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FIXED POINT RESULTS IN GENERALIZED CONE METRIC SPACES

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Abstract. Some fixed point theorems are obtained in the set up of generalized cone metric spaces. These results generalize several well known comparable results in the literature.

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1. Introduction and Preliminaries

To overcome fundamental flaws in Dhage's [5] theory of generalized metric spaces, Mustafa and Sims [13] introduced a more appropriate generalization of metric spaces, that of $G-$ metric spaces. Afterwards, Mustafa *et al.* [14, 15] obtained fixed point for mappings satisfying different contractive conditions in $G-$ metric spaces. Recently Abbas and Rhoades [2] obtained some fixed point results for noncommuting mappings without continuity in generalized metric spaces. Guang and Xian [7] generalized the concept of a metric space, replacing the set of real numbers by an ordered Banach space and obtained some fixed point theorems for mappings satisfying different contractive conditions (see also [1]). Rezapour and Hamlbarani [8, 9, 10, 11] showed the existence of a non normal cone metric space and obtained some fixed point results in cone metric spaces. Recently, Beg *et al.* [3] introduced a notion of generalized cone metric spaces, replacing the set of real numbers in the definition of $G-$ metric by an ordered Banach space (see also [4]). It is noted the concept of a generalized cone metric space is more general than that of a $G-$ metric spaces and cone metric spaces. In continuation of [3], some fixed point theorems that satisfying certain contractive conditions are obtained. It is worth mentioning that we did not use the normality property of cone to obtain our results. Our results have several consequences including generalizations/extensions of comparable results in the literature (see [5, 6, 12, 15] and the references therein).
Consistent with Beg *et al.* ([3], [4]) the following definitions and results will be needed in the sequel.
Let E be a real Banach space. A subset P of E is called a *cone* if and only if:

(a) P is closed, non empty and $P \neq \{0\}$;

(b) $a, b \in R$, $a, b \geq 0$, $x, y \in P$ imply that $ax + by \in P$;

(c) $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. A cone P is said to be normal if there is a number $K > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \text{ implies } \|x\| \leq K \|y\|.$$

The least positive number satisfying the above inequality is called the *normal constant* of P, while $x \ll y$ stands for $y - x \in intP$ (interior of P).
Rezapour [8] proved that there is no normal cones with normal constants $K < 1$ and for each $k > 1$ there are cones with normal constants $K > k$.
Definition 1.1. Let X be a nonempty set. Suppose that the mapping $G : X \times X \times X \to E$ satisfies:

(G_1) $0 \leq G(x, y, z)$ for all $x, y, z \in X$ and $G(x, y, z) = 0$ if and only if $x = y = z$,

(G_2) $0 < G(x, x, y)$ for all $x, y \in X$, with $x \neq y$,

(G_3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$, with $y \neq z$,

(G_4) $G(x, y, z) = G(x, z, y) = G(y, z, x)$ (symmetric in all three variables),

(G_5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.

Then G is called a generalized cone metric on X or $G-$ cone metric on X and (X, G) is called a $G-$ cone *metric space*.

The concept of a $G-$ cone metric space is more general than that of $G-$ metric spaces and cone metric spaces.
Definition 1.2. Let X be a $G-$ cone metric space and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is:

(a) a *Cauchy* sequence if, for every $c \in E$ with $0 \ll c$, there is N such that for all $n, m, l > N$, $G(x_n, x_m, x_l) \ll c$.

(b) a *convergent* sequence if, for every $c \in E$ with $0 \ll c$, there is N such that for all $n, m > N$, $G(x_n, x_m, x) \ll c$ for some fixed x in X. Here x is called the limit of a sequence $\{x_n\}$ and is denoted by $\lim_{n\to\infty} x_n = x$.

A $G-$ cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X.

Remarks 1.3. Let X be a $G-$ cone metric space.

(a) If $x \ll y \ll z$, then $x \ll z$.

(b) If $x \ll y \leq z$, then $x \ll z$.

(c) If $x \leq y \ll z$, then $x \ll z$.

(d) If E is a real Banach space with cone P and if $a \leq \lambda a$ where $a \in P$ and $\lambda \in [0, 1)$, then $a = 0$.

For sake of completeness, we now state some basic facts in a $G-$ cone metric space.

Lemma 1.4. Let X be a $G-$ cone metric space, then the following are equivalent.

(i) $\{x_n\}$ is converges to x.

(ii) $G(x_n, x_n, x) \ll c$, as $n \to \infty$.

(iii) $G(x_n, x, x) \ll c$, as $n \to \infty$.

(iv) $G(x_n, x_m, x) \ll c$, as $m, n \to \infty$.

Lemma 1.5. Let X be a $G-$ cone metric space. If

(i) $\{x_m\}$, $\{y_n\}$ and $\{z_l\}$ be sequences in X such that $x_m \to x$, $y_n \to y$ and $z_l \to z$, then $G(x_m, y_n, z_l) \to G(x, y, z)$ as m, n, $l \to \infty$.

(ii) $\{x_n\}$ be a sequence in X and $x \in X$, such that $\{x_n\}$ converges to x, and $\{x_n\}$ converges to y, then $x = y$.

(iii) $\{x_n\}$ be a sequence in X and $x \in X$, such that $\{x_n\}$ converges to x, then $G(x_m, x_n, x) \to 0$ as m, $n \to \infty$.

(iv) $\{x_n\}$ be a sequence in X and $x \in X$. If $\{x_n\}$ converges to $x \in X$, then $\{x_n\}$ is a Cauchy sequence.

(v) $\{x_n\}$ be a sequence in X, and $\{x_n\}$ is a Cauchy sequence in X, then $G(x_m, x_n, x_l) \to 0$, as $m, n, l \to \infty$.

Definition 1.6. Let (X, G) be a $G-$ cone metric space. Then for $x_0 \in X$, $r \gg 0$, the $G-$ *ball* with center x_0 and radius r is

$$B_G(x_0, r) = \{y \in X : G(x_0, y, y) < r\}.$$

Proposition 1.7. Let X be a $G-$ cone metric space. Then for any $x_0 \in X$ and $r \gg 0$,

1) if $G(x_0, x, y) < r$, then $x, y \in B_G(x_0, r)$,

2) if $y \in B_G(x_0, r)$, then there exists a $\delta \gg 0$ such that $B_G(y, \delta) \subseteq B_G(x_0, r)$.

Thus the family of all $G-$balls, $\beta = \{B_G(x, r) : x \in X, r \gg 0\}$, is the base of a topology $\tau(G)$ on X, the $G-$ cone metric topology.
Definition 1.8. Let (X, G) and (Y, G') be two generalized cone metric spaces and let $f : X \to Y$ be a function, then f is said to be $G-$ *continuous* at a point $a \in X$ if for given $\varepsilon \gg 0$, there exists $\delta \gg 0$ such that $x, y \in X$ and $G(a, x, y) < \delta$ implies that $G'(f(a), f(x), f(y)) < \varepsilon$. A function f is $G-$ continuous at X if and only if it is $G-$continuous at all $a \in X$.
Following propositions given in [13] can also be prove in generalized cone metric spaces. For sake of completeness we state them in setting of $G-$ cone metric spaces.
Proposition 1.9. Let (X, G) and (Y, G') be two generalized cone metric spaces. Then a function $f : X \to Y$ is $G-$ continuous at a point $x \in X$ if and only if it is $G-$ sequentially continuous at x, that is, whenever x_n is $G-$ convergent to x, then $\{f(x_n)\}$ is $G-$ convergent to $f(x)$.
Proposition 1.10. Let X be a $G-$ cone metric space. Then the function G is jointly continuous in all three of its variables.
Definition 1.11. A $G-$ cone metric is said to be *symmetric* if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.
Proposition 1.12. Let X be a $G-$ cone metric space. Define $d_G : X \times X \to E$ by $d_G(x, y) = G(x, y, y) + G(y, x, x)$ is the cone metric on X.
Note that (X, d_G) is a cone metric space. If (X, G) is symmetric $G-$ cone metric space, then

$$d_G(x, y) = 2G(x, y, y)$$

for all $x, y \in X$.

2.Fixed Point Theorems

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In this section, fixed point theorem for maps, without completeness property of $G-$ cone metric space are obtained.
Theorem 2.1. Let X be a $G-$ cone metric space. Suppose that the mapping $f : X \longrightarrow X$ satisfies:

$$G(fx, fy, fz) \leq \alpha G(x, fx, fx) + \beta G(y, fy, fy) + \gamma G(z, fz, fz), \qquad (2.1)$$

for all $x, y, z \in X$ and $\alpha, \beta, \gamma \geq 0$ with $\alpha + \beta + \gamma < 1$. If f is $G-$ continuous at a point $u \in X$ and $\{f^n(x_0)\}$ has a subsequence $\{f^{n_k}(x_0)\}$ which is $G-$convergent to u for some $x_0 \in X$. Then u is the unique fixed point of f.

Proof. For every $x, y \in X$, we have

$$G(fx, fy, fy) \leq \alpha G(x, fx, fx) + \beta G(y, fy, fy) + \gamma G(y, fy, fy),$$

and

$$G(fy, fx, fx) \leq \alpha G(y, fy, fy) + \beta G(x, fx, fx) + \gamma G(x, fx, fx).$$

If G is symmetric, then by adding above two inequalities, we obtain

$$\begin{aligned} d_G(fx, fy) &\leq \frac{\alpha+\beta+\gamma}{2} d_G(x, fx) + \frac{\alpha+\beta+\gamma}{2} d_G(y, fy) \\ &= \frac{(\alpha+\beta+\gamma)}{2}(d_G(x, fx) + d_G(y, fy)). \end{aligned} \tag{1}$$

Since f is $G-$ continuous at a point $u \in X$ and $\{f^{n_k}(x_0)\}$ is $G-$ convergent to u therefore $f^{n_k+1}(x_0) \to f(u)$. For $k > n_0$ in $\mathbb{N}$, we have

$$\begin{aligned} d_G(f^{n_k+1}x_0, f^{n_k+2}x_0) \leq \ & \frac{(\alpha+\beta+\gamma)}{2}(d_G(f^{n_k}x_0, f^{n_k+1}x_0) \\ & + d_G(f^{n_k+1}x_0, f^{n_k+2}x_0)), \end{aligned}$$

then

$$d_G(f^{n_k+1}x_0, f^{n_k+2}x_0) \leq \lambda d_G(f^{n_k}x_0, f^{n_k+1}x_0),$$

where $\lambda = \dfrac{\alpha+\beta+\gamma}{2-\alpha-\beta-\gamma}$. Obviously $0 \leq \lambda < 1$. In a similar way for $l > k > n_0$, we have

$$\begin{aligned} d_G(f^{n_l}x_0, f^{n_l+1}x_0) &\leq \lambda d_G(f^{n_l-1}x_0, f^{n_l}x_0) \\ &\leq \lambda^2 d_G(f^{n_l-2}x_0, f^{n_l-1}x_0) \\ &\leq \cdots \leq \lambda^{n_l-n_k} d_G(f^{n_k}x_0, f^{n_k+1}x_0). \end{aligned}$$

Hence $d_G(u, fu) \leq 0$ as $k \to \infty$, that is $-d_G(u, fu) \in P$. But $d_G(u, fu) \in P$ gives that $d_G(u, fu) = 0$, and $fu = u$.
In case, if G is not symmetric, then (2.2) gives no information about the maps, as in this case, the contractive constant need not be less that 1. In this case, for $k > n_0$ in $\mathbb{N}$, we have

$$\begin{aligned} G(f^{n_k+1}x_0, f^{n_k+2}x_0, f^{n_k+3}x_0) &\leq \alpha G(f^{n_k}x_0, f^{n_k+1}x_0, f^{n_k+1}x_0) \\ &\quad + \beta G(f^{n_k+1}x_0, f^{n_k+2}x_0, f^{n_k+2}x_0) \\ &\quad + \gamma G(f^{n_k+2}x_0, f^{n_k+3}x_0, f^{n_k+3}x_0) \\ &\leq \alpha G(f^{n_k}x_0, f^{n_k+1}x_0, f^{n_k+1}x_0) \\ &\quad + \beta G(f^{n_k+1}x_0, f^{n_k+2}x_0, f^{n_k+3}x_0) \\ &\quad + \gamma G(f^{n_k+1}x_0, f^{n_k+2}x_0, f^{n_k+3}x_0), \end{aligned}$$

then

$$G(f^{n_k+1}x_0, f^{n_k+2}x_0, f^{n_k+3}x_0) \leq \lambda G(f^{n_k}x_0, f^{n_k+1}x_0, f^{n_k+1}x_0),$$

where $\lambda = \dfrac{\alpha}{1-\beta-\gamma}$. Obviously $0 \leq \lambda < 1$. In similar way for $l > k > n_0$, we have

$$\begin{aligned} G(f^{n_l}x_0, f^{n_l+1}x_0, f^{n_l+1}x_0) &\leq \lambda G(f^{n_l-1}x_0, f^{n_l}x_0, f^{n_l}x_0) \\ &\leq \lambda^2 G(f^{n_l-2}x_0, f^{n_l-1}x_0, f^{n_l-1}x_0) \\ &\leq \cdots \leq \lambda^{n_l-n_k} G(f^{n_k}x_0, f^{n_k+1}x_0, f^{n_k+1}x_0). \end{aligned}$$

Hence $G(u, fu, fu) \leq 0$ as $k \to \infty$, that is $-G(u, fu, fu) \in P$. But $G(u, fu, fu) \in P$ gives that $G(u, fu, fu) = 0$ and hence $fu = u$. Suppose that there is another v in X such that $fv = v$, then

$$\begin{aligned} G(u, v, v) &= G(fu, fv, fv) \\ &\leq \alpha G(u, fu, fu) + (\beta+\gamma)G(v, fv, fv) \\ &= \alpha G(u, u, u) + (\beta+\gamma)G(v, v, v), \end{aligned}$$

which implies that $-G(u, v, v) \in P$. But $G(u, v, v) \in P$ gives that $G(u, v, v) = 0$ and hence $u = v$, which proves the uniqueness of fixed point of f.

Example 2.2. Let $X = [0, 1)$. The mapping $G : X \times X \times X \to R^2$ given by

$$G(x, y, z) = (|x-y| + |y-z| + |z-x|, h(|x-y| + |y-z| + |z-x|)),$$

where h is a positive constant, is a $G-$ cone metric on X. Define a discontinuous map $f : X \to X$ as

$$f(x) = \begin{cases} \dfrac{x}{9} & \text{for } x \in [0, \frac{1}{2}), \\ \dfrac{x}{6} & \text{for } x \in [\frac{1}{2}, 1). \end{cases}$$

Note that for $x, y, z \in [0, \frac{1}{2})$,

$$\begin{aligned} G(fx, fy, fz) &= \frac{1}{9}((|x-y| + |y-z| + |z-x|, h(|x-y| + |y-z| + |z-x|)), \\ G(x, fx, fx) &= \frac{16}{9}(x, hx),\ G(y, fy, fy) = \frac{16}{9}(y, hy), \\ \text{and } G(z, fz, fz) &= \frac{16}{9}(z, hz). \end{aligned}$$

Now

$$\begin{aligned} G(fx, fy, fz) &\leq \frac{2}{9}(x+y+z, h(x+y+z)) \\ &\leq \frac{2}{9}\left(\frac{16}{9}(x, hx)\right) + \frac{2}{9}\left(\frac{16}{9}(y, hy)\right) + \frac{2}{9}\left(\frac{16}{9}(z, hz)\right) \\ &= \alpha G(x, fx, fx) + \beta G(y, fy, fy) + \gamma G(z, fz, fz). \end{aligned}$$

Thus

$$G(fx, fy, fz) \leq \alpha G(x, fx, fx) + \beta G(y, fy, fy) + \gamma G(z, fz, fz),$$

is satisfied for $\alpha = \beta = \gamma = \dfrac{2}{9}$, where $\alpha + \beta + \gamma = \dfrac{2}{3} < 1$.

For $x, y, z \in [\frac{1}{2}, 1)$, then

$$\begin{aligned} G(fx, fy, fz) &= \frac{1}{6}((|x-y| + |y-z| + |z-x|\,, h(|x-y| + |y-z| + |z-x|)), \\ G(x, fx, fx) &= \frac{5}{3}(x, hx),\ G(y, fy, fy) = \frac{5}{3}(y, hy), \\ \text{and } G(z, fz, fz) &= \frac{5}{3}(z, hz). \end{aligned}$$

Now

$$\begin{aligned} G(fx, fy, fz) &< \frac{1}{3}((x+y+z), h(x+y+z)) \\ &\leq \frac{2}{9}\left(\frac{5}{3}(x, hx)\right) + \frac{2}{9}\left(\frac{5}{3}(y, hy)\right) + \frac{2}{9}\left(\frac{5}{3}(z, hz)\right) \\ &= \alpha G(x, fx, fx) + \beta G(y, fy, fy) + \gamma G(z, fz, fz). \end{aligned}$$

Hence

$$G(fx, fy, fz) \leq \alpha G(x, fx, fx) + \beta G(y, fy, fy) + \gamma G(z, fz, fz),$$

is satisfied for $\alpha = \beta = \gamma = \dfrac{2}{9}$.

For $x \in [0, \frac{1}{2})$, $y, z \in [\frac{1}{2}, 1)$, we have

$$\begin{aligned} G(fx, fy, fz) &= (\left|\frac{x}{9} - \frac{y}{6}\right| + \left|\frac{y}{6} - \frac{z}{6}\right| + \left|\frac{z}{6} - \frac{x}{9}\right|, h(\left|\frac{x}{9} - \frac{y}{6}\right| \\ &\quad + \left|\frac{y}{6} - \frac{z}{6}\right| + \left|\frac{z}{6} - \frac{x}{9}\right|) \\ &= \frac{1}{6}((y + z + |y-z| - \frac{4x}{3}), h(y + z + |y-z| - \frac{4x}{3})) \\ G(x, fx, fx) &= \frac{16}{9}(x, hx),\ G(y, fy, fy) = \frac{5}{3}(y, hy), \\ \text{and } G(z, fz, fz) &= \frac{5}{3}(z, hz). \end{aligned}$$

Now for

$$\begin{aligned} G(fx, fy, fz) &\leq \frac{1}{3}(x+y+z, h(x+y+z)) \\ &\leq \frac{2}{9}\left(\frac{16}{9}(x, hx)\right) + \frac{2}{9}\left(\frac{5}{3}(y, hy)\right) + \frac{2}{9}\left(\frac{5}{3}(z, hz)\right) \\ &= \alpha G(x, fx, fx) + \beta G(y, fy, fy) + \gamma G(z, fz, fz). \end{aligned}$$

Thus
$$G(fx,fy,fz) \leq \alpha G(x,fx,fx) + \beta G(y,fy,fy) + \gamma G(z,fz,fz),$$
is satisfied for $\alpha = \beta = \gamma = \dfrac{2}{9}$.

Finally for $x, y \in [0, \frac{1}{2})$, $z \in [\frac{1}{2}, 1)$,

$$\begin{aligned}
G(fx,fy,fz) &= (\left|\frac{x}{9}-\frac{y}{9}\right| + \left|\frac{y}{9}-\frac{z}{6}\right| + \left|\frac{z}{6}-\frac{x}{9}\right|, h(\left|\frac{x}{9}-\frac{y}{9}\right| \\
&\quad + \left|\frac{y}{9}-\frac{z}{6}\right| + \left|\frac{z}{6}-\frac{x}{9}\right|) \\
&= \frac{1}{6}((\frac{2}{3}\left|x-y\right| + 2z - \frac{2y}{3} - \frac{2x}{3}), h(\frac{2}{3}\left|x-y\right| + 2z - \frac{2y}{3} - \frac{2x}{3})) \\
G(x,fx,fx) &= \frac{16}{9}(x,hx),\ G(y,fy,fy) = \frac{16}{9}(y,hy), \\
\text{and } G(z,fz,fz) &= \frac{5}{3}(z,hz).
\end{aligned}$$

And

$$\begin{aligned}
G(fx,fy,fz) &\leq \frac{1}{3}(x+y+z, h(x+y+z)) \\
&\leq \frac{2}{9}\left(\frac{16}{9}(x,hx)\right) + \frac{2}{9}\left(\frac{5}{3}(y,hy)\right) + \frac{2}{9}\left(\frac{5}{3}(z,hz)\right) \\
&= \alpha G(x,fx,fx) + \beta G(y,fy,fy) + \gamma G(z,fz,fz).
\end{aligned}$$

Thus
$$G(fx,fy,fz) \leq \alpha G(x,fx,fx) + \beta G(y,fy,fy) + \gamma G(z,fz,fz),$$
is satisfied for $\alpha = \beta = \gamma = \dfrac{2}{9}$.

Also observe that f is $G-$continuous at $0 \in X$ and for each $u \in X$, $\{f^n u\}$ converges to 0 and hence every subsequence of $\{f^n(u)\}$ is $G-$convergent to 0. Hence, the axioms of Theorem 2.1 are satisfied. Moreover, 0 is a fixed point of f.

Corollary 2.3. Let X be a $G-$ cone metric space and $f : X \to X$ satisfies the following condition:

$$G(fx,fy,fz) \leq h\{G(x,fx,fx) + G(y,fy,fy) + G(z,fz,fz)\}, \tag{2.3}$$

for all $x, y, z \in X$ where $0 \leq h < \dfrac{1}{3}$. If f is $G-$ continuous at a point $u \in X$ and $\{f^n(x_0)\}$ has a subsequence $\{f^{n_k}(x_0)\}$ which is $G-$ convergent to u for some $x_0 \in X$. Then u is the unique fixed point of f.

Corollary 2.4. Let X be $G-$ cone metric space and $f : X \to X$ be a mapping which satisfies

$$G(fx, fy, fy) \leq aG(x, fx, fx) + bG(y, fy, fy), \tag{2.4}$$

for all $x, y \in X$, where $a, b \geq 0$ and $a + b < 1$. If f is $G-$continuous at a point $u \in X$ and $\{f^n(x_0)\}$ has a subsequence $\{f^{n_k}(x_0)\}$ which is $G-$convergent to u for some $x_0 \in X$. Then u is the unique fixed point of f.

Corollary 2.5. Let X be a $G-$ cone metric space and $f : X \to X$ satisfies the following condition:

$$G(fx, fy, fy) \leq k\{G(x, fx, fx) + G(y, fy, fy)\}, \tag{2.5}$$

for all $x, y, z \in X$ where $0 \leq k < \dfrac{1}{6}$. If f is $G-$continuous at a point $u \in X$ and $\{f^n(x_0)\}$ has a subsequence $\{f^{n_k}(x_0)\}$ which is $G-$convergent to u for some $x_0 \in X$. Then u is the unique fixed point of f.

Corollary 2.6. Let X be a $G-$ cone metric space and $f : X \to X$ satisfies the following

$$G(fx, fy, fy) \leq kG(x, y, y), \tag{2.6}$$

for all $x, y, z \in X$ where $0 \leq k < \dfrac{1}{4}$. If f is $G-$ continuous at a point $u \in X$ and $\{f^n(x_0)\}$ has a subsequence $\{f^{n_k}(x_0)\}$ which is $G-$ convergent to u for some $x_0 \in X$. Then u is the unique fixed point of f.

Proof. Note that

$$\begin{aligned} G(fx, fy, fy) &\leq kG(x, y, y) \\ &\leq kG(x, fx, fx) + kG(fx, fy, fy) + 2kG(y, fy, fy), \end{aligned}$$

implies that

$$G(fx, fy, fy) \leq aG(x, fx, fx) + bG(y, fy, fy),$$

for all $x, y \in X$, with $a = \dfrac{k}{1-k}$, $b = \dfrac{2k}{1-k}$. Since $a + b < 1$. The result follows from *Corollary 2.4*.

Theorem 2.7. Let X be a $G-$ cone metric space. Suppose that the mapping $f : X \longrightarrow X$ satisfies:

$$G(fx, fy, fz) \leq hu(x, y, z), \tag{2.7}$$

where

$$u(x, y, z) \in \{G(x, fx, fx), G(y, fy, fy), G(z, fz, fz)\},$$

for all $x, y, z \in X$ and $\alpha, \beta, \gamma \geq 0$ with $0 < \alpha + \beta + \gamma < 1$. If f is $G-$continuous at a point $u \in X$ and $\{f^n(x_0)\}$ has a subsequence $\{f^{n_k}(x_0)\}$ which is $G-$ convergent to u for some $x_0 \in X$. Then u is the unique fixed point of f.

Proof. For all $x, y \in X$, we obtain

$$G(fx, fy, fy) \leq hu(x, y, y)$$

where

$$\begin{aligned} u(x, y, y) &\in \{G(x, fx, fx), G(y, fy, fy), G(y, fy, fy)\} \\ &= \{G(x, fx, fx), G(y, fy, fy)\}, \end{aligned}$$

and

$$G(fy, fx, fx) \leq hu(y, x, x),$$

where

$$u(y, x, x) \in \{G(y, fy, fy), G(x, fx, fx)\}.$$

If G is a symmetric, then we obtain

$$\begin{aligned} d_G(fx, fy) &\leq hu_G(x, y), \text{ where} \\ u_G(x, y) &\in \{d_G(x, fx), d_G(y, fy), \frac{1}{2}[d_G(x, fx) + d_G(y, fy)]\}. \end{aligned} \tag{2}$$

Since f is $G-$ continuous at a point $u \in X$ and $\{f^{n_k}(x_0)\}$ is $G-$ convergent to u therefore $f^{n_k+1}(x_0) \to f(u)$. For $k > n_0$ in $\mathbb{N}$, $u_G(x, y) = d_G(x, fx)$ implies that

$$d_G(f^{n_k+1}x_0, f^{n_k+2}x_0) \leq hd_G(f^{n_k}x_0, f^{n_k+1}x_0),$$

$u_G(x, y) = d_G(y, fy)$ implies that

$$d_G(f^{n_k+1}x_0, f^{n_k+2}x_0) \leq hd_G(f^{n_k+1}x_0, f^{n_k+1}x_0),$$

which by remark 1.3(d) implies $d_G(f^{n_k+1}x_0, f^{n_k+2}x_0) = 0$, $f^{n_k+1}x_0 = f^{n_k+2}x_0$.and result follows.

Finally for $u_G(x, y) = \frac{1}{2}[d_G(x, fx) + d_G(y, fy)]$, we obtain

$$d_G(f^{n_k+1}x_0, f^{n_k+2}x_0) \leq \frac{h}{2}[d_G(f^{n_k}x_0, f^{n_k+1}x_0) + d_G(f^{n_k+1}x_0, f^{n_k+2}x_0)],$$

$$d_G(f^{n_k+1}x_0, f^{n_k+2}x_0) \leq hd_G(f^{n_k}x_0, f^{n_k+1}x_0).$$

In similar way for $l > k > n_0$, we have

$$\begin{aligned} d_G(f^{n_l}x_0, f^{n_l+1}x_0) &\leq \lambda d_G(f^{n_l-1}x_0, f^{n_l}x_0) \\ &\leq \lambda^2 d_G(f^{n_l-2}x_0, f^{n_l-1}x_0) \\ &\leq \cdots \leq \lambda^{n_l-n_k} d_G G(f^{n_k}x_0, f^{n_k+1}x_0). \end{aligned}$$

Hence $d_G(u, fu) \leq 0$ as $k \to \infty$, that is $-d_G(u, fu) \in P$. But $d_G(u, fu) \in P$ gives that $d_G(u, fu) = 0$, and $fu = u$.
However, if (X, G) is not symmetric, then (2.8) gives no information about the maps, as in this case, the contractive constant need not be less that 1. In this case, since f is $G-$ continuous at a point $u \in X$ and $\{f^{n_k}(x_0)\}$ is $G-$ convergent to u therefore $f^{n_k+1}(x_0) \to f(u)$. For $k > n_0$ in $\mathbb{N}$, we have

$$G(f^{n_k+1}x_0, f^{n_k+2}x_0, f^{n_k+3}x_0) \leq hu(f^{n_k}x_0, f^{n_k+1}x_0, f^{n_k+2}x_0),$$

where

$$\begin{aligned} & u(f^{n_k}x_0, f^{n_k+1}x_0, f^{n_k+2}x_0) \\ \in\ & \{G(f^{n_k}x_0, f^{n_k+1}x_0, f^{n_k+1}x_0), G(f^{n_k+1}x_0, f^{n_k+2}x_0, f^{n_k+2}x_0), \\ & G(f^{n_k+2}x_0, f^{n_k+3}x_0, f^{n_k+3}x_0)\}. \end{aligned}$$

Now $u(f^{n_k}x_0, f^{n_k+1}x_0, f^{n_k+2}x_0) = G(f^{n_k}x_0, f^{n_k+1}x_0, f^{n_k+1}x_0)$ implies that

$$G(f^{n_k+1}x_0, f^{n_k+2}x_0, f^{n_k+3}x_0) \leq hG(f^{n_k}x_0, f^{n_k+1}x_0, f^{n_k+1}x_0).$$

When $u(f^{n_k}x_0, f^{n_k+1}x_0, f^{n_k+2}x_0) = G(f^{n_k+1}x_0, f^{n_k+2}x_0, f^{n_k+2}x_0)$, we obtain

$$\begin{aligned} G(f^{n_k+1}x_0, f^{n_k+2}x_0, f^{n_k+3}x_0) &\leq hG(f^{n_k+1}x_0, f^{n_k+2}x_0, f^{n_k+2}x_0) \\ &\leq hG(f^{n_k+1}x_0, f^{n_k+2}x_0, f^{n_k+3}x_0). \end{aligned}$$

So by remark 1.3(d) implies that $G(f^{n_k+1}x_0, f^{n_k+2}x_0, f^{n_k+3}x_0) = 0$, which gives $f^{n_k+1}x_0 = f^{n_k+2}x_0 = f^{n_k+3}x_0$ and result follows. In case

$$u(f^{n_k}x_0, f^{n_k+1}x_0, f^{n_k+2}x_0) = G(f^{n_k+2}x_0, f^{n_k+3}x_0, f^{n_k+3}x_0),$$

we get

$$\begin{aligned} G(f^{n_k+1}x_0, f^{n_k+2}x_0, f^{n_k+3}x_0) &\leq hG(f^{n_k+2}x_0, f^{n_k+3}x_0, f^{n_k+3}x_0) \\ &\leq hG(f^{n_k+1}x_0, f^{n_k+2}x_0, f^{n_k+3}x_0), \end{aligned}$$

agian implies $f^{n_k+1}x_0 = f^{n_k+2}x_0 = f^{n_k+3}x_0$ and the result follows. Thus

$$G(f^{n_k+1}x_0, f^{n_k+2}x_0, f^{n_k+3}x_0) \leq hG(f^{n_k}x_0, f^{n_k+1}x_0, f^{n_k+1}x_0).$$

In a similar way for $l > k > n_0$ we have

$$\begin{aligned} G(f^{n_l}x_0, f^{n_l+1}x_0, f^{n_l+1}x_0) &\leq hG(f^{n_l-1}x_0, f^{n_l}x_0, f^{n_l}x_0) \\ &\leq h^2G(f^{n_l-2}x_0, f^{n_l-1}x_0, f^{n_l-1}x_0) \\ &\leq \cdots \leq h^{n_l-n_k}G(f^{n_k}x_0, f^{n_k+1}x_0, f^{n_k+1}x_0). \end{aligned}$$

Hence $G(u, fu, fu) \leq 0$ as $k \to \infty$, that is $-G(u, fu, fu) \in P$. But $G(u, fu, fu) \in P$ gives that $G(u, fu, fu) = 0$, and $fu = u$. Suppose that there is v in X such that $fv = v$, then use given contractive condition to obtain

$$G(u, v, v) = G(fu, fv, fv) \leq hu(u, v, v),$$

where

$$u(u, v, v) \in \{G(u, fu, fu), G(v, fv, fv), G(v, fv, fv)\} = \{0\}.$$

Which implies that $-G(u, v, v) \in P$. But $G(u, v, v) \in P$ gives that $G(u, v, v) = 0$ and hence $u = v$, which prove the uniqueness of fixed point of f.

Theorem 2.8. Let X be a $G-$ cone metric space and $f : X \to X$ be a mapping satisfying one of the following conditions

$$\begin{aligned} G(fx, fy, fz) \quad \leq \quad & aG(x, y, z) + bG(x, fx, fx) \\ & + cG(y, fy, fy) + dG(z, fz, fz), \end{aligned} \tag{3}$$

or

$$\begin{aligned} G(fx, fy, fz) \quad \leq \quad & aG(x, y, z) + bG(x, fx, x) \\ & + cG(y, y, fy) + dG(z, z, fz), 2.10 \end{aligned} \tag{4}$$

for all $x, y, z \in X$, where $a, b, c, d \geq 0$ with $a + b + c + d < 1$. If f is $G-$ continuous at a point $u \in X$ and $\{f^n(x_0)\}$ has a subsequence $\{f^{n_k}(x_0)\}$ which is $G-$convergent to u for some $x_0 \in X$, then u is the unique fixed point of f.

Proof. Suppose that f satisfies (2.9), then for all $x, y \in X$

$$\begin{aligned} & G(fx, fy, fy) \\ \leq \quad & aG(x, y, y) + bG(x, fx, fx) + (c + d)G(y, fy, fy), \end{aligned} \tag{5}$$

and

$$\begin{aligned} & G(fy, fx, fx) \\ \leq \quad & aG(y, x, x) + bG(y, fy, fy) + (c + d)G(x, fx, fx). \end{aligned} \tag{6}$$

Suppose X is a symmetric $G-$ cone metric space. Adding (2.11) and (2.12), we have

$$\begin{aligned} & d_G(fx, fy) \\ \leq \quad & ad_G(x, y) + (\frac{b + c + d}{2})d_G(x, fx) + (\frac{b + c + d}{2})d_G(y, fy) \\ = \quad & \alpha d_G(x, y) + \beta d_G(x, fx) + \gamma d_G(y, fy), \end{aligned} \tag{7}$$

for all x, $y \in X$, where $\alpha = a$, $\beta = \gamma = \dfrac{b+c+d}{2}$ and $0 \leq \alpha + \beta + \gamma < 1$.
Since f is $G-$ continuous at a point $u \in X$ and $\{f^{n_k}(x_0)\}$ is $G-$ convergent to u therefore $f^{n_k+1}(x_0) \to f(u)$. For $k > n_0$ in $\mathbb{N}$, we have

$$\begin{aligned}
& d_G(f^{n_k+1}x_0, f^{n_k+2}x_0) \\
\leq\ & \alpha d_G(f^{n_k}x_0, f^{n_k+1}x_0) + \beta d_G(f^{n_k}x_0, f^{n_k+1}x_0) + \gamma d_G(f^{n_k+1}x_0, f^{n_k+2}x_0).
\end{aligned}$$

Which further implies that

$$d_G(f^{n_k+1}x_0, f^{n_k+2}x_0) \leq \lambda d_G(f^{n_k}x_0, f^{n_k+1}x_0),$$

where $\lambda = \dfrac{\alpha+\beta}{1-\gamma}$. Obviously $0 \leq \lambda < 1$. In similar way for $l > k > n_0$, we have

$$\begin{aligned}
d_G(f^{n_l}x_0, f^{n_l+1}x_0) &\leq \lambda d_G(f^{n_l-1}x_0, f^{n_l}x_0) \\
&\leq \lambda^2 d_G(f^{n_l-2}x_0, f^{n_l-1}x_0) \\
&\leq \cdots \leq \lambda^{n_l-n_k} d_G(f^{n_k}x_0, f^{n_k+1}x_0).
\end{aligned}$$

Hence $d_G(u, fu) \leq 0$ as $k \to \infty$, that is $-d_G(u, fu) \in P$. But $d_G(u, fu) \in P$ gives that $d_G(u, fu) = 0$, and $fu = u$.
Suppose that there is v in X such that $fv = v$, then using (2.13) one obtains

$$\begin{aligned}
d_G(u, v) &= d_G(fu, fv) \\
&\leq \alpha d_G(u, v) + \beta d_G(u, fu) + \gamma d_G(v, fv) \\
&= \alpha d_G(u, v) + \beta d_G(u, u) + \gamma d_G(v, v) \\
&= \alpha d_G(u, v).
\end{aligned}$$

Remark 2.3(d) implies that $d_G(u, v) = 0$ and hence $u = v$ which prove the uniqueness of fixed point of f.
However, If G is not a symmetric cone, then adding (2.11) and (2.12), we obtain the following

$$d_G(fx, fy) \leq a d_G(x, y) + \frac{2(b+c+d)}{3} d_G(x, fx) + \frac{2(b+c+d)}{3} d_G(y, fy),$$

for all $x, y \in X$. Here, $0 \leq a + \dfrac{2(b+c+d)}{3} + \dfrac{2(b+c+d)}{3}$ which may not be less than 1. So that in this case, for $k > n_0$ in $\mathbb{N}$, consider

$$\begin{aligned}
& G(f^{n_k+1}x_0, f^{n_k+2}x_0, f^{n_k+3}x_0) \\
\leq\ & aG(f^{n_k}x_0, f^{n_k+1}x_0, f^{n_k+2}x_0) + bG(f^{n_k}x_0, f^{n_k+1}x_0, f^{n_k+1}x_0) \\
& + cG(f^{n_k+1}x_0, f^{n_k+2}x_0, f^{n_k+2}x_0) + dG(f^{n_k+2}x_0, f^{n_k+3}x_0, f^{n_k+3}x_0) \\
\leq\ & aG(f^{n_k}x_0, f^{n_k+1}x_0, f^{n_k+2}x_0) + bG(f^{n_k}x_0, f^{n_k+1}x_0, f^{n_k+2}x_0)) \\
& + cG(f^{n_k+1}x_0, f^{n_k+2}x_0, f^{n_k+3}x_0) + dG(f^{n_k+1}x_0, f^{n_k+2}x_0, f^{n_k+3}x_0).
\end{aligned}$$

Which further implies that

$$G(f^{n_k+1}x_0, f^{n_k+2}x_0, f^{n_k+3}x_0) \leq \lambda G(f^{n_k}x_0, f^{n_k+1}x_0, f^{n_k+2}x_0)$$

where $\lambda = \dfrac{a+b}{1-c-d}$. Obviously $0 \leq \lambda < 1$. In similar way for $l > k > n_0$, we have

$$\begin{aligned} G(f^{n_l}x_0, f^{n_l+1}x_0, f^{n_l+1}x_0) &\leq \lambda G(f^{n_l-1}x_0, f^{n_l}x_0, f^{n_l}x_0) \\ &\leq \lambda^2 G(f^{n_l-2}x_0, f^{n_l-1}x_0, f^{n_l-1}x_0) \\ &\leq \cdots \leq \lambda^{n_l-n_k} G(f^{n_k}x_0, f^{n_k+1}x_0, f^{n_k+1}x_0). \end{aligned}$$

Hence $G(u, fu, fu) \leq 0$ as $k \to \infty$, that is $-G(u, fu, fu) \in P$. But $G(u, fu, fu) \in P$ gives that $G(u, fu, fu) = 0$, and $fu = u$. Suppose that there is v in X such that $fv = v$, then

$$\begin{aligned} G(u, v, v) &= G(fu, fv, fv) \\ &\leq aG(u, v, v) + bG(u, fu, fu) + (c+d)G(v, fv, fv) \\ &= aG(u, v, v). \end{aligned}$$

and remark 1.3(d) give that $G(u, v, v) = 0$ and hence $u = v$, which prove the uniqueness of fixed point.

Theorem 2.9. Let X be a $G-$ cone metric space and $f : X \to X$ be a mapping satisfying one of the following conditions

$$G(fx, fy, fy) \leq \alpha\{G(x, fy, fy) + G(y, fx, fx)\}, \tag{2.14}$$

or

$$G(fx, fy, fy) \leq \alpha\{G(x, x, fy) + G(y, y, fx)\}, \tag{2.15}$$

for all $x, y \in X$ where $\alpha \in [0, \frac{1}{2})$. If f is $G-$continuous at a point $u \in X$ and $\{f^n(x_0)\}$ has a subsequence $\{f^{n_k}(x_0)\}$ which is $G-$convergent to u for some $x_0 \in X$, then u is the unique fixed point of f.

Proof. Suppose that f satisfies condition (2.14), then for all $x, y \in X$

$$G(fx, fy, fy) \leq \alpha\{G(x, fy, fy) + G(y, fx, fx)\}, \tag{2.16}$$

and

$$G(fy, fx, fx) \leq \alpha\{G(y, fx, fx) + G(x, fy, fy)\}. \tag{2.17}$$

Now if X is a symmetric $G-$ cone metric space, then above two inequalities give

$$d_G(fx, fy) \leq \alpha\{d_G(x, fy) + d_G(y, fx)\},$$

for all $x, y \in X$. Since f is $G-$continuous at a point $u \in X$ and $\{f^{n_k}(x_0)\}$ is $G-$convergent to u therefore $f^{n_k+1}(x_0) \to f(u)$. For $k > n_0$ in $\mathbb{N}$, we have

$$\begin{aligned} d_G(f^{n_k+1}x_0, f^{n_k+2}x_0) &\leq \alpha\{d_G(f^{n_k}x_0, f^{n_k+2}x_0) + d_G(f^{n_k+1}x_0, f^{n_k+1}x_0)\} \\ &\leq \alpha\{d_G(f^{n_k}x_0, f^{n_k+1}x_0) + d_G(f^{n_k+1}x_0, f^{n_k+2}x_0)\}, \end{aligned}$$

which implies that

$$d_G(f^{n_k+1}x_0, f^{n_k+2}x_0) \leq \lambda G(f^{n_k}x_0, f^{n_k+1}x_0),$$

where $\lambda = \dfrac{\alpha}{1-\alpha}$. Obviously $0 \leq \lambda < 1$. The result follows using similar arguments to those given in *Theorem 2.8.*
Now, if G is not symmetric, then by adding (2.10) and (2.11), we obtain

$$\begin{aligned} d_G(fx, fy) &= G(fx, fy, fy) + G(fy, fx, fx) \\ &\leq 2\alpha\{G(y, fx, fx) + G(x, fy, fy)\} \\ &\leq \frac{4\alpha}{3}\{d_G(y, fx) + d_G(x, fy)\}, \end{aligned}$$

for all $x, y \in X$. Here, contractivity constant $\dfrac{4\alpha}{3}$ may not be less than 1. So in this case, for $k > n_0$ in $\mathbb{N}$, consider

$$\begin{aligned} & G(f^{n_k+1}x_0, f^{n_k+2}x_0, f^{n_k+2}x_0) \\ \leq\; & \alpha\{G(f^{n_k}x_0, f^{n_k+2}x_0, f^{n_k+2}x_0) + G(f^{n_k+1}x_0, f^{n_k+1}x_0, f^{n_k+1}x_0)\} \\ \leq\; & \alpha\{G(f^{n_k}x_0, f^{n_k+1}x_0, f^{n_k+1}x_0) + G(f^{n_k+1}x_0, f^{n_k+1}x_0, f^{n_k+2}x_0)\} \\ \leq\; & \alpha\{G(f^{n_k}x_0, f^{n_k+1}x_0, f^{n_k+1}x_0) + G(f^{n_k+1}x_0, f^{n_k+2}x_0, f^{n_k+2}x_0) \\ & +G(f^{n_k+1}x_0, f^{n_k+2}x_0, f^{n_k+2}x_0)\}, \end{aligned}$$

and

$$G(f^{n_k+1}x_0, f^{n_k+2}x_0, f^{n_k+2}x_0) \leq \lambda G(f^{n_k}x_0, f^{n_k+1}x_0, f^{n_k+1}x_0),$$

where $\lambda = \dfrac{\alpha}{1-2\alpha}$. Obviously $0 \leq \lambda < 1$. In similar way for $l > k > n_0$, we have

$$\begin{aligned} G(f^{n_l}x_0, f^{n_l+1}x_0, f^{n_l+1}x_0) &\leq \lambda G(f^{n_l-1}x_0, f^{n_l}x_0, f^{n_l}x_0) \\ &\leq \lambda^2 G(f^{n_l-2}x_0, f^{n_l-1}x_0, f^{n_l-1}x_0) \\ &\leq \cdots \leq \lambda^{n_l-n_k} G(f^{n_k}x_0, f^{n_k+1}x_0, f^{n_k+1}x_0). \end{aligned}$$

Hence $G(u, fu, fu) \leq 0$ as $k \to \infty$, that is $-G(u, fu, fu) \in P$. But $G(u, fu, fu) \in P$ gives that $G(u, fu, fu) = 0$, and $fu = u$. Suppose that there is v in X such that $fv = v$, then

$$\begin{aligned} G(u, v, v) &= G(fu, fv, fv) \\ &\leq \alpha\{G(u, v, v) + G(v, u, u)\}, \end{aligned}$$

implies that

$$G(u,v,v) \leq kG(v,u,u),$$

where $k = \dfrac{\alpha}{1-\alpha}$. Obviously, $k < 1$ and remark 1.3(d) give that $G(u,v,v) = 0$ and hence $u = v$, which prove the uniqueness of fixed point.

Example 2.10. Let $X = [0,1)$, $E = C^1_R$ and let $P = \{x \in E : x(t) \geq 0\}$. The mapping $G : X \times X \times X \to X$ is defined by

$$G(x,y,y) = (|x-y| + |y-z| + |z-x|)\,\psi(t),$$

where $\psi(t) \in P$ is a fixed function, for example, (i) $\psi(t) = e^t$, (ii) $\psi(t) = 2^t$, (ii) $\psi(t) = \lambda t$, $\lambda \in [0,1)$, $t \in P$. Clearly, the metric given above is $G-$ cone metric on X. Let f be the self map on X defined by $f(x) = \frac{1}{3}e^{-x}$.

Note that

$$\begin{aligned}
G(fx,fy,fy) &= \frac{1}{3}\left|e^{-x} - e^{-y}\right|\psi(t)\\
G(x,fy,fy) &= \left|x - \frac{1}{3}e^{-y}\right|\psi(t),\ G(y,fx,fx) = \left|y - \frac{1}{3}e^{-x}\right|\psi(t).
\end{aligned}$$

Now, for $x, y \in X$,

$$\begin{aligned}
G(fx,fy,fy) &\leq \frac{1}{3}\left|e^{-x} - e^{-y}\right|\psi(t)\\
&\leq \frac{1}{3}\left|x - \frac{1}{3}e^{-x} + y - \frac{1}{3}e^{-y} + \frac{2}{3}(2e^{-x} - e^{-y})\right|\psi(t)\\
&\leq \frac{1}{3}\left[\left|x - \frac{1}{3}e^{-y}\right| + \left|y - \frac{1}{3}e^{-x}\right| + \frac{2}{3}\left|2e^{-x} - e^{-y}\right|\right]\psi(t)\\
&\leq \frac{1}{3}\left[\left|x - \frac{1}{3}e^{-y}\right| + \left|y - \frac{1}{3}e^{-x}\right|\right]\psi(t)\\
&= \frac{1}{3}[G(x,fy,fy) + G(y,fx,fx)].
\end{aligned}$$

Therefore

$$G(fx,fy,fy) \leq \alpha\{G(x,fy,fy) + G(y,fx,fx)\} \text{ for all } x,y \in X,$$

is satisfied for all $x,y,z \in X$, where $\alpha = \dfrac{1}{3} < \dfrac{1}{2}$. Also note that, for each $x \in X$, $\{f^n(x)\} \to 0$. Therefore, f satisfy all conditions of *Theorem 2.9* and 0 is the unique fixed point of f.

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