \\ 0, i \notin I_{r(j)} \end{cases}$$

for all $t \in \mathbb{N}$. Then $x = (x_i) \in [\hat{c}, p]_0^\theta(\Delta_u^m, q, s)$ but (4) implies that $x = (x_i) \notin [\hat{c}, M, p]_\infty^\theta(\Delta_u^m, q, s)$ which contradicts (b). Hence (c) must hold.

(c)⇒(a) : Let (c) hold and $x = (x_i) \in [\hat{c}, p]_\infty^\theta(\Delta_u^m, q, s)$. Suppose that $x = (x_i) \notin [\hat{c}, M, p]_\infty^\theta(\Delta_u^m, q, s)$. Then

$$\sup_{r,t} \frac{1}{h_r} \sum_{i \in I_r} i^{-s}[M(q(\frac{\mid \Delta_u^m(x_{i+t}) \mid}{\rho}))]^{p_i} = \infty. \tag{5}$$

Let $l = \mid \Delta_u^m(x_{i+t}) \mid$, for each i and fixed t, then using (5), we get that :

$$\sup_r \frac{1}{h_r} \sum_{i \in I_r} i^{-s}[M(q(\frac{l}{\rho}))]^{p_i} = \infty$$

which contradicts (c). Hence (a) must hold.

Theorem 6. Let $1 \leq p_i \leq \sup p_i < \infty$ and M be an Orlicz function, then the following are equivalent :

(a) $[\hat{c}, M, p]_0^\theta(\Delta_u^m, q, s) \subset [\hat{c}, p]_0^\theta(\Delta_u^m, q, s)$,
(b) $[\hat{c}, M, p]_0^\theta(\Delta_u^m, q, s) \subset [\hat{c}, p]_\infty^\theta(\Delta_u^m, q, s)$,
(c) $\inf_r \frac{1}{h_r} \sum_{i \in I_r} i^{-s}[M(q(\frac{l}{\rho}))]^{p_i} > 0 \ (l, \rho > 0)$.

Proof. (a)⇒(b) : It is obvious.

(b)⇒(c) : Let (b) hold. Suppose that (c) does not hold. Then for some $l, \rho > 0$,

$$\inf_r \frac{1}{h_r} \sum_{i \in I_r} i^{-s}[M(q(\frac{l}{\rho}))]^{p_i} = 0$$

and therefore we can find a subinterval $I_{r(j)}$ of the set of intervals I_r such that

$$\frac{1}{h_{r(j)}} \sum_{i \in I_{r(j)}} i^{-s}[M(q(\frac{j^{-1}}{\rho}))]^{p_i} < j^{-1}, \ j = 1, 2, \cdots \tag{6}$$

Define the sequence $x = (x_i)$ by

$$\Delta_u^m(x_{i+t}) = \begin{cases} j, i \in I_{r(j)} \\ 0, i \notin I_{r(j)} \end{cases}$$

for all $t \in \mathbb{N}$. Then $x = (x_i) \in [\hat{c}, M, p]_0^\theta(\Delta_u^m, q, s)$ but (6) implies that $x = (x_i) \notin [\hat{c}, p]_\infty^\theta(\Delta_u^m, q, s)$ which contradicts (b). Hence (c) must hold.(c)⇒(a) : Let (c) hold and $x = (x_i) \in [\hat{c}, M, p]_0^\theta(\Delta_u^m, q, s)$. Then

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{i \in I_r} i^{-s}[M(q(\frac{\mid \Delta_u^m(x_{i+t}) \mid}{\rho}))]^{p_i} = 0, \text{ uniformly in } t, \text{ for some } \rho > 0. \tag{7}$$

Now, suppose that $x = (x_i) \notin [\hat{c}, p]_0^\theta(\Delta_u^m, q, s)$.Then for some $\varepsilon > 0$ and a subinterval $I_{r(j)}$ of the set of intervals I_r, we have $| \Delta_u^m(x_{i+t}) | \geq \varepsilon$ for each i and some $t \geq t_0$. Then from the properties of Orlicz functions we can write

$$[M(q(\frac{| \Delta_u^m(x_{i+t}) |}{\rho}))]^{p_i} \geq [M(q(\frac{\varepsilon}{\rho}))]^{p_i}$$

and consequently by (7) we see that :

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} i^{-s}[M(q(\frac{\varepsilon}{\rho}))]^{p_i} = 0,$$

which contradicts (c). Hence (a) must hold.

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