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ABOUT NATURAL SPLINE FUNCTIONS OF INTERPOLATION AND THEIR APPLICATIONS

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Abstract. Spline functions have proved to be very useful in numerical analysis, in numerical treatment of differential, integral and partial differential equations, in statistics, and have founded applications in science, engineering, economics, biology, medecine, etc. The aim of this paper is to prove a sequence of theorems and results on spline functions of interpolation and their applications.

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1. Historic Introduction

Historically, the term "spline function" was used for the first time by I.J. Schoenberg ([3], 1946) to indicate a function formed by fragments of polynomials which link as well as some of their derivatives with junction points.
We obtain in particular functions of this type when we minimize the functional

$$\int_a^b (f^{(q)}(s))^2 ds$$

on the set of all the functions which verify conditions of interpolation of the type

$$f^{(j)}(s_i) = y_{ij}$$

In broad outline, we can divide into two periods the researches made in the field of the spline functions.
Before 1964, articles are especially dedicated to the properties of the functions formed by pieces of polynomials.
After 1964 (functional approaches on Hilbert space), numerous authors tried hard to generalize these notions by using the functional analysis. Taking place generally in Hilbert spaces, they considered problems of minimization similar to those who intervene in the elementary theory of spline functions, still giving the name of spline

to their solutions. This tactics showed itself extremely fruitful and increased considerably the number of the examples and the applications.
It is necessary to distinguish an important exception for our classification in two periods.
In 1958, M. Golom and H. Weinberger, by studying the optimal estimate of linear continuous functional spaces, were brought to introduce the notion of spline function in an Hilbert space "without giving her however this name and without establishing explicitly all the connections with the properties known for spline functions as pieces of polynomials".
The importance of this fundamental article was underestimated for a long time.
The first appearance, for the spline functions of adjustment, is owing to M. Atteia ([1], 1965).
A decisive stage was the discovery by I.J. Schoenberg ([5], 1964) relations between the spline functions and the best formulae of estimate in the sense of Sard.
This relation was generalized at once by M. Atteia ([1], 1965) in the case of Hilbert spaces. Let us note however that these relations were already implicitly contained, under a slightly different form, in M. Golomb and H. Weinberger's article ([2], 1958).
Since 1967-1968 and 1968-2009 an impressive number of mathematicians was interested in the theory and in the applications of the spline functions.

2.Main Results

2.1 Problem

Let $I = [a, b]$ be a given finite interval of the real line $\mathbb{R}$.We define a partition Π_n of I by:

$$\Pi_n : a < x_1 < x_2 < ... < x_N < b$$

Let assume that f is defined on I. Let us given N ditinct abscissas x_i on the interval I, with

$$a < x_1 < x_2 < ... < x_N < b$$

Being given N real numbers y_i (who can be the ordinates in x_i of a certain function f,($f(x_i) = y_i$), we propose to find a function $s(x)$ such that

$$s(x_i) = y_i$$

and which is formed by polynomials by fragments linking, as well as some of their derivatives in knots x_i.

2.2. Natural Spline Functions

Definition 2.1 Let $a < x_1 < x_2 < ... < x_N < b$ and k is an integer such that $1 \leq k \leq N$. We call natural spline function of degree $(2k-1)$ a function $s : [a, b] \to \mathbb{R}$ such that:

(a) s is reduced to a polynomial of degree $\leq (k-1)$ on [a,x_1] and on [x_N,b];

(b) on every interval [x_i,x_{i+1}], $1 \leq i \leq N-1$, s is reduced to a polynomial of degree $\leq (2k-1)$;

(c) in every knot x_i, the defining polynomials s link continuously until their derivatives $(2k-1)$ included (hence $s \in C^{2k-2}([a, b])$).

Proposition 2.1 Denote by $\mathfrak{F}_k$ the set of the functions belonging to $C^{2k-2}([a, b])$.

1) Let s be a function of $\mathfrak{F}_k$ and f be a function of $C^k([a, b])$ such that

$$f(x_i) = s(x_i) \text{ for } i = 1, ..., N.$$

Then

$$\int_a^b (s^{(k)}(x)(s^{(k)}(x) - f^{(k)}(x)))dx = 0.$$

2) Let s be a spline function of Definition 2.1 (that is verifying (a)-(c)). Then for all $f \in C^k([a, b])$ verifying

$$f(x_i) = y_i \text{ for } i = 1, ..., N, f \not\equiv s.$$

one has

$$\int_a^b (s^{(k)}(x))^2 dx < \int_a^b (f^{(k)}(x))^2 dx.$$

Proof. Indeed,

1) put $r = f - s$.

To calculate, $\int_a^b (s^{(k)}(x)r^{(k)}(x))dx$, we integrate $(k-1)$ time by parts what gives (check by recurrence) :

$$\int_a^b (s^{(k)}(x)r^{(k)}(x))dx = (\sum_{j=0}^{k-2} (-1)^j (r^{(k-j-1)}(x)s^{k+j}(x))|_a^b + (-1)^{(k-1)} \int_a^b (r'(x)s^{(2k-1)}(x))dx$$

But

$$s^{(k+j)}(a) = s^{(k+j)}(b) = 0 \text{ for } j = 0, ..., k-2$$

because s is a polynomial of degree $(k-1)$ in [a,x_1] and [x_N,b]. Furthermore, $s^{(2k-1)}(x)$ is a constant c_i in $[x_i, x_{i+1}]$ and it is zero in [a,x_1] and [x_N,b], hence

$$\int_a^b (r'(x)s^{(2k-1)}(x))dx = \sum_{i=1}^{N-1} c_i \int_{x_i}^{x_{i+1}} r'(x)dx = 0$$

because $r(x_i) = 0$ for $i = 1, ..., N$. 2) It is enough to apply 1). We have

$$\int_a^b (f^{(k)}(x))^2 dx - \int_a^b (s^{(k)}(x))^2 dx = \int_a^b (f^{(k)}(x))^2 dx - 2\int_a^b f^{(k)}(x)s^{(k)}(x)dx + \int_a^b s^{(k)}(x)dx$$
$$= \int_a^b (f^{(k)}(x) - s^{(k)}(x))^2 dx \geq 0$$

Furthermore this quantity is not zero that if the integrated function, which is continuous and positive, is zero in any point; thus if

$$f^{(k)}(x) = s^{(k)}(x), \forall x \in [a, b]$$

or still, there is a polynomial p_{k-1} of degree $(k-1)$ such that

$$f(x) = s(x) + p_{k-1}(x), \forall x \in [a, b].$$

But we should to have $p_{k-1}(x_i) = 0$ for $i = 1, ..., N$ which implies that $p_{k-1} = 0$ because $N \geq k$. **Definiton 2.2** If e is a real arithmetical expression, we have

$$e_+^m = \begin{cases} e^m & if \quad e \geq 0 \\ 0 & else \end{cases}$$

Lemma 2.2 The following two properties are equivalent :
(p_1): s is a natural spline function of degree $(2k-1)$;
(p_2): one has

$$(2.1) \begin{cases} s(x) = p_{k-1}(x) + \sum_{i=1}^{n} \lambda_i (x - x_i)_+^{2k-1} & \\ and & with \quad \lambda_i \in \mathbb{R} \\ \sum_{i=1}^{N} \lambda_i x_i^r = 0 & for \quad 0 \leq r \leq k-1 \end{cases}$$

where p_{k-1} is a polynomial of degree $\leq (k-1)$.
Proof. We have
$(p_1) \Rightarrow (p_2)$: for $x \in [a, x_1]$ it is necessary that

$$s(x) = p_{k-1}(x).$$

Let $x \in [x_1,x_2[$, since $s \in C^{2k-2}([a, b])$, it is necessary that in the neighborhood of x_1 we have

$$s(x) \simeq \lambda_1(x-x_1)^{2k-1}$$

Consider the expression

$$s(x) = p_{k-1}(x) + \lambda_1(x-x_1)_+^{2k-1} \text{ for } x \in [a, x_2[,$$

then

$$s(x) = p_{k-1}(x) \quad for \quad x \in [a, x_1]$$

$$s(x) = p_{k-1}(x) + \lambda_1(x-x_1)^{2k-1},$$

polynomial of degree $(2k-1)$, for $x \in [x_1, x_2[$, and

$$s^{(i)}(x) = p_{k-1}^{(i)}(x) + \lambda_1(2k-1)(2k-2)...(2k-i)(x-x_1)^{2k-1-i} \text{ for } x \in [x_1,x_2[;$$

the derivatives link well at x_1, the $(k-1)$ first ones are

$$p'(x_1), p''(x_1), ..., p^{(k-1)}(x_1)$$

and others 0. More generally, for $x \in]x_j, x_{j+1}[$, we consider the expression

$$s(x) = p_{k-1}(x) + \sum_{i=1}^{j} \lambda_i(x-x_i)_+^{2k-1}.$$

We envisage three cases:
• if $x \leq x_j$, then

$$s(x) = p_{k-1}(x) + \sum_{i=1}^{j-1} \lambda_i(x-x_i)^{2k-1}$$

and

$$s^{(\alpha)}(x) = p_{k-1}^{(\alpha)}(x) + (2k-1)(2k-2)...(2k-\alpha) \sum_{i=1}^{j-1} \lambda_i(x-x_i)^{2k-1-\alpha};$$

• if $x \geq x_j$, then

$$s(x) = p_{k-1}(x) + \sum_{i=1}^{j} \lambda_i(x-x_i)^{2k-1}$$

and

$$s^{(\alpha)}(x) = p_{k-1}^{(\alpha)}(x) + (2k-1)(2k-2)...(2k-\alpha)\lambda_j(x-x_j)^{2k-1-\alpha};$$

• for $x \doteq x_j$, then there is equality of the derivatives so much that $\alpha \leq (2k-2)$. For $\alpha = 2k-1$, they differ by $(2k-1)!\lambda_j$;

• to finish, let $x \in [x_n, b]$, by what precedes :

$$s(x) = p_{k-1}(x) + \sum_{i=1}^{N} \lambda_i (x - x_i)^{2k-1}$$

has to be reduced to a polynomial of degree $\leq (k-1)$, it will be the case if, and only if, $s^{(k)}(x) \equiv 0$, but

$$s^{(k)}(x) = (2k-1)(2k-2)...(k) \sum_{i=1}^{N} \lambda_i (x - x_i)^{k-1};$$

it is necessary and it is enough that

$$\sum_{i=1}^{N} \lambda_i \sum_{r=0}^{k-1} (-1)^r C_{k-1}^r x^{k-r-1} x_i^r = 0, \ \forall x \in [x_N, b] \Leftrightarrow \sum_{i=1}^{N} \lambda_i x_i^r = 0, 0 \leq r \leq k-1.$$

$(p_2) \Rightarrow (p_1)$: It is clear that the function s defined by (2.1) verifies the three conditions of a Definition 2.1.

Remark 2.1 The derivative

$$s^{(2k-2)}(x) = (2k-1)! \sum_{i=1}^{N} \lambda_i (x - x_i)_+$$

is a polygonal line, not derivable. The distribution derivative $s^{(2k-1)}$ is a function in staircase, the jumps of which are proportional in λ_i.

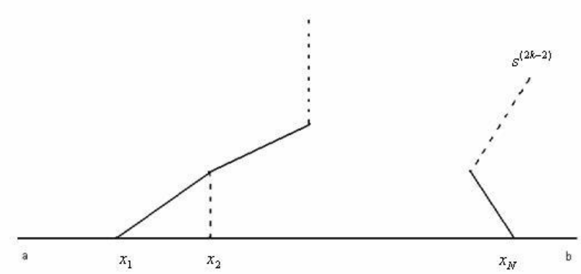


Fig.1.

2.3. Natural Spline Functions of Interpolation

Lemma 2.3 Let us given N numbers y_i, there exists a natural spline function of order $(2k-1)$, unique if $k \leq N$, such that

$$s(x_i) = y_i, 1 \leq i \leq N.$$

Proof. we can see [1].

Remark 2.2 There is to determine k coefficients $a_0, a_1, ..., a_{k-1}$ of p_{k-1} and the N coefficients λ_i, but now we have $(N+k)$ conditions leading to $(N+k)$ linear equations, they are :

$$(2.2) \begin{cases} s(x_i) = y_i, & 1 \leq i \leq N, \\ \forall x \in]x_N, b], s^{(p)} = 0, & p = k, k+1, ..., 2k-1. \end{cases}$$

The proof consists in verifying that the linear system described below is Cramerian

$$(2.1) \Leftrightarrow \begin{cases} y_i = \sum_{p=0}^{k-1} a_p x_i^p + \sum_{p=1}^{i-1} \lambda_p (x_i - x_p)^{2k-1}, & 1 \leq i \leq N, \\ \sum_{i=1}^{N} \lambda_i x_i^r = 0, & 0 \leq r \leq k-1. \end{cases}$$

Or explicitly (see below)

$$\begin{pmatrix} 1 & x_1 & x_1^2 & . & x_1^{k-1} & 0 & 0 & 0 & 0 & 0 \\ 1 & x_2 & x_2^2 & . & x_2^{k-1} & (x_2-x_1)^{2k-1} & 0 & 0 & 0 & 0 \\ . & . & . & . & . & (x_3-x_1)^{2k-1} & (x_3-x_2)^{2k-1} & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ 1 & x_N & x_N^2 & . & x_N^{k-1} & (x_N-x_1)^{2k-1} & (x_N-x_2)^{2k-1} & (x_N-x_3)^{2k-1} & . & (x_N-x_{N-1})^{2k-1} \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & . & 1 \\ 0 & 0 & 0 & 0 & 0 & x_1 & x_2 & x_3 & . & x_N \\ 0 & 0 & 0 & 0 & 0 & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & x_1^{k-1} & x_2^{k-1} & x_3^{k-1} & . & x_N^{k-1} \end{pmatrix}$$

$$\begin{pmatrix} a_0 \\ a_1 \\ . \\ a_{k-1} \\ \lambda_1 \\ . \\ 0 \\ . \\ . \\ . \\ . \\ \lambda_n \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ . \\ . \\ . \\ . \\ y_N \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Such a system, if it allows to resolve in theory the problem, is often very badly conditioned in fact ($x_i \neq x_{i+1}$ for example) where from the necessity of finding different methods for the calculation of the coefficients.

Theorem 2.4 *(fundamental property)* Among all the functions $u \in C^{k-1}([a,b])$, such that

$$\begin{cases} I_k(u) = \int_a^b (u^{(k)}(x))^2 dx < +\infty \\ \quad and \\ u(x_i) = y_i, \qquad 1 \leq i \leq N. \end{cases}$$

the natural spline function s of interpolation of degree $(2k-1)$ is the one which minimizes $I_k(u)$.
Reciprocally, if v is a function such that

$$\begin{cases} v(x_i) = y_i, \quad 1 \leq i \leq N \\ \quad and \\ I_k(v) \leq I_k(u) \qquad \forall u, \end{cases}$$

then $v = s$ $(k \leq N)$.
Proof. To prove this theorem see ([1]).

In other words, the spline function of interpolation is characterized by a property of minimization which is essential when we generalize the problem.

2.4. Generalized Problem

Let X and Y be two Hilbert spaces. Let $T : X \to Y$ be a linear continuous mapping from X into Y.

Let $\mathfrak{A} : X \to \mathbb{R}^N$ be a linear continuous mapping from X into $\mathbb{R}^N$. $\mathfrak{N}(T)$ and $\mathfrak{N}(\mathfrak{A})$ are Kernels of the mappings T and $\mathfrak{A}$.

Theorem 2.5 *(Atteia) Let us given N reals $y_i, 1 \leq i \leq N$, let Y be a corresponding vector of $\mathbb{R}^N$ and*

$$\mathfrak{T}_Y := \{x \in X : \mathfrak{A}(x) = Y\}.$$

if $\mathfrak{N}(T)$ is of finite dimension and if $\mathfrak{N}(T) \bigcap \mathfrak{N}(\mathfrak{A}) = \{0\}$, then there exists an unique s in $\mathfrak{T}_Y$ which minimizes $\|Tx\|_Y$. Moreover

$$\|Ts\|_Y \leq \|Tx\|_Y, \forall x \in \mathfrak{T}_Y.$$

The problem which we treat is a particular case. Indeed ; take

$X = H^k([a,b]) := \{u \in C^{k-1}([a,b]) : \|u\|_X^2 = \sum_{i=1}^{k} \int_a^b (u^{(i)}(x))^2 dx < +\infty\}$

$Y = H^0([a,b]) = \mathbb{L}^2([a,b]) := \{u : [a,b] \to \mathbb{R}; \|u\|_Y^2 = \int_a^b u^2(x)dx < +\infty\}$

$T = D^k$, *the k^{th} derivative operator ;*

$$\mathfrak{A}(u) = \begin{pmatrix} u(x_1) \\ . \\ . \\ . \\ u(x_N) \end{pmatrix}$$

with

$$a < x_1 < x_2 < ... < x_N < b.$$

We are then brought to look for a function u such that $u(x_i) = y_i, 1 \leq i \leq N$ and

$$\int_a^b (u^k(x))^2 dx$$

is minimum. Since $\mathfrak{N}(T) = \mathcal{P}_{k-1}$ is a finite dimension k and

$$\mathfrak{N}(T)\bigcap\mathfrak{N}(\mathfrak{A}) = \{ \text{ polynomials of degree } (k-1) \text{ that nullify } N \text{ time } \} \\ = \{0\}, N \geq k,$$

we have the existence and uniqueness of a solution (that we know besides how to be the spline function s defined previously).

2.5. Projection Method for Building $s(x)$

We have, to define s, conditions at once in $X : s(x_i) = y_i$ and in $Y : \|Ts\|_Y$ minimum.

The method consists in throwing the first conditions in Y (namely $s^{(k)}$) by means of Peano's theorem a simplified version of which we give :

Theorem 2.6 *(Peano) Let $L : C([a,b]) \to \mathbb{R}$ be a continuous linear application, that we can take, to simplify, of the form*

$$L(f)(x) = \sum_{j=1}^{p} b_j f(x_j), x_j \in [a,b]$$

and which has for kernel $\mathfrak{N}(L) = \mathcal{P}_m$, the set of polynomials of degree $\leq m$, then for all f such that $f^{(m+1)} \in \mathbb{L}^2(]a,b[)$:

$$L(f)(y) = (\tfrac{1}{m!}) \int_a^b L_y[(y-x)_+^m] f^{(m+1)}(x)dx.$$

Proof. *Write Taylor's formula with remainder integral and apply L to both members. Let us consider in particular divided differences functional of order k on points $x_i, x_{i+1}, ..., x_{i+k}$.*

This functional is zero on $\mathcal{P}_{k-1}$, then for $u \in C^{k-1}([a,b]), u^{(k)} \in \mathbb{L}^2([a,b])$, one has

$$L_u = [u(x_i), u(x_{i+1}), ...u(x_{i+k})] \qquad \text{with} \quad 1 \leq i \leq N-k$$

$$= \tfrac{1}{(k-1)!} \int_a^b [(x_i - x)_+^{k-1} ... (x_{i+k} - x)_+^{k-1}] u^{(k)}(x)dx$$

because we work on abscissas $x_1, x_2, ..., x_N$. Let us remind that

$$[u(x_i), u(x_{i+1}), ...u(x_{i+k})] = \sum_{j=0}^{k} \tfrac{u(x_{i+j})}{w_i'(x_{i+j})}$$

where

$$w_i(x) = \prod_{j=0}^{k} (x - x_{i+j}) \text{ and } w_i(x_1) = \prod_{j=i, j\neq 1}^{i+k} (x_1 - x_j).$$

This formula is convenient for the reasoning, but for the effective calculation, we shall use the algorithm of calculation of the divide differences by recurrence. •*Put*

$$(2.3)\ \varphi_i(x) = \frac{1}{(k-1)!}[(x_i - x)_+^{k-1} ... (x_{i+k} - x)_+^{k-1}] = \frac{1}{(k-1)!}\sum_{j=0}^{k} \frac{(x_{i+j}-x)_+^{k-1}}{w_i'(x_{i+j})}.$$

We have the following properties of functions $\varphi_i(x)$.

Proposition 2.7 $\varphi_i(x)$ *is* > 0 *on* $]x_i, x_{i+k}[$ *and has* $[x_i, x_{i+k}]$ *for support.*

Proof. Indeed,

- *for* $x \leq x_i$, *we can remove all the signs* $+$ *in the expression (2.3), but then* $\varphi_i(x)$ *expresses himself as divided difference of order* k *of a polynomial of degree* $(k-1)$, *which is zero;*
- *for* $x > x_{i+k}$, *all the terms* $(x - x_j)_+^{k-1}$ *are zero by definition. On the other hand, as*

$$\varphi_i(x_i) = \varphi_i(x_{i+k}) = 0,$$

by Rolle's theorem :

$$\varphi_i' \text{ nullifies at least } 1 \text{ time on }]x_i,\, x_{i+k}[\ ;$$

$$\varphi_i'' \text{ nullifies at least } 2 \text{ time on }]x_i,\, x_{i+k}[\ ;$$

$$\varphi_i^{(k-2)} \text{ nullifies at least } (k-2) \text{ time on }]x_i,\, x_{i+k}[\ .$$

But

$$\varphi_i^{(k-2)}(x) = (-1)^{k-2}\sum_{j=i}^{i+k}\frac{(x_j - x)_+}{w_i'(x_j)}$$

nullifies exactly $(k-2)$ *time on* $]x_i,\, x_{i+k}[$.

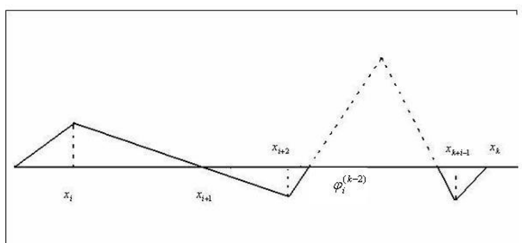


Fig.2.

Thus in fact $\varphi_i^{'}$ nullifies exactly 1 time on $]x_i,\, x_{i+k}[$. On the other hand, for $x \in [x_{i+k-1},\, x_{i+k}[$, one has

$$\varphi_i(x) = \frac{1}{(k\,-\,1)!}\,\frac{(x_{i+k}\,-\,x)_+^{k-1}}{w_i^{'}(x_{i+k})} > 0$$

because

$$w_i^{'}(x_{i+k}) = \prod_{j=i,\, j\neq i+k}^{i+k} (x_{i+k} - x_j) > 0$$

therefore

$$\varphi_i(x) > 0 \quad on \quad]x_i,\ x_{i+k}[\,.$$

Proposition 2.8 *$\varphi_i(x), \quad 1 \leq i \leq N-k,$ are Linearly independents.*
Proof. Indeed, let

$$\sum_{i=1}^{N-k} \alpha_i\,\varphi_i(x) = 0 \quad \forall x,$$

then

- for $x \in [x_1,\ x_2[$ we have $\alpha_1\,\varphi_1(x) = 0 \quad \forall x,$ but

$$\varphi_1(x) > 0 \quad \Longrightarrow \alpha_1 = 0\,;$$

- for $x \in [x_2,\ x_3[$ we have $\alpha_1\,\varphi_1(x) + \alpha_2\,\varphi_2(x) = 0\,\forall x$, but

$$\varphi_2(x) > 0 \quad \Longrightarrow \alpha_2 = 0$$

... etc

$$\alpha_1 = \alpha_2 = ... = \alpha_{N-k} = 0.$$

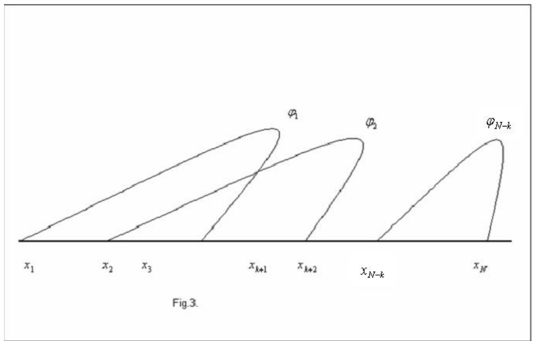
It holds the following theorem :

Theorem 2.9 *If $f \in \mathcal{C}([a,\, b])$ is a function such that*

$$f(x_i) = y_i \quad 1 \leq i \leq N,$$

the spline function s of interpolation of f at points (x_i, f_i) is such that

$$\int_{x_i}^{x_{i+k}} \varphi_i(x)\, s^{(k)}(x)dx = z_i, \quad 1 \leq i \leq (N-k)$$



where

$$z_i = [f(x_i), ..., f(x_{i+k})] = [s(x_i), ..., s(x_{i+k})] .$$

Proof. Apply the Theorem 2.6 and properties of φ_i.

Then, we have back all the conditions in $Y = H^0([a, b]) = L^2([a, b])$. We look for an element $v^* = \{s^{(k)}\} \in Y$ such that :

1) one has

$$(\varphi_i, v^*)_Y = \int_{x_i}^{x_{i+h}} \varphi_i(x)\, v^*(x)\, dx = z_i, \quad 1 \leq i \leq (N-k) ;$$

2) and $\|v^*\|_Y$ is minimum.

To resolve this problem, we have the following theorem :

Theorem 2.10 *Let Y be a Hilbert space, φ_i $1 \leq i \leq m$, m linearly independent elements of Y engendering the linear subspace W of dimension m. Let*

$$\Delta_z = \{v \in Y : (\varphi_i, v)_Y = z_i, \ 1 \leq i \leq m\} .$$

Then element $v^* \in \Delta_z$ (which is of a minimum norm) belongs to W.

Proof. Indeed, let

$$W^\perp = \{v \in Y : \forall w \in W, \ (v, w)_Y = 0\} = \{v \in Y : \forall i = 1, ..., m, \ (v, \varphi_i)_Y = 0\}$$

it is the orthogonal complementary of W in Y. But Δ_z is a translation of $W^\perp$, namely, $\exists \rho \in \Delta_z$ such that

$$\Delta_z = \rho + W^\perp$$

and any element v of Δ_z can be written $v = \rho + \lambda$ with $\lambda \in W^\perp$.

The element $v^* \in \Delta_z$ which is of a minimum norm, it is the one whose distance to O is minimum. It is the unique orthogonal projection of O on Δ_z, then

$$v^* \perp W^{\perp} \Longrightarrow v^* \in W \Longrightarrow \exists c_i \in \mathbb{R}\,;\ v^* = \sum_{i=1}^{m} c_i \varphi_i$$

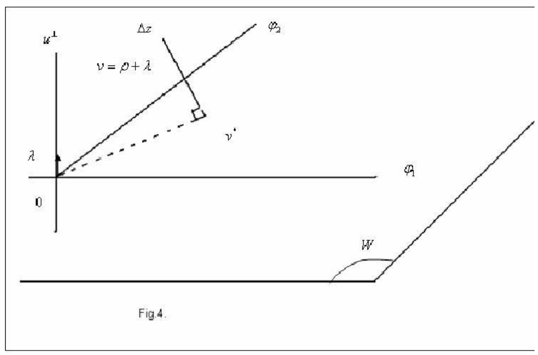


Fig 4.

2.6. Some Applications

We have as applications :

Proposition 2.11 *For* $Y = \mathbb{L}^2([a,\, b]),\ m = N - k,\ \varphi_i$ *defined by* (2.3), *it holds that*

(2.4)

$$s^{(h)}(x) = \sum_{j=1}^{N-k} c_j\, \varphi_j(x)$$

and by Theorem 2.9, the coefficients c_j *are solutions of linear system :*

(2.5)

$$\sum_{j=1}^{N-k} c_j \int_a^b \varphi_i(x)\, \varphi_j(x)\, dx = z_i, \quad 1 \le i \le N - k.$$

To obtain $s(x)$ it will be enough to integrate k time $s^{(k)}$ but previously, it is necessary to calculate explicitly the integral (2.5), *namely* $(\varphi_i, \varphi_j)_Y$.

- **Calculus of** $(\varphi_i, \varphi_j)_Y$:

Let us notice at first that

$$e_+^m = e^m + (-1)^{m+1}(-e)_+^m$$

because for $e < 0$, one has

$$(-1)^m |e|^m + (-1)^{m+1} |e|^m = 0,$$

then

$$\varphi_i(x) = \frac{1}{(k-1)!}\sum_{p=i}^{i+k}\frac{(x_p - x)_+^{k-1}}{w_i'(x_p)}$$

and

$$\varphi_j(x) = \frac{1}{(k-1)!}\sum_{r=j}^{j+k}\frac{(x-x_r)^{k-1}}{w_j'(x_r)} = \frac{(-1)^k}{(k-1)!}\sum_{r=j}^{j+k}\frac{(x-x_r)_+^{k-1}}{w_j'(x_r)},$$

because

$$(y-x)_+^{k-1} = (y-x)^{k-1} + (-1)^k(x-y)_+^{k-1}$$

and the first term gives 0. Therefore

$$(\varphi_i, \varphi_j)_Y = \frac{(-1)^k}{((k-1)!)^2}\sum_{p=i}^{i+k}\sum_{r=j}^{j+k}\frac{1}{w_i'(x_p)\, w_j'(x_r)}\int_a^b (x_p - x)_+^{k-1}(x-x_r)_+^{k-1}dx$$

By definition of e_+^m, we have :

$$I = \int_a^b (x_p - x)_+^{k-1}(x-x_r)_+^{k-1}dx = \begin{cases} \int_{x_r}^{x_p} (x_p - x)^{k-1}(x-x_r)^{k-1}dx & \text{if } r < p, \\ 0 & \text{else.} \end{cases}$$

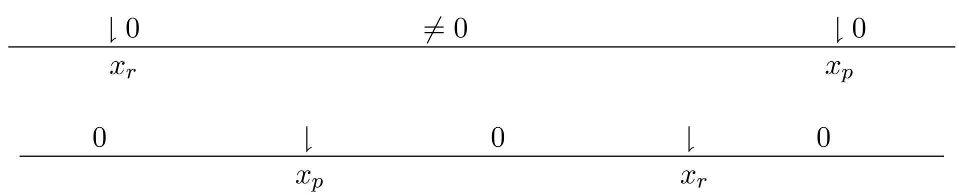


Fig.5.

For the case $r < p$,

- Let us calculate first time by parts :

  $u = (x_p - x)^{k-1} \Longrightarrow du = -(k-1)(x_p - x)^{k-2}dx$

  $dv = (x - x_r)^{k-1}dx \Longrightarrow v = \frac{1}{k}(x - x_r)^k.$

  $I = \underbrace{uv|_{x_r}^{x_p}}_{=0} + \frac{k-1}{k}\int\limits_{x_r}^{x_p}(x - x_p)^{k-2}(x - x_r)^k dx = \frac{k-1}{k}\int\limits_{x_r}^{x_p}(x - x_p)^{k-2}(x - x_r)^k dx\,;$

- second time by parts :

$$I = \frac{(k-1)(k-2)}{k(k+1)}\int\limits_{x_r}^{x_p}(x_p - x)^{k-3}(x - x_r)^{k+1}dx\,;$$

- later $(k-1)$ integrations by parts:

$$I = \frac{(k-1)!}{k(k+1)...(2k-2)}\int\limits_{x_r}^{x_p}(x - x_r)^{2k-2}dx,$$

and finally

$$I = \begin{cases} \frac{((k-1)!)^2}{(2k-1)!}(x_p - x_r)^{2k-1} & \text{if} \quad p > r, \\ 0 & \text{else,} \end{cases}$$

then

$$(\varphi_i, \varphi_j)_Y = \frac{(-1)^k}{(2k-1)!}\sum_{p=i}^{i+k}\frac{1}{w_i'(x_p)}\sum_{r=j}^{j+k}\frac{(x_p - x_r)_+^{2k-1}}{w_j'(x_r)}.$$

If we put

$$h_j(y) = \left[(y - x_j)_+^{2k-1}, ..., (y - x_{j+k})_+^{2k-1}\right],$$

we deduce that

(2.6)

$$(\varphi_i, \varphi_j)_Y = \frac{(-1)^k}{(2k-1)!}\left[h_j(x_i), ..., h_j(x_{i+k})\right].$$

Remark 2.3 (concerning the system (2.5)) Because of the supports of φ_i and φ_j, one has

$$(\varphi_i, \varphi_j)_Y = 0 \text{ if } |j-i| \geq k.$$

The matrix is symmetric, $(2k-1)$ diagonal, regular and generally very well conditioned.

- **Calculus of** $s(x)$:

We had by the formula (2.4)

$$s^{(k)}(x) = \sum_{j=1}^{N-k} C_j \varphi_j(x).$$

If we put

$$C_j^* = \frac{(-1)^k}{(2k-1)!} C_j,$$

the C_j^* are given by the formula (2.5) as a solution of system

$$\sum_{j=1}^{N-k} C_j^* \left[h_j(x_i), ..., h_j(x_{i+k}) \right] = z_i, \qquad 1 \leq i \leq N-k.$$

Besides, we saw that

$$\varphi_j(x) = \frac{(-1)^k}{(k-1)!} \sum_{r=j}^{j+k} \frac{(x-x_r)_+^{k-1}}{w_j'(x_r)},$$

then, we can write

$$s^{(k)}(x) = \frac{(2k-1)!}{(k-1)!} \sum_{j=1}^{N-k} C_j^* \sum_{r=j}^{j+k} \frac{(x-x_r)_+^{k-1}}{w_j'(x_r)}.$$

The k^{th} primitive of $s^{(k)}$ is in the form

$$s(x) = P_{k-1}(x) + \frac{(2k-1)!}{(k-1)!} \frac{1}{k(k+1)...(2k-1)} \sum_{j=1}^{N-k} C_j^* \sum_{r=j}^{j+k} \frac{(x-x_r)_+^{2k-1}}{w_j'(x_r)},$$

or still, by definition of h_j :
(2.7)

$$s(x) = P_{k-1}(x) + \sum_{j=1}^{N-k} C_j^* h_j(x).$$

To determine the coefficients a_0, a_1, ..., a_{k-1} of P_{k-1} (which are for the moment the arbitrary constants of integration), it is enough to write that :

$$s(x_i) = y_i \text{ for } k \text{ values of } i \text{ taken in } \{1, 2, ..., N\}.$$

For example, the k first ones, we have then the system :

(2.8)
$$\sum_{j=0}^{k-1} (a_j(x_i))^j = y_i - \sum_{j=1}^{N-k} C_j^* h_j(x_i), \qquad 1 \le i \le k.$$

To show the sufficiency of the expression (2.7) to resolve the initial problem it is enough to verify that then we have

$$s(x_i) = y_i \text{ for } i = k+1, ..., N.$$

Indeed, one has

$$\int_a^b \varphi_i(x)\, s^{(k)}(x)\, dx = z_i = [s(x_i), ..., s(x_{i+k})] = [f(x_i), ..., f(x_{i+k})], \quad 1 \le i \le N-k$$

and we suppose that

$$s(x_i) = y_i \quad \text{for } 1 \le i \le k.$$

Then,

- for $i = 1$,
$$\sum_{j=1}^{k+1} \frac{s(x_j)}{w_1'(x_j)} = \sum_{j=1}^{k+1} \frac{y_j}{w_1'(x_j)} \Longrightarrow s(x_{i+1}) = y_{i+1}$$
... etc ;
- for $i = N-k$,
$$\sum_{j=N-k}^{N} \frac{s(x_j)}{w_{N-k}'(x_j)} = \sum_{j=N-k}^{N} \frac{y_j}{w_{N-k}'(x_j)} \Longrightarrow s(x_N) = y_N.$$

Remark 2.4: *a*) For $k = 1$, the function $s(x)$ is the joining broken line of (x_i, y_i) with

$$\begin{cases} s(x) = y_1 \text{ on } [a, x_1] \\ \text{and} \\ s(x) = y_N \text{ on } [x_N, b]. \end{cases}$$

b) The value $k = 2$ is most used ($\min \int_a^b (s''(x))^2\,dx$ corresponds to a minimization of energy), the spline is then formed by polynomials of the 3^{rd} degree linking with the order 2 (cubic splines), and of the 1^{st} degree in the extremities.
c) If $N = k$, $s(x)$ is reduced to the polynomial of interpolation $\Pi_{N-1}(x)$ on points $(x_i,\, y_i)$, then we have

$$\int_a^b (\Pi_{N-1}^{(k)}(x))^2\,dx = 0.$$

d) If $k > N$, $s(x)$ is not defined any more in a unique way :

$$s(x) = \Pi_{N-1}(x) + \text{polynomial of degree } (2k-1),$$

where polynomial of degree $(2k-1)$ is zero in x_i.

e) It is evident, by decomposing

$$\sum_{j=1}^{N-k} C_j^* \sum_{r=j}^{j+k} \frac{(x-x_r)_+^{2k-1}}{w_j'(x_r)}$$

that $s(x)$ is a natural spline function of degree $(2k-1)$.
Theorem 2.12 *(numerical calculus of spline functions of order 2) According to Definition 2.1, the looked function is a polynomial of degree 3 on every interval $[x_{n-1},\, x_n]$, $n = 2, ..., N$. Let us note p_n this polynomial and take as unknowns the values m_n of p_n'' in x_n and m_{n-1} of p_n'' in x_{n-1}. One hase*

$$p_n''(x) = m_n\,\frac{x_{n-1}-x}{-h_n} + m_{n-1}\frac{x_n - x}{h_n} \qquad \textit{with } h_n = x_n - x_{n-1}.$$

Then we have

1)
$$p_n(x) = m_n\,\tfrac{(x_{n-1}-x)^3}{-6h_n} + m_{n-1}\,\tfrac{(x_n-x)^3}{6h_n} + \tfrac{x-x_{n-1}}{h_n}\,(y_n - m_n\,\tfrac{h_n^2}{6}\,) + \tfrac{x_n-x}{h_n}(y_{n-1} - m_{n-1}\,\tfrac{h_n^2}{6}).$$

2) *By writing the conditions of connecting of the first derivatives in points x_n, $n = 1, ..., N$, then the numbers m_n, $n = 2, ..., N-1$ are the solutions of the linear system $AM = k$, where A is a tridiagonal symmetric matrix. It has the property of strictly diagonally dominant matrix :*

α) *diagonal terms a_{ii} are positive ;*

β) *one has*

$$|a_{ii}| = \frac{h_i + h_{i+1}}{3} > \sum_{j\neq i,\, j=2}^{N-1} |a_{ij}| = \frac{h_i + h_{i+1}}{6}.$$

It is thus invertible.
Proof. Indeed,
1) one has

(2.9)
$$p_n^{''}(x) = m_n\,\frac{x_{n-1}-x}{-h_n} + m_{n-1}\frac{x_n - x}{h_n}$$
and $h_n = x_n - x_{n-1}$. Namely
$$p_n^{''}(x) = \frac{m_n}{h_n}\,x - \frac{m_n}{h_n}x_{n-1} - \frac{m_{n-1}}{h_n}x + m_{n-1}\frac{x_n}{h_n}$$
By integration of (2.9) once, then

(2.10)
$$p_n^{'}(x) = \frac{m_n}{2h_n}\,x^2 - m_n\frac{x_{n-1}}{h_n}x - \frac{m_{n-1}}{2h_n}x^2 + m_{n-1}\frac{x_n}{h_n}x + c$$
By integration of (2.10) one, then
$$p_n(x) = \frac{m_n}{6h_n}\,x^3 - m_n\frac{x_{n-1}}{2h_n}x^2 - \frac{m_{n-1}}{6h_n}x^3 + m_{n-1}\frac{x_n}{2h_n}x^2 + cx + d.$$
Or
$$(a-b)^3 = a^3 - 2a^2b + 2ab^2 - b^3$$
namely
$$(x_{n-1}-x)^3 = x_{n-1}^3 - 2x_{n-1}^2x + 2x_{n-1}x^2 - x^3.$$
Hence
$$p_n(x) = m_n\frac{(x_{n-1}-x)^3}{-6h_n} + m_{n-1}\frac{(x_n-x)^3}{6h_n} + C_n(x_{n-1}-x) + D_n(x_n-x),$$
and
$$p_n^{'}(x) = m_n\tfrac{(x_{n-1}-x)^2}{2h_n} - m_{n-1}\tfrac{(x_n-x)^2}{2h_n} - C_n - D_n,$$
$$p_n^{''}(x) = m_n\tfrac{(x_{n-1}-x)}{-h_n} + m_{n-1}\tfrac{(x_n-x)}{h_n}.$$
By integrating twice this relation, we obtain
$$p_n(x) = m_n\frac{(x_{n-1}-x)^3}{-6h_n} + m_{n-1}\frac{(x_n-x)^3}{6h_n} + C_n(x_{n-1}-x) + D_n(x_n-x).$$
Determine C_n and D_n. If we put $x = x_n$, then
$$p_n(x_n) = m_n\frac{(x_{n-1}-x_n)^3}{-6h_n} + m_{n-1}\frac{(x_n-x_n)^3}{6h_n} + C_n(x_{n-1}-x_n) + D_n(x_n-x_n),$$

therefore

$$x_n = m_n \frac{h_n^2}{6} - C_n h_n.$$

If we take $x = x_{n-1}$, then

$$p_n(x_{n-1}) = m_n \tfrac{(x_{n-1}-x_{n-1})^3}{-6h_n} + m_{n-1} \tfrac{(x_n-x_{n-1})^3}{6h_n} + C_n(x_{n-1} - x_{n-1}) + D_n(x_n - x_{n-1}),$$

$$= m_{n-1} \tfrac{(x_n-x_{n-1})^3}{6h_n} + D_n(x_n - x_{n-1})$$

therefore

$$y_{n-1} = m_{n-1} \frac{h_n^2}{6} + D_n h_n \,;\; C_n = \frac{1}{h_n}(m_n \frac{h_n^2}{6} - y_n) \,;\; D_n = \frac{1}{h_n}(y_{n-1} - m_{n-1} \frac{h_n^2}{6}).$$

It holds that

$$p_n(x) = m_n \tfrac{(x_{n-1}-x)^3}{-6h_n} + m_{n-1} \tfrac{(x_n-x)^3}{6h_n} + \tfrac{x-x_{n-1}}{h_n}(y_n - \tfrac{m_n}{6} h_n^2)$$

$$+ \tfrac{x_n-x}{h_n}(y_{n-1} - m_{n-1} \tfrac{h_n^2}{6}).$$

2) We notice that by construction the second derivatives link in x_n, $n = 1, ..., N$. We write then the conditions of connecting of the first derivatives in points x_n, $n = 1, ..., N$: $p_n^{'}(x_n) = p_{n+1}^{'}(x_n)$. We have

$$p_n(x) = m_n \tfrac{(x_{n-1}-x)^3}{-6h_n} + m_{n-1} \tfrac{(x_n-x)^3}{6h_n} + \tfrac{x-x_{n-1}}{h_n}(y_n - \tfrac{m_n}{6} h_n^2) + \tfrac{x_n-x}{h_n}(y_{n-1} - m_{n-1} \tfrac{h_n^2}{6}).$$

$$p_n^{'}(x) = m_n \tfrac{(x_{n-1}-x)^2}{2h_n} - m_{n-1} \tfrac{(x_n-x)^2}{2h_n} + \tfrac{1}{h_n}(y_n - m_n \tfrac{h_n^2}{6}) - \tfrac{1}{h_n}(y_{n-1} - m_{n-1} \tfrac{h_n^2}{6}).$$

and

$$p_{n+1}(x) = m_{n+1} \tfrac{(x_n-x)^3}{-6h_{n+1}} + m_n \tfrac{(x_{n+1}-x)^3}{6h_{n+1}}$$

$$+ \tfrac{x-x_n}{h_{n+1}}(y_{n+1} - m_{n+1} \tfrac{h_{n+1}^2}{6}) + \tfrac{x_{n+1}-x}{h_{n+1}}(y_n - m_n \tfrac{h_{n+1}^2}{6}).$$

$$p_{n+1}^{'}(x) = m_{n+1} \tfrac{(x_n-x)^2}{2h_{n+1}} - m_n \tfrac{(x_{n+1}-x)^2}{2h_{n+1}} + \tfrac{1}{h_{n+1}}(y_{n+1} - m_{n+1} \tfrac{h_{n+1}^2}{6}) - \tfrac{1}{h_{n+1}}(y_n - m_n \tfrac{h_{n+1}^2}{6})$$

and

$$p_n^{'}(x_n) = m_n \tfrac{(x_{n-1}-x_n)^2}{2h_n} - m_{n-1} \tfrac{(x_n-x_n)^2}{2h_n}$$

$$+ \tfrac{1}{h_n}(y_n - m_n \tfrac{h_n^2}{6}) - \tfrac{1}{h_n}(y_{n-1} - m_{n-1} \tfrac{h_n^2}{6})$$

$$= m_n \tfrac{(x_{n-1}-x_n)^2}{2h_n} + \tfrac{1}{h_n}(y_n - m_n \tfrac{h_n^2}{6}) - \tfrac{1}{h_n}(y_{n-1} - m_{n-1} \tfrac{h_n^2}{6}).$$

But

$$h_n = x_n - x_{n-1}\,; \quad h_{n+1} = x_{n+1} - x_n,$$

hence

$$p_n^{'}(x_n) = m_n \tfrac{h_n}{2} + \tfrac{1}{h_n}(y_n - m_n \tfrac{h_n^2}{6}) - \tfrac{1}{h_n}(y_{n-1} - m_{n-1}\tfrac{h_n^2}{6})$$

$$p_{n+1}^{'}(x_n) = -m_n \tfrac{h_{n+1}}{2} + \tfrac{1}{h_{n+1}}(y_{n+1} - m_{n+1}\tfrac{h_{n+1}^2}{6}) - \tfrac{1}{h_{n+1}}(y_n - m_n \tfrac{h_{n+1}^2}{6}).$$

Now

$$p_n^{'}(x_n) = p_{n+1}^{'}(x_n), \text{ for } n = 1, ..., N,$$

therefore

$$m_n \tfrac{h_n}{2} + \tfrac{y_n}{h_n} - m_n \tfrac{h_n}{6} - \tfrac{y_{n-1}}{h_n} + m_{n-1}\tfrac{h_n}{6} = -m_n \tfrac{h_{n+1}}{2} + \tfrac{y_{n+1}}{h_{n+1}} - m_{n+1}\tfrac{h_{n+1}}{6} - \tfrac{y_n}{h_{n+1}} + m_n \tfrac{h_{n+1}}{6}$$

$$\Longrightarrow m_{n-1}\tfrac{h_n}{6} + m_n \tfrac{h_n + h_{n+1}}{3} + m_{n+1}\tfrac{h_{n+1}}{6} = \tfrac{y_{n+1} - y_n}{h_{n+1}} - \tfrac{y_n - y_{n-1}}{h_n} \text{ for } n = 2, ..., N-1.$$

By hypothesis, we have $m_1 = m_N = 0$.

Therefore the m_n, $n = 2, ...N-1$ are solutions of linear system $AM = k$, with

$$M = (m_2, \, ..., \, m_{N-1})^t, \;\; k = (k_2, \, ..., \, k_{N-1})^t$$

where

$$k_n = \frac{y_{n+1} - y_n}{h_{n+1}} - \frac{y_n - y_{n-1}}{h_n}, \;\; n = 2, \, ..., \, N-1$$

and

$$A = \begin{pmatrix} \frac{h_2+h_3}{3} & \frac{h_3}{6} & 0 & . & . & 0 \\ \frac{h_3}{6} & \frac{h_3+h_4}{3} & \frac{h_4}{6} & 0 & . & 0 \\ 0 & . & . & . & 0 & . \\ . & . & . & . & . & . \\ . & . & . & \frac{h_{N-2}}{6} & \frac{h_{N-2}+h_{N-1}}{3} & \frac{h_{N-1}}{6} \\ 0 & . & . & 0 & \frac{h_{N-1}}{6} & \frac{h_{N-1}+h_N}{3} \end{pmatrix}.$$

A matrix A is tridiagonal and symmetric. It has a property of strictly diagonally dominant matrix, namely :

α) diagonal terms a_{ii} are positive ;

β) $|a_{ii}| = \frac{h_i + h_{i+1}}{3} > \sum\limits_{j=2,\, j\neq i}^{N-1} |a_{ij}| = \frac{h_i + h_{i+1}}{6}$.

therefore it is invertible because A is a matrix with strict diagonal dominance implies A invertible.

Indeed, else there existe $x \neq 0$ such that $\sum\limits_{j\neq i} a_{ij}x_j + a_{ii}x_i = 0$.

Take i such that $|x_i| \geq |x_j|$, $\forall j \neq i$. Then

$$1 \leq \sum_{j \neq i} \frac{|a_{ij}|}{|a_{ii}|} \frac{|x_j|}{|x_i|} \leq \sum_{j \neq i} \frac{|a_{ij}|}{|a_{ii}|} < 1$$

contradiction.

Definition 2.3 We call cubic spline function all function f of class $\mathcal{C}^2$ on $[a, b]$ such that its restriction to each of intervals $[x_{i-1}, x_i]$ is a polynomial of degree ≤ 3.

Theorem 2.13 *Let f be a function of class $\mathcal{C}^4$ on $[a, b]$. Put*

$$M = \sup_{a \leq x \leq b} \left| f^{(4)}(x) \right|.$$

Then

$$|f(x) - s(x)| \leq \frac{7}{8} M h^4,$$

where s is a cubic spline function such that

$$s(x_i) = f(x_i) \ \ \text{for } i = 0, ..., n\,; \ s''(a) = 0 \ ; \ s''(b) = 0.$$

To prove this theorem, we need to the following results :

Lemma 2.14 *Let f be a function defined on $[a, b]$. Put*

$$x_i = a + ih, \quad h = \frac{b-a}{n}\,; \quad x_0 = a, \ x_n = b.$$

Let s be a cubic spline function such that

$$s(x_i) = f(x_i) \text{ for } i = 0, ..., n\,; \quad s''(a) = 0 \ ; \ s''(b) = 0.$$

Let $x \in \mathbb{R}^{n-1}$. Put $\|x\| = \sup_i |x_i|$. If $Ax = y$, where

$$A = \begin{pmatrix} 4 & 1 & 0 & 0 & . & . & 0 \\ 1 & 4 & 1 & 0 & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & 1 & 4 & 1 \\ 0 & . & . & . & . & 1 & 4 \end{pmatrix},$$

then

$$\|x\| \leq \frac{1}{2} \|y\| \ \ \text{and} \ \ \left\|A^{-1}\right\| \leq \frac{1}{2}.$$

Proof. Let $x \in \mathbb{R}^{n-1}$. Put $\|x\| = \sup_i |x_i|$. If $Ax = y$, then $\|x\| \leq \frac{1}{2}\|y\|$ and $\left\|A^{-1}\right\| \leq \frac{1}{2}$, where

$$A = \begin{pmatrix} 4 & 1 & 0 & 0 & . & . & 0 \\ 1 & 4 & 1 & 0 & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & 1 & 4 & 1 \\ 0 & . & . & . & . & 1 & 4 \end{pmatrix}?$$

Indeed, there exists j such that $\|x\| = |x_j|$ and

$$y_j = x_{j-1} + 4x_j + x_{j+1}$$

$$|y_j| \geq 4|x_j| - |x_{j-1}| - |x_{j+1}| \geq 2|x_j| = 2\|x\|$$

But $\|y\| \geq |y_j|$, hence $\|x\| \leq \frac{1}{2}\|y\|$, it holds that $\|A\| \leq \frac{1}{2}$.

A is a tridiagonal and symmetric matrix. It has a strict diagonal dominant therefore it is invertible. It holds that $\left\|A^{-1}\right\| \leq \frac{1}{2}$.

Lemma 2.15 *With the same notations of previous Lemma 2.14, let* $z_i = s''(x_i)$. *Put*

$$z_{i-1} + 4z_i + z_{i+1} = \tfrac{6}{h^2}(f(x_{i-1}) - 2f(x_i) + f(x_{i+1})) = u_i$$

$$R_i = f''(x_{i-1}) + 4f''(x_i) + f''(x_{i+1}) - \tfrac{6}{h^2}(f(x_{i-1}) - 2f(x_i) + f(x_{i+1})) = w_i - u_i$$

$$Az = u, \quad R = Aw - u.$$

Then we have

$$\|R\| \leq \frac{3}{2} M h^2.$$

Proof. Indeed, let $z_i = s''(x_i)$ and

$$z_{i-1} + 4z_i + z_{i+1} = \frac{6}{h^2}(f(x_{i-1}) + f(x_{i+1}) - 2f(x_i)) = u_i.$$

Put

$$R_i = f''(x_{i-1}) + f''(x_{i+1}) + 4f''(x_i) - \frac{6}{h^2}(f(x_{i-1}) + f(x_{i+1}) - 2f(x_i)) = w_i - u_i$$

We have

$$Az = u, \quad R = Aw - u \Longrightarrow A(w - z) = R.$$

Estimate R_i by means of Taylor's formula :

$f(a+h) = f(a) + hf^{'}(a) + \frac{h^2}{2!}f^{''}(a) + ... + \frac{h^n}{n!}f^{(n)}(a) + \frac{h^{n+1}}{(n+1)!}f^{(n+1)}(c),$

with $c \in \,]a, \, a+h[\, , \, h > 0.$

We have

$$f^{''}(x_{i+1}) = f^{''}(x_i + h) = f^{''}(x_i) + hf^{(3)}(x_i) + \frac{h^2}{2}f^{(4)}(\xi_1)$$

and

$$f^{''}(x_{i-1}) = f^{''}(x_i - h) = f^{''}(x_i) - hf^{(3)}(x_i) + \frac{h^2}{2}f^{(4)}(\xi_2).$$

Therefore

$$f^{''}(x_{i-1}) + f^{''}(x_{i+1}) + 4f^{''}(x_i) = w_i = 6f^{''}(x_i) + \frac{h^2}{2}(f^{(4)}(\xi_1) + f^{(4)}(\xi_2))$$

and

$$f(x_{i+1}) = f(x_i + h) = f(x_i) + hf^{'}(x_i) + \tfrac{h^2}{2}f^{''}(x_i) + \tfrac{h^3}{6}f^{(3)}(x_i) + \tfrac{h^4}{24}f^{(4)}(\xi_3)$$

$$f(x_{i-1}) = f(x_i - h) = f(x_i) - hf^{'}(x_i) + \tfrac{h^2}{2}f^{''}(x_i) - \tfrac{h^3}{6}f^{(3)}(x_i) + \tfrac{h^4}{24}f^{(4)}(\xi_4)$$

Then

$$u_i = 6f^{''}(x_i) + \frac{h^2}{4}(f^{(4)}(\xi_3) + f^{(4)}(\xi_4))$$

and

$$R_i = w_i - u_i = \frac{h^2}{4}(2f^{(4)}(\xi_1) + 2f^{(4)}(\xi_2) - f^{(4)}(\xi_3) - f^{(4)}(\xi_4)).$$

It holds that

$$|R_i| \leq \frac{6Mh^2}{4} = \frac{3}{2}Mh^2 \Longrightarrow \|R\| \leq \frac{3}{2}Mh^2,$$

where $M = \sup \left| f^{(4)}(\xi_i) \right|$

Lemma 2.16 *Using the same notations of previous Lemma 2.14 and Lemma 2.15, we have*

$$a) \ \left| f^{''}(x_i) - s^{''}(x_i) \right| \leq \frac{3}{4}M\,h^2.$$

$$b) \ \left\| f^{(3)}(x) - s^{(3)}(x) \right\| \leq 2M\,h.$$

$$c) \ \left\| f^{''}(x) - s^{''}(x) \right\| \leq \frac{7}{4}M\,h^2.$$

Proof. Indeed,

a) Prove that

$$\left|f^{''}(x_i) - s^{''}(x_i)\right| \leq \frac{3}{4}M\,h^2?$$

We have

$$A(w-z) = R \Longrightarrow \|R\| = \|A(w-z)\| \leq \|A\| \,.\, \|w-z\|$$

and (according to Lemma 2.14 and Lemma 2.15) we have

$$(\|w-z\| \leq \frac{1}{2}\|R\|\;;\;\; \|R\| \leq \frac{3}{2}Mh^2) \Longrightarrow \|w-z\| \leq \frac{3}{4}Mh^2.$$

It holds that

$$\left|f^{''}(x_i) - s^{''}(x_i)\right| \leq \frac{3}{4}M\,h^2.$$

b) Prove that

$$\left\|f^{(3)}(x) - s^{(3)}(x)\right\| \leq 2M\,h\,?$$

Indeed, let $x \in [a,\, b]$, there exists i such that $x \in [x_{i-1},\, x_i]$.

Suppose $x \in]x_{i-1},\, x_i[$ such that $z_i = z_{i-1} + hs^{(3)}(x)$, namely

$$s^{(3)}(x) = \frac{z_i - z_{i-1}}{h}.$$

Then

$$f^{(3)}(x) - s^{(3)}(x) = f^{(3)}(x) - \tfrac{z_i - z_{i-1}}{h}$$

$$= f^{(3)}(x) - \tfrac{f^{''}(x_i) - f^{''}(x_{i-1})}{h} + \tfrac{1}{h}(f^{''}(x_i) - z_i - f^{''}(x_{i-1}) + z_{i-1})$$

But

$$\left|f^{''}(x_i) - z_i - f^{''}(x_{i-1}) + z_{i-1}\right| \leq \frac{3}{2}Mh^2$$

and

$$f^{''}(x_i) = f^{''}(x) + (x_i - x)f^{(3)}(x) + \tfrac{1}{2}(x_i - x)^2 f^{(4)}(\xi_1)$$

$$f^{''}(x_{i-1}) = f^{''}(x) + (x_{i-1} - x)f^{(3)}(x) + \tfrac{1}{2}(x_{i-1} - x)^2 f^{(4)}(\xi_2).$$

Therefore

$$f^{''}(x_i) - f^{''}(x_{i-1}) - hf^{(3)}(x) = \tfrac{1}{2}(x_i - x)^2 f^{(4)}(\xi_1) - \tfrac{1}{2}(x_{i-1} - x)^2 f^{(4)}(\xi_2)$$

$$\Longrightarrow \left|f^{''}(x_i) - f^{''}(x_{i-1}) - hf^{(3)}(x)\right| \leq \tfrac{1}{2}\sup(f^{(4)}(\xi_1), f^{(4)}(\xi_2))((x_i - x)^2 + (x_{i-1} - x)^2)$$

$$\leq \tfrac{M}{2}((x_i - x)^2 + (x_{i-1} - x)^2) \leq \tfrac{M}{2}h^2.$$

It holds that

$$\left\|f^{(3)}(x)-s^{(3)}(x)\right\| \leq \frac{M}{2}\frac{h^2}{h}+\frac{3M}{2}\frac{h^2}{h}=2Mh.$$

c) Prove that

$$\left\|f^{''}(x)-s^{''}(x)\right\| \leq \frac{7}{4}M\,h^2?$$

Indeed, let $x \in [a,\, b]$; there exists i such that $|x-x_i| \leq \frac{h}{2}$. We have

$$f^{''}(x)-s^{''}(x)=f^{''}(x_i)-s^{''}(x_i)+\int_{x_i}^{x}(f^{^{(3)}}(t)-s^{^{(3)}}(t))dt$$

and then

$$\left\|f^{''}(x)-s^{''}(x)\right\| \leq \frac{3}{4}Mh^2+\frac{h}{2}2Mh=\frac{7}{4}Mh^2.$$

Lemma 2.17 *Using the same notations of previous Lemma 2.14, Lemma 2.15 and Lemma 2.16, we have*

d) $\left\|f^{'}(x)-s^{'}(x)\right\| \leq \frac{7}{4}M\,h^3;$

e) $\|f(x)-s(x)\| \leq \frac{7}{8}M\,h^4.$

Proof. Indeed,

d) Prove that

$$\left\|f^{'}(x)-s^{'}(x)\right\| \leq \frac{7}{4}M\,h^3.$$

Let

$$f(x)-s(x)=0 \ \text{ for } \ x=x_0,\, x_1,\, ...,\, x_n$$

According to Rolle's theorem, for all i, there exists $\xi_i \in\,]x_{i-1},\, x_i[$ such that $f^{'}(\xi_i)-s^{'}(\xi_i)=0$.

Indeed, let $x \in [a,\, b]$, there exists i such that $x \in [x_{i-1},\, x_i]$ and we have

$$\left\|f^{'}(x)-s^{'}(x)\right\| = \left\|\int_{\xi_i}^{x}(f^{''}(t)-s^{''}(t))dt\right\| \leq \int_{\xi_i}^{x}\left\|f^{''}(t)-s^{''}(t)\right\|dt \leq \frac{7}{4}M\,h^3.$$

e) Prove that

$$\|f(x)-s(x)\| \leq \frac{7}{8}M\,h^4$$

Indeed, let $x \in [a, b]$, there exists i such that $|x - x_i| \leq \frac{h}{2}$ and we have

$$\|f(x) - s(x)\| = \left\| \int_{x_i}^{x} (f^{'}(t) - s^{'}(t))dt \right\| \leq \int_{x_i}^{x} \left\| f^{'}(t) - s^{'}(t) \right\| dt \leq \frac{7}{8} M\, h^4.$$

Proof. (**Theorem 2.13**) The proof results from Lemma 2.14, Lemma 2.15, Lemma 2.16 and Lemma 2.17.

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