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COMMON FIXED POINT THEOREMS FOR OCCASIONALLY WEAKLY COMPATIBLE MAPPINGS IN COMPACT METRIC SPACES

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Abstract. We prove common fixed point theorems for four mappings in compact metric spaces satisfying implicit relations using the concept of occasionally weak compatibility which generalize Theorems of [1], [18] and [19].

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1. Introduction

Let S and T be self-mappings of a metric space (X, d). S and T are commuting if $STx = TSx$ for all $x \in X$. Sessa [22] defined S and T to be weakly commuting if for all $x \in X$

$$d(STx, TSx) \leq d(Tx, Sx)$$

Jungck [6] defined S and T to be compatible as a generalization of weakly commuting if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.
It is easy to show that commuting implies weakly commuting implies compatible and there are examples in the literature verifying that the inclusions are proper, see [6] and [22].
Jungck et.al [7] defined S and T to be compatible mappings of type (A) if

$$\lim_{n\to\infty} d(STx_n, T^2x_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(TSx_n, S^2x_n) = 0.$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Examples are given to show that the two concepts of compatibility are independent, see [7].

Recently, Pathak and Khan [13] defined S and T to be compatible mappings of type (B) as a generalization of compatible mappings of type (A) if

$$\lim_{n\to\infty} d(TSx_n, S^2x_n) \leq \frac{1}{2}[\lim_{n\to\infty} d(TSx_n, Tt) + \lim_{n\to\infty} d(Tt, T^2x_n)] \text{ and}$$
$$\lim_{n\to\infty} d(STx_n, T^2x_n) \leq \frac{1}{2}[\lim_{n\to\infty} d(STx_n, St) + \lim_{n\to\infty} d(St, S^2x_n)]$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Clearly, compatible mappings of type (A) are compatible mappings of type (B), but the converse is not true, see [13]. However, compatibility, compatibility of type (A) and compatibility of type (B) are equivalent if S and T are continuous, see [13]. Pathak et al [14] defined S and T to be compatible mappings of type (P) if

$$\lim_{n\to\infty} d(S^2x_n, T^2x_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

However, compatibility, compatibility of type (A) and compatibility of type (P) are equivalent if S and T are continuous, see [14]. Pathak et al [15] defined S and T to be compatible mappings of type (C) as a generalization of compatible mappings of type (A) if

$$\lim_{n\to\infty} d(TSx_n, S^2x_n) \leq \frac{1}{3}[\lim_{n\to\infty} d(TSx_n, Tt) + \lim_{n\to\infty} d(Tt, S^2x_n) + \lim_{n\to\infty} d(Tt, T^2x_n)] \text{ and}$$
$$\lim_{n\to\infty} d(STx_n, T^2x_n) \leq \frac{1}{3}[\lim_{n\to\infty} d(STx_n, St) + \lim_{n\to\infty} d(St, T^2x_n) + \lim_{n\to\infty} d(St, S^2x_n)]$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Compatibility, compatibility of type (A) and compatibility of type (C) are equivalent if S and T are continuous, see [15].

2. Preliminaries

Let A and S be self-mappings of a metric space (X, d) and $C(A, S)$ the set of coincidence points of A and S.

Definition 2.1 [8]. A and S are said to be weakly compatible if $SAu = ASu$ for all $u \in C(A, S)$.

Lemma 2.2. [6, 7, 13, 14, 15]. If A and S are compatible, or compatible of type (A), or compatible of type (P), or compatible of type (B), or compatible of type (C), then they are weakly compatible.

The converse is not true in general, see [1].

Definition 2.3 [11]. A and S are said to be $R-$weakly commuting if there exists $R > 0$ such that

$$d(SAx, ASx) \leq Rd(Ax, Sx) \text{ for all } x \in X. \tag{2.1}$$

Definition 2.4 [12]. A and S are said to be pointwise $R-$weakly commuting if for all $x \in X$, there exists an $R > 0$ such that (2.1) holds.

It was proved in [12] that $R-$weak commutativity is equivalent to commutativity at coincidence points; i.e., A and S are pointwise $R-$weakly commuting if and only if they are weakly compatible.

Definition 2.5 [3]. A and S are said to be occasionally weakly compatible if $SAu = ASu$ for some $u \in C(A, S)$.

Remark 2.6 [3]. If A and S are weakly compatible, then they are occasionally weakly compatible, but the following example shows that the converse is not true in general.

Example 2.7. Let $X = [1, \infty)$ with the usual metric. Define A, $S : X \to X$ by: $Ax = 3x - 2$ and $Sx = x^2$. We have $Ax = Sx$ iff $x = 1$ or $x = 2$ and $AS(1) = SA(1) = 1$, but $AS(2) \neq SA(2)$. Therefore, A and S are occasionally weakly compatible, but they are not weakly compatible.

Lemma 2.8 [9]. If A and S have a unique coincidence point $w = Ax = Sx$, then w is the unique common fixed point of A and S.

In [18], a general common fixed point theorem for four mappings in a compact metric space was proved and this theorem was generalized by [1].

An altering distance is a mapping $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ which satisfies:

(ϕ_1) : Φ is increasing and continuous,

(ϕ_2) : $\Phi(t) = 0$ if and only if $t = 0$.

In [10], [20] and [21] fixed points theorems involving an altering distance have been introduced.

In [19], a fixed point theorem for weakly compatible mappings in compact metric spaces was proved which extend main results of [4] and [20].

Theorem 2.9 [19]. *Let f, g, S and T be self-mappings of a compact metric space (X, d) such that*

(a) $f(X) \subset T(X)$ and $g(X) \subset S(X)$.

(b) *The pair (f, S) is compatible or compatible of type (A) or compatible of type (P) and the pair (g, T) is weakly compatible.*

(c) *f and S are continuous.*
(d)

$$\begin{aligned}\Psi(d(fx,gy)) &\leq a(\Psi(d(fx,Sx))+\Psi(d(gy,Ty)))+b(\Psi(d(Sx,Ty))+\\ &\quad c(\Psi(d(Sx,gy)\cdot\Psi(d(fx,Ty)))^{\frac{1}{2}}\end{aligned}$$

for all $x,y\in X$, $a,b,c\geq 0$, $2a+b<1$, $b+c<1$ *and* Ψ *is an altering distance. Then,* f,g,S *and* T *have a unique common fixed point in* X.

In [16] and [17], the study of fixed points for mappings satisfying an implicit relation was initiated.

It is our purpose in this paper to extend Theorem 2.9 and Theorem 2 of [1] for occasionally weakly compatible mappings satisfying implicit relations in compact metric spaces without decreasing assumption, see [1] and [2].

3. Implicit relations

Let F_6 the family of functions $F(t_1,t_2,t_3,t_4,t_5,t_6):\mathbb{R}_+^6\to\mathbb{R}$ satisfying the following conditions:

(C_1) : For all $u\geq 0, v>0$ and $w\geq 0$ with
(C_a) : $F(u,v,v,u,w,0)\leq 0$ or
(C_b) : $F(u,v,u,v,0,w)\leq 0$
we have $u<v$ and $u=0$ if $v=0$.
(C_2) : For all $u>0$, $F(u,u,0,0,u,u)>0$.

Example 3.1. $F(t_1,t_2,t_3,t_4,t_5,t_6)=t_1-bt_2-a(t_3+t_4)-c(t_5t_6)^{\frac{1}{2}}$, $a,b,c\geq 0$, $2a+b<1$ and $b+c<1$

(C_1) : Let $u,v>0$ and $w\geq 0$ and $F(u,v,v,u,w,0)=u-bv-a(u+v)\leq 0$. Then $u\leq\dfrac{a+b}{1-a}v$.

Similarly, if $F(u,v,u,v,0,w)\leq 0$ then $u<v$.

If $u=0$, $v>0$ and $w\geq 0$, then $u<v$.

If $v=0$ then $u=0$.

(C_2) : $F(u,u,0,0,u,u)=2bu>0$ for all $u>0$.

Example 3.2. $F(t_1,t_2,t_3,t_4,t_5,t_6)=t_1-h\max\{t_2,t_3,t_4\}+b(t_5+t_6)$, where $0\leq h<1$ and $b>0$.

(C_1) : Let $u,v>0$ and $w\geq 0$. We have
$F(u,v,v,u,w,0)=u-h\max\{v,u\}+bw\leq 0$.
If $v\leq u$, then $u<u$ which is a contradiction. Therefore, $u<v$. Similarly, if
$F(u,v,u,v,0,w)\leq 0$ then $u<v$.
If $u=0$, $v>0$ and $w\geq 0$, then $u<v$.

If $v = 0$ then $u = 0$.
$(C_2) : F(u, u, 0, 0, u, u) = 2bu > 0$ for all $u > 0$.

Example 3.3. $F(t_1, t_2, t_3, t_4, t_5, t_6) = (1 + pt_2)t_1 - pt_3t_4 - h\max\{t_2, t_3, t_4\} + b(t_5 + t_6)$,
$0 \leq h < 1$, $b > 0$ and $p \geq 0$.
(C_1) and (C_2) as in Example 3.2.

Example 3.4. $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^2 - at_2^2 - b\dfrac{t_3^2 + t_4^2}{t_5 + t_6 + 1}$, $0 < a, b < 1$ and $a + 2b < 1$.
(C_1) : Let $u, v > 0$, $w \geq 0$ and $F(u, v, v, u, w, 0) = u^2 - av^2 - b\dfrac{(u^2 + v^2)}{w + 1} \leq 0$.
Then, $u^2 \leq \dfrac{a + b}{1 - b}v^2 = v^2$. Hence, $u < v$. Similarly, if $F(u, v, u, v, 0, w) \leq 0$, then $u < v$.
If $u = 0$, $v > 0$ and $w \geq 0$ then $u < v$.
If $v = 0$ then $u = 0$.
(C_2) : For all $u > 0$, $F(u, u, 0, 0, u, u) = (1 - a)u^2 > 0$.

Example 3.5. $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^2 - at_2^2 - b\dfrac{t_3^2 + t_4^2}{t_5t_6 + 1}$, $0 < a, b < 1$ and $a + 2b < 1$.
(C_1) and (C_2) as in Example 3.4.

Example 3.6. $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^3 - \dfrac{t_3^2t_4^2}{t_2 + t_5 + t_6 + 1}$.
(C_1) : Let $u, v > 0$, $w \geq 0$ and $F(u, v, v, u, w, 0) = u^3 - \dfrac{u^2v^2}{v + w + 1} \leq 0$. Then $u \leq \dfrac{v^2}{v + w + 1} < v$. Similarly, if $F(u, v, u, v, 0, w) \leq 0$ then $u < v$.
If $u = 0$, $v > 0$ and $w \geq 0$ then $u < v$.
If $v = 0$ then $u = 0$.
(C_2) : $F(u, u, 0, 0, u, u) = u^3 > 0$ for all $u > 0$.

Example 3.7. $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^3 - \dfrac{t_3^2t_4^2}{t_2 + t_5t_6 + 1}$.
(C_1) and (C_2) as in Example 3.6.

Example 3.8. $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - at_2 - bt_3 - c\dfrac{t_4t_5}{t_5 + t_6 + 1}$,
$0 < a, b, c < 1$ and $a + b + c < 1$.
(C_1) : Let $u, v > 0$, $w \geq 0$ and $F(u, v, v, u, w, 0) = u - av - bv - c\dfrac{uw}{w + 1} \leq 0$.
Then, $u \leq \dfrac{a + b}{1 - c}v < v$. Similarly, if $F(u, v, u, v, 0, w) \leq 0$ then $u < v$.
If $u = 0$, $v > 0$ and $w \geq 0$ then $u < v$.
If $v = 0$ then $u = 0$.

$(C_2) : F(u,u,0,0,u,u) = (1-a)u > 0$ for all $u > 0$.

Example 3.9. $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - at_2 - b\dfrac{t_3t_6}{t_5+t_6+1} - ct_4$, $0 < a,b,c < 1$ and $a+b+c < 1$.

(C_1) and (C_2) as in Example 3.8.

Example 3.10. $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1^2 - at_2^2 - b\dfrac{\min\{t_5^2,t_6^2\}}{1+t_3+t_4}$, $0 < a, b \geq 0$ and $a+b < 1$.

Example 3.11. $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - bt_2 - a(t_3+t_4) - c\min\{t_5,t_6\}$, $a,b,c \geq 0$, $2a+b < 1$ and $b+c < 1$.

Let F_6^* the family of functions $F^*(t_1,t_2,t_3,t_4,t_5,t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:

(C_1^*) : For all $u \geq 0, v > 0$ and $w \geq 0$ with

(C_a^*) : $F^*(u,v,v,u,w,0) < 0$ or

(C_b^*) : $F^*(u,v,u,v,0,w) < 0$

we have $u < v$ and $u = 0$ if $v = 0$.

(C_2^*) : For all $u > 0$, $F^*(u,u,0,0,u,u) \geq 0$.

Example 3.12. $F^*(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - \max\{t_2,t_3,t_4\} + b(t_5+t_6)$, where $b > 0$.

Example 3.13. $F^*(t_1,t_2,t_3,t_4,t_5,t_6) = (1+pt_2)t_1 - pt_3t_4 - \max\{t_2,t_3,t_4\} + b(t_5+t_6)$, $b > 0$ and $p \geq 0$.

Example 3.14. $F^*(t_1,t_2,t_3,t_4,t_5,t_6) = t_1^2 - at_2^2 - b\dfrac{t_3^2+t_4^2}{t_5+t_6+1}$, $0 < a,b < 1$ and $a+2b = 1$.

Example 3.15. $F^*(t_1,t_2,t_3,t_4,t_5,t_6) = t_1^2 - at_2^2 - b\dfrac{t_3^2+t_4^2}{t_5t_6+1}$, $0 < a,b < 1$ and $a+2b = 1$.

Example 3.16. $F^*(t_1,t_2,t_3,t_4,t_5,t_6) = t_1^3 - \dfrac{t_3^2t_4^2}{t_2+t_5+t_6+1}$.

Example 3.17. $F^*(t_1,t_2,t_3,t_4,t_5,t_6) = t_1^3 - \dfrac{t_3^2t_4^2}{t_2+t_5t_6+1}$.

Example 3.18. $F^*(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - at_2 - bt_3 - c\dfrac{t_4t_5}{t_5+t_6+1}$,

$0 < a,b,c < 1$ and $a+b+c = 1$.

Example 3.19. $F^*(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - at_2 - b\dfrac{t_3t_6}{t_5+t_6+1} - ct_4$, $0 < a,b,c < 1$

and $a+b+c = 1$.

Example 3.20. $F^*(t_1,t_2,t_3,t_4,t_5,t_6) = t_1^2 - at_2^2 - b\dfrac{\min\{t_5^2,t_6^2\}}{1+t_3+t_4}$, $0 < a,b \geq 0$ and $a+b < 1$.

Example 3.21. $F^*(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - bt_2 - a(t_3+t_4) - c\min\{t_5,t_6\}$, $a,b,c \geq 0$, $2a+b=1$ and $b+c \leq 1$.

Example 3.22. $F^*(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - bt_2 - a(t_3+t_4) - c(t_5t_6)^{\frac{1}{2}}$, $a,b,c \geq 0$, $2a+b=1$ and $b+c \leq 1$.

4. Main Results

A weakly altering distance is a mapping $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ which satisfies:

Φ is increasing and $\Phi(t) = 0$ if and only if $t = 0$.

Theorem 4.1. *Let f, g, S and T be self-mappings of a compact metric space (X,d) satisfying the following conditions:*

$$f(X) \subset T(X) \text{ and } g(X) \subset S(X) \tag{4.1}$$

$$\begin{aligned} &F(\Psi(d(fx,gy)),\Psi(d(Sx,Ty)),\Psi(d(fx,Sx)),\\ &\Psi(d(gy,Ty)),\Psi(d(Sx,gy)),\Psi(d(fx,Ty))) \leq 0 \end{aligned} \tag{1}$$

for all $x, y \in X$, $F \in F_6$ and Ψ is a weakly altering distance. Assume that f and S are continuous *and the pairs (f,S) and (g,T) are occasionally weakly compatible. Then, f, g, S and T have a unique common fixed point in X.*

Let $m = \inf\{d(fx,Sx), x \in X\}$.Since X is a compact metric space, there is a convergent sequence $\{x_n\}$ with limit x_0 in X such that $\lim_{n\to\infty} d(fx_n,Sx_n) = m$. As $d(fx_0,Sx_0) \leq d(fx_0,fx_n) + d(fx_n,Sx_n) + d(Sx_n,Sx_0)$. By the continuity of f and S and $\lim_{n\to\infty} x_n = x_0$, we get $d(fx_0,Sx_0) \leq m$ and so $d(fx_0,Sx_0) = m$. Since $f(X) \subset T(X)$, there exists $v \in X$ such that $fx_0 = Tv$ and $d(Sx_0,Tv) = m$. Suppose that $m > 0$. Using (4.2) we have

$$\begin{aligned} &F(d(\Psi(fx_0,gv)),\Psi(d(Sx_0,Tv)),\Psi(d(fx_0,Sx_0)),\\ &\Psi(d(gv,Tv)),\Psi(d(Sx_0,gv)),\Psi(d(fx_0,Tv)))\\ = \ &F(\Psi(d(gv,Tv)),\Psi(m),\Psi(m),\Psi(d(gv,Tv)),\Psi(d(Sx_0,gv)),0) \leq 0. \end{aligned}$$

By (C_a) we get $\Psi(d(gv,Tv)) < \Psi(m)$. Since $g(X) \subset S(X)$, there exists $u \in X$ such that $Su = gv$ and so $\Psi(d(Su,Tv)) < \psi(m)$. Since $d(fu,Su) \geq m > 0$. Applying (4.2) we get

$$\begin{aligned} &F(\Psi(d(fu,gv)),\Psi(d(Su,Tv)),\Psi(d(fu,Su)),\\ &\Psi(d(gv,Tv)),\Psi(d(Su,gv)),\Psi(d(fu,Tv)))\\ = \ &F(\Psi(d(fu,Su)),\Psi(d(gv,Tv)),\Psi(d(fu,Su)),\\ &\Psi(d(gv,Tv)),0,\Psi(d(fu,Tv))) \leq 0. \end{aligned}$$

If $\Psi(d(gv,Tv)) = 0$ then $\Psi(d(fu,Su)) = 0$ and so $fu = Su$ which is a contradiction. Therefore, $\Psi(d(gv,Tv)) > 0$ and by (C_b) we get

$$\begin{aligned}\Psi(m) &\leq \Psi(d(fu,Su)) \\ &< \Psi(d(gv,Tv)) < \Psi(m).\end{aligned}$$

which is a contradiction and so $m = 0$ which implies that $fx_0 = Sx_0 = Tv$. On the other hand, using (4.2) we obtain

$$\begin{aligned}&F(\Psi(d(fx_0,gv)),\Psi(d(Sx_0,Tv)),\Psi(d(fx_0,Sx_0)),\\ &\Psi(d(gv,Tv)),\Psi(d(Sx_0,gv)),\Psi(d(fx_0,Tv)))\\ = \;&F(\Psi(d(gv,Tv)),0,0,\Psi(d(gv,Tv)),\Psi(d(gv,Tv)),0) \leq 0\end{aligned}$$

which is a contradiction of (C_a). Therefore, $z = fx_0 = Sx_0 = gv = Tv$. Hence x_0 is a coincidence point of f and S and v is a coincidence point of g and T. If there is a point x_1 such that $fx_1 = Sx_1$, using (4.2) we have

$$\begin{aligned}&F(\Psi(d(fx_1,gv)),\Psi(d(Sx_1,Tv)),\Psi(d(fx_1,Sx_1)),\\ &\Psi(d(gv,Tv)),\Psi(d(Sx_1,gv)),\Psi(d(fx_1,Tv)))\\ = \;&F(\Psi(d(gv,Tv)),\Psi(d(gv,Tv)),0,0,\Psi(d(gv,Tv)),\Psi(d(gv,Tv))) \leq 0\end{aligned}$$

which is a contradiction of (C_2). Therefore, $z = fx_1 = Sx_1$ and so z is the unique coincidence point of f and S . In a similar manner, z is the unique coincidence point of g and T . By Lemma 2.8, z is the unique common fixed point of f, g, S and T .

If $\Psi(t) = t$ in Theorem 4.1 we get the following Theorem.

Theorem 4.2. *Let f, g, S and T be self-mappings of a compact metric space (X, d) satisfying* (4.1) *and the following inequality*

$$F(d(fx,gy),d(Sx,Ty),d(fx,Sx),d(gy,Ty),d(Sx,gy),d(fx,Ty)) \leq 0$$

for all $x, y \in X$ and $F \in F_6$. Assume that f and S are continuous *and the pairs (f, S) and (g, T) are occasionally weakly compatible. Then, f, g, S and T have a unique common fixed point in X.*

Corollary 4.3. Theorem 2.9.

Proof. It follows from Example 3.1 and the fact that weak compatibility implies occasionally weak compatibility.

Theorem 4.4. *Let f, g, S and T be self-mappings of a compact metric space (X, d) satisfying the inequality* (4.2) *for all $x, y \in X$, $F \in F_6$ and Ψ is a weakly altering distance. Then*

$$(Fix(S) \cap Fix(T)) \cap Fix(f) = (Fix(S) \cap Fix(T)) \cap Fix(g),$$

where $Fix(f) = \{x \in X : fx = x\}$.

Proof. Let $x \in (Fix(S) \cap Fix(T)) \cap Fix(f)$, then by (4.2) we have for $x = y$

$$F(\Psi(d(x, gx)), 0, 0, \Psi(d(x, gx)), \Psi(d(x, gx), 0) \leq 0.$$

By (C_a) we obtain $gx = x$ and so $Fix(S) \cap Fix(T)) \cap Fix(f) \subset (Fix(S) \cap Fix(T)) \cap Fix(g)$.

Similarly, we can prove that $Fix(S) \cap Fix(T)) \cap Fix(g) \subset (Fix(S) \cap Fix(T)) \cap Fix(f)$.

Theorems 4.1 and 4.4 imply the following one.

Theorem 4.5. *Let $\{f_i\}_{i \in \mathbb{N}^*}, S$ and T be self-mappings of a compact metric space (X, d) satisfying the following conditions:*

$$f_1(X) \subset T(X) \text{ and } f_2(X) \subset S(X),\ i \geq 1.$$

$$F(\Psi(d(f_ix, f_{i+1}y)), \Psi(d(Sx, Ty)), \Psi(d(f_ix, Sx)),$$
$$\Psi(d(f_{i+1}y, Ty)), \Psi(d(Sx, f_{i+1}y)), \Psi(d(f_ix, Ty))) \leq 0$$

for all $x, y \in X$, $F \in F_6$ and Ψ is a weakly altering distance. Assume that f_1 and S are continuous *and the pairs (f_1, S) and (f_2, T) are occasionally weakly compatible. Then, $\{f_i\}_{i \in \mathbb{N}^*}, S$ and T have a unique common fixed point in X.*

As in Theorem 4.1, we can prove the following Theorem.

Theorem 4.6. *Let f, g, S and T be self-mappings of a compact metric space (X, d) satisfying (4.1) and*

$$\begin{aligned} & F^*(\Psi(d(fx, gy)), \Psi(d(Sx, Ty)), \Psi(d(fx, Sx)), \\ & \Psi(d(gy, Ty)), \Psi(d(Sx, gy)), \Psi(d(fx, Ty))) \\ < \ & 0 \end{aligned}$$

for all $x, y \in X$, $F^* \in F_6^*$ and Ψ is a weakly altering distance. Assume that f and S are continuous and the pairs (f, S) and (g, T) are occasionally weakly compatible. Then, f, g, S and T have a unique common fixed point in X.

If $\Psi(t) = t$ in Theorem 4.6 we get the following Theorem which generalizes theorems of [1] and [18].

Theorem 4.7. *Let f, g, S and T be self-mappings of a compact metric space (X, d) satisfying (4.1) and the following inequality*

$$F^*(d(fx, gy), d(Sx, Ty), d(fx, Sx), d(gy, Ty), d(Sx, gy), d(fx, Ty)) < 0$$

for all $x, y \in X$ and $F^* \in F_6^*$. Assume that f and S are continuous and the pairs (f, S) and (g, T) are occasionally weakly compatible. Then, f, g, S and T have a unique common fixed point in X.

5. Applications

Let

$$\Phi = \left\{ \begin{array}{c} \varphi : \mathbb{R}_+ \to \mathbb{R}_+ \text{ such that } \varphi \text{ is a Lebesgue integral mapping} \\ \text{which is summable and satisfies} \\ \int_0^{\epsilon} \varphi(t)t > 0 \text{ for all } \epsilon > 0. \end{array} \right\},$$

see [5].

Example 5.1.

$F(t_1, t_2, t_3, t_4, t_5, t_6) = \int_0^{t_1} \varphi(t)dt - b\int_0^{t_2} \varphi(t)dt - a(\int_0^{t_3} \varphi(t)dt + \int_0^{t_4} \varphi(t)dt) - c(\int_0^{t_5} \varphi(t)dt \cdot \int_0^{t6} \varphi(t)dt)^{\frac{1}{2}}$, $a, b, c \geq 0$, $2a + b < 1$ and $b + c < 1$.

Example 5.2.

$F(t_1, t_2, t_3, t_4, t_5, t_6) = \int_0^{t_1} \varphi(t)dt - h\max\{\int_0^{t_2} \varphi(t)dt, \int_0^{t_3} \varphi(t)dt, \int_0^{t_4} \varphi(t)dt\} + b(\int_0^{t_5} \varphi(t)dt + \int_0^{t_6} \varphi(t)dt)$, $0 \leq h < 1$ and $b > 0$.

Example 5.3.

$F(t_1, t_2, t_3, t_4, t_5, t_6) = \int_0^{t_1} \varphi(t)dt - b\int_0^{t_2} \varphi(t)dt - a(\int_0^{t_3} \varphi(t)dt + \int_0^{t_4} \varphi(t)dt) - c(\int_0^{t_5} \varphi(t)dt \cdot \int_0^{t6} \varphi(t)dt)^{\frac{1}{2}}$, $a, b, c \geq 0$, $2a + b = 1$ and $b + c \leq 1$.

By Theorem 4.1 and Example 5.1 and Theorem 4.6 and Example 5.3, we get the following Theorems.

Theorem 5.4. *Let f, g, S and T be self-mappings of a compact metric space (X, d) satisfying (4.1) and the following inequality*

$$\begin{aligned} \int_0^{d(fx,gy)} \varphi(t)dt \leq b \int_0^{d(Sx,Ty)} \varphi(t)dt + a(\int_0^{d(fx,Sx)} \varphi(t)dt + \int_0^{d(gy,Ty)} \varphi(t)dt) \\ +c(\int_0^{d(Sx,gy)} \varphi(t)dt \cdot \int_0^{d(fx,Ty)} \varphi(t)dt)^{\frac{1}{2}} \end{aligned}$$

for all $x, y \in X$, $a, b, c \geq 0$, $2a + b < 1$, $b + c < 1$ and *$\varphi \in \Phi$. Assume that f and S* are continuous *and the pairs (f, S) and (g, T) are occasionally weakly compatible. Then, f, g, S and T have a unique common fixed point in X.*

Theorem 5.5. *Let f, g, S and T be self-mappings of a compact metric space (X, d) satisfying* (4.1) *and the following inequality*

$$\int_0^{d(fx,gy)} \varphi(t)dt < b \int_0^{d(Sx,Ty)} \varphi(t)dt + a(\int_0^{d(fx,Sx)} \varphi(t)dt + \int_0^{d(gy,Ty)} \varphi(t)dt)$$
$$+c(\int_0^{d(Sx,gy)} \varphi(t)dt \cdot \int_0^{d(fx,Ty)} \varphi(t)dt)^{\frac{1}{2}}$$

for all $x, y \in X$, $a, b, c \geq 0$, $2a + b = 1$, $b + c \leq 1$ and $\varphi \in \Phi$. Assume that f and S are continuous *and the pairs (f, S) and (g, T) are occasionally weakly compatible. Then, f, g, S and T have a unique common fixed point in X.*

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