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A NOTE ON A NONLINEAR BACKWARD HEAT EQUATION: STABILITY AND ERROR ESTIMATES

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Abstract. We consider the problem of finding, from the final data $u(x,y,T) = \varphi(x,y)$, the initial data $u(x,y,0)$ of the temperature function $u(x,y,t),\ (x,y) \in I \equiv (0,\pi)\times(0,\pi),\ t \in [0,T]$ satisfying the following nonlinear system

$$u_t = u_{xx} + u_{yy} + g(x,y,t,u(x,y,t)), \qquad (x,y,t) \in I \times (0,T),$$
$$u(0,y,t) = u(\pi,y,t) = u(x,0,t) = u(x,\pi,t) = 0, \qquad t \in (0,T).$$

The problem is nonlinear and severely ill-posed. Using the eigenfunction expansion method we shall improve the results of some recent papers [18, 19, 20] and get some new error estimates. A numerical example also shows that the method works effectively.

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1. Introduction

Let T be a positive number. We consider the problem of finding the tempereture $u(x,y,t),\ (x,y,t) \in I \times [0,T]$ such that the following system

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + g(x,y,t,u(x,y,t)) \quad (x,y,t) \in I \times (0,T), \\ u(x,y,t) = 0 \quad (x,y,t) \in \partial I \times [0,T], \\ u(x,y,T) = \varphi(x,y) \quad (x,y) \in I, \end{cases} \tag{1}$$

where $I = (0,\pi)\times(0,\pi)$, ∂I is the boundary of I and $\varphi(x,y), g(z)$ are given. The problem is called the backward heat problem, the backward Cauchy problem or the final value problem.

As we known, the problem is severely ill-posed, i.e., solutions do not always exist, and in the case of existence, these do not depend continuously on the given data. In fact, from small noise contaminated physical measurements, the corresponding solutions have large errors. It makes difficult to numerical calculations. Hence, a regularization is in order. The linear case was studied extensively in the last four decades by many methods. The literature related to the problem is impressive (see, e.g. [2, 5, 8] and the references therein). In the pioneering work [8] in 1967, the authors present, in a heuristic way, the quasi-reversibility method. They approximated the problem by adding a "corrector" into the main equation. In fact, they considered the problem construct explicitly the adjoint A^* of the operator A

$$\begin{aligned} u_t + Au - \epsilon A^* Au &= 0, \quad t \in [0, T], \\ u(T) &= \varphi. \end{aligned}$$

The stability magnitude of the method are of order $e^{c\epsilon^{-1}}$. In [1], the problem is approximated with

$$\begin{aligned} u_t + Au + \epsilon A u_t &= 0, \quad t \in [0, T], \\ u(T) &= \varphi. \end{aligned}$$

The method is useful if we cannot construct clearly the operator A^*. However, the stability order in the case are quite large as in the original quasi-reversibility methods. In [15], using the method, so-called, of stabilized quasi reversibility, the author approximated the problem with

$$\begin{aligned} u_t + f(A)u &= 0, \quad t \in [0, T], \\ u(T) &= \varphi. \end{aligned}$$

He shown that, with appropriate conditions on the "corrector" $f(A)$, the stability magnitude of the method is of order $c\epsilon^{-1}$.

Sixteen years after the pioneering work by Lattes-Lions,in 1983, Showalter [12]presented the quasi-boundary method. He considered the problem

$$\begin{aligned} u_t - Au(t) &= Bu(t), \quad t \in [0, T], \\ u(0) &= \varphi, \end{aligned}$$

and approximated the problem with

$$\begin{aligned} u_t - Au(t) &= Bu(t), \quad t \in [0, T], \\ u(0) + \epsilon u(T) &= \varphi. \end{aligned}$$

He introduced a better stability estimate than the other discussed methods. Clark and Oppenheimer, in their paper [5], used the quasi-boundary method to regularize the backward problem with

$$\begin{aligned} u_t + Au(t) &= 0, \quad t \in [0, T], \\ u(T) + \epsilon u(0) &= \varphi. \end{aligned}$$

The authors shown that the stability estimate of the method is of order ϵ^{-1}. In [6], the quasi-boundary method was used to solve a backward heat equation with integral boundary condition.

For two dimensional homogeneous backward heat, we refer the reader to [4, 9, 10]. Very recently, in [11], J.Liu and his coauthors applied the Tikhonov method to regularized the homogeneous 2-D backward heat. Although we have many works on the linear homogeneous case of the backward heat problem, the literature on the linear nonhomogeneous case and the nonlinear case of the problem are quite scarce. To our knowledge, there are rarely results of treating the 2-D nonhomogeneous and nonlinear cases of the backward problem until now. In 2009, Trong and Tuan [20] regularized the nonhomogeneous 2-D backward heat problem by using the quasi-boundary value method. Very recently, Trong et al [21] established the error estimates in H^2 norm by using the truncation method.

In the present paper, we apply the eigenfunction expansion method to regularize the problem (1) from a new point of view. To form the approximate problem, we don't follow the way of Clark and Oppenheimer. We introduce a new regularized problem in the following integral problem. Our idea is as follows: first, we transform the problem (1) into a integral equation. Then, we approximate the exact solution by replace the instability terms by the stability terms. Finally, some new error estimates are established. Especially, the convergence of the approximate solution at t = 0 is also proved. This is an improvement of many previous results [16, 18, 19, 20].

2. Regularization and error estimate

For the system (1) we have no guarantee that the solutions exists. In the simplest case $g = 0$, the problem (1) has a unique solution if and only if

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} e^{2T(i^2+j^2)}\varphi_{ij}^2 < \infty$$

where $\varphi_{ij} = \frac{4}{\pi^2}\int_0^\pi\int_0^\pi \varphi(x,y)\sin(ix)\sin(jy)$(see [5]). If $g = g(x,y,t)$, (See [22], p.43, Lemma 1) then the problem (1) has a unique solution if and only if

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(e^{T(i^2+j^2)}\varphi_{ij} - \int_0^T e^{s(i^2+j^2)}g_{ij}(s)ds\right)^2 < \infty,$$

where $g_{ij}(s) = \frac{4}{\pi^2}\int_0^\pi\int_0^\pi g(x,y,s)\sin(ix)\sin(jy)dxdy$. When $g = g(x,y,t,u)$, we do not know any general condition under which the problem (1) is solvable. In [18], we present a simple way to check the existence of problem (1)(See Theorem 3.2a, page 239). The main purpose of this paper is to find a stable computation method to approximate the exact solution when it exists. Hence, the regularization techniques are required. Informally, problem (1) can be transformed to the following integral equation (See,e.g., [3],chapter 4)

$$u(x,y,t) = \frac{4}{\pi^2}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(e^{(T-t)(i^2+j^2)}\varphi_{ij} - \int_t^T e^{(s-t)(i^2+j^2)} g_{ij}(u)(s)ds\right)\sin(ix)\sin(jy).$$

The terms $e^{(T-t)(i^2+j^2)}$ and $e^{(s-t)(i^2+j^2)}$ are the unstability cause. Hence, in order to regularize the problem, we have to replace these terms by the better terms. Naturally, we shall replace these terms by $\frac{e^{(T-t)(i^2+j^2)}}{1+\beta(i^2+j^2)e^{T(i^2+j^2)}}$ and $\frac{e^{(s-t)(i^2+j^2)}}{1+\beta(i^2+j^2)e^{T(i^2+j^2)}}$ respectively. Thus, we shall approximate problem (4) by the following integral equation

$$u^\beta(x,y,t) = \frac{4}{\pi^2}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\frac{e^{(T-t)(i^2+j^2)}}{1+\beta(i^2+j^2)e^{T(i^2+j^2)}}\varphi_{ij} - \int_t^T \frac{e^{(s-t)(i^2+j^2)}}{1+\beta(i^2+j^2)e^{T(i^2+j^2)}} g_{ij}(u^\beta)(s)ds\right)\sin(ix)\sin(jy).$$

For a short, we rewrite the equation (4) and (5) respectively as follows

$$u(x,y,t) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(A(i,j,t)\varphi_{ij} - \int_t^T G_{ij}(u)(t,s)ds\right)X_i(x)X_j(y).$$

$$u^\beta(x,y,t) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(A_\beta(i,j,t)\varphi_{ij} - \int_t^T G^\beta_{ij}(u^\beta)(t,s)ds\right)X_i(x)X_j(y) \qquad (2)$$

where we denote for $i,j \in N$, $x,y \in [0,\pi]$

$$X_i(x) = \frac{2}{\pi}\sin(ix), X_j(y) = \frac{2}{\pi}\sin(jy), \varphi_{ij} = \int_I \varphi(x,y)X_i(x)X_j(y)dxdy, \lambda_{ij} = (i^2+j^2).$$

$$\begin{aligned}
A(i,j,t) &= \exp\{(T-t)\lambda_{ij}\}.\\
A_\beta(i,j,t) &= \frac{e^{(T-t)\lambda_{ij}}}{1+\beta(i^2+j^2)e^{T\lambda_{ij}}}.\\
C_\beta(i,j,t,s) &= \frac{\exp\{(s-t-T)(i^2+j^2)\}}{\beta(i^2+j^2)+e^{-T(i^2+j^2)}}.\\
G_{ij}(w)(t,s) &= e^{(s-t)\lambda_{ij}}g_{ij}(w)(s).\\
G^\beta_{ij}(w)(t,s) &= C_\beta(i,j,t,s)g_{ij}(w)(s).
\end{aligned}$$

For $\lambda > 0$, we have the following inequality

$$\frac{1}{\beta\lambda + e^{-T\lambda}} \leq \frac{T}{\beta\left(1+\ln(\frac{T}{\beta})\right)}.$$

The proof of the above inequality can be found on page 4, [20]. Applying this inequality and (3)-(3), we obtain

$$\begin{aligned}
C_\beta(i,j,t,s) &= \frac{\exp\{(s-t-T)(i^2+j^2)\}}{\beta(i^2+j^2)+e^{-T(i^2+j^2)}} \\
&= \frac{e^{(s-t-T)(i^2+j^2)}}{\left(\beta(i^2+j^2)+e^{-T(i^2+j^2)}\right)^{\frac{s-t}{T}}\left(\beta(i^2+j^2)+e^{-T(i^2+j^2)}\right)^{\frac{T+t-s}{T}}} \\
&\leq \frac{e^{(s-t-T)\lambda_{ij}}}{(e^{-T(i^2+j^2)})^{\frac{T+t-s}{T}}}\frac{1}{(\beta\lambda_{ij}+e^{-T(i^2+j^2)})^{\frac{s}{T}-\frac{t}{T}}} \\
&\leq \left(\frac{T}{\beta\left(1+\ln(\frac{T}{\beta})\right)}\right)^{\frac{s}{T}-\frac{t}{T}} = \\
&= \beta^{\frac{t}{T}-\frac{s}{T}}\left(\frac{T}{1+\ln(\frac{T}{\beta})}\right)^{\frac{s}{T}-\frac{t}{T}} \\
&= \beta^{\frac{t}{T}-\frac{s}{T}}(M_\beta)^{\frac{s}{T}-\frac{t}{T}}. \qquad (3)
\end{aligned}$$

where

$$M_\beta = T\left(1+\ln(\frac{T}{\beta}\right)^{-1}.$$

Let $s = T$ in (3) , we get

$$\begin{aligned}
C_\beta(i,j,t,T) &= A_\beta(i,j,t) \\
&= \frac{e^{-t\lambda_{ij}}}{\beta\lambda_{ij}+e^{-T\lambda_{ij}}} \\
&\leq \beta^{\frac{t}{T}-1}(M_\beta)^{1-\frac{t}{T}}. \qquad (4)
\end{aligned}$$

Throught out this paper, denote $\|.\|$ is the norm of $L^2(I)$.
In the section, we shall study the existence, the uniqueness and the stability of a solution of Problem (2). In fact, one has

Theorem 1
Let $\varphi \in L^2(I)$ and let $g \in L^\infty([0,\pi]\times[0,\pi]\times[0,T]\times R)$ satisfy

$$|g(w) - g(v)| \leq k|w - v|$$

for a $k > 0$ independent of w, v. Then Problem (2) *has a unique solution* $u^\beta \in C([0,T]; H_0^1(I)) \cap C^1((0,T); L^2(I))$.

Theorem 2
The solution of the problem (2) *depends continuously on* φ *in* $L^2(I)$.
Theorem 3

Let φ, g be as in Theorem 1. Suppose problem (1) has a unique solution $u \in C([0,T]; H_0^1(I)) \cap C^1((0,T); L^2(I))$ *which satisfies*

$$P = 2 \sup_{0\le t\le T} \left(\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \lambda_{ij}^2 e^{2t\lambda_{ij}} | < u(x,y,t), X_i(x)X_i(y) > |^2 \right) < \infty.$$

Then

$$|||u(.,.,t) - u^\beta(.,.,t) \quad \le \quad \sqrt{P} e^{k^2 T(T-t)} \beta^{\frac{t}{T}} \left(\frac{T}{1 + \ln(\frac{T}{\beta})} \right)^{1-\frac{t}{T}} \tag{5}$$

for every $t \in [0,T]$.
Remark.
1) In [18], the stability estimates is order of $\beta^{\frac{t}{T}}$. *If the time t is close to the original time $t = 0$, the convergence rates here are very slow. This implies that the methods studied in [16, 18] are not useful to derive the error estimations in the case t is near zero. Comparing* (5) *with the previous results obtained in [16, 18], we realize that this estimate is sharp and good estimate. This is also among of strong point of our method. If $t = 0$ then the error* (5) *becomes*

$$|||u(.,.,0) - u^\beta(.,.,0) \quad \le \quad \sqrt{P} e^{k^2 T^2} \left(\frac{T}{1 + \ln(\frac{T}{\beta})} \right). \tag{6}$$

Noting that (6) *is not given in [16, 18]. These estimates, as noted above, are very seldom in the theory of ill-posed problems.*

2)We also note that the condition of solution u in (5) *depend on the nonlinear term g and therefore $g_p, g_p(u)$ are very difficult to be valued. Such an obscurity makes this theorem hard to be used for numerical computations. To improve this, in Theorem 3, we only require the assumption on u, depending not on the function $g(u)$. Infact, in the simplest case $g(x,y,t,u(x,y,t)) = 0$, then*

$$P = 2 \sup_{0\le t\le T} \left(\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \lambda_{ij}^2 e^{2t\lambda_{ij}} | < u(x,y,t), X_i(x)X_i(y) > |^2 \right) = 2\|u_{xx} + u_{yy}\|^2.$$

Hence, this condition is natural and acceptable.

Theorem 4
Let u be the exact solution of (1) corresponding to φ *. Let* φ_β *be a measured data such that*

$$|||\varphi_\beta - \varphi \leq \beta.$$

Then there exists a function w^β *satisfying*

$$|||u(.,.,t) - w^\beta(.,.,t) \quad \leq \quad (2+\sqrt{P})e^{k^2T(T-t)}\beta^{\frac{t}{T}}\left(\frac{T}{1+\ln(\frac{T}{\beta})}\right)^{1-\frac{t}{T}} \tag{7}$$

for every $t \in [0,T]$.

3. Proof of the main results

Proof of Theorem 1.
The existence and the uniqueness of solution of (2).
Now we consider the operator

$$K : C([0,T];L^2(I)) \to C([0,T];L^2(I))$$

defined by

$$K(w)(x,y,t) = \Psi(x,y,t) - \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\int_t^T G_{ij}^{\beta}(w)(t,s)ds\right)X_i(x)X_j(y)$$

where

$$\Psi(x,y,t) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} A_\beta(i,j,t)\varphi_{ij}X_i(x)X_j(y).$$

By induction, we shall prove the following inequality

$$\|K^p(u)(.,.,t) - K^p(v)(.,.,t)\|^2 \leq \left(\frac{k}{\beta}\right)^{2p}\frac{(T-t)^pC^p}{p!}|||u-v|||^2 \tag{8}$$

for every $p \geq 1$, where $C = \max\{T,1\}$ and $|||.|||$ is sup norm in $C([0,T];L^2(I))$.

Thus, for $p = 1$, we have

$$\begin{aligned}
\|K(u)(.,.,t) &- K(v)(.,.,t)\|^2 \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left[\int_t^T \left(G_{ij}^{\beta}(u)(t,s) - G_{ij}^{\beta}(v)(t,s)\right) ds\right]^2 \\
&\le \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_t^T (C_\beta(i,j,t,s))^2\, ds \int_t^T (g_{ij}(u)(s) - g_{ij}(v)(s))^2 ds \\
&\le \frac{1}{\beta^2}(T-t)\int_t^T\int_0^\pi\int_0^\pi (g(u(x,y,s)) - g(v(x,y,s))^2 dxdyds \\
&\le \frac{k^2}{\beta^2}(T-t)\int_t^T\int_0^\pi\int_0^\pi |u(x,y,s) - v(x,y,s)|^2 dxdyds \\
&\le C\frac{k^2}{\beta^2}(T-t)|||u-v|||^2.
\end{aligned}$$

Hence, (8) holds. Let (8) holds for $p = m$. We prove that (8) holds for $p = m + 1$. We have

$$\begin{aligned}
\|K^{m+1}(u)(.,.,t) &- K^{m+1}(v)(.,.,t)\|^2 = \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left[\int_t^T C_\beta(i,j,t,s)\left(G_{ij}(K^m(u))(t,s) - G_{ij}(K^m(v))(t,s)\right) ds\right]^2 \\
&\le \frac{1}{\beta^2}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left[\int_t^T |G_{ij}(K^m(u))(t,s) - G_{ij}(K^m(v))(t,s)| ds\right]^2 \\
&\le \frac{1}{\beta^2}(T-t)k^2\int_t^T \|K^m(u)(.,.,s) - K^m(v)(.,.,s)\|^2 ds \\
&\le \frac{1}{\beta^2}(T-t)k^2\left(\frac{k}{\beta}\right)^{2m}\int_t^T \frac{(T-s)^m}{m!} ds C^m|||u-v|||^2 \\
&\le \left(\frac{k}{\beta}\right)^{2(m+1)}\frac{(T-t)^{m+1}}{(m+1)!}C^{m+1}|||u-v|||^2.
\end{aligned}$$

Therefore

$$|||K^p(u) - K^p(v)||| \le \left(\frac{k}{\beta}\right)^p \frac{T^{p/2}}{\sqrt{p!}}C^p|||u-v|||$$

for all $u, v \in C([0,T]; L^2(I))$.

Since $\lim\limits_{p\to\infty} \left(\frac{k}{\beta}\right)^p \frac{T^{p/2}C^p}{\sqrt{p!}} = 0$, there exists a positive integer number p_0, such that K^{p_0} is a contraction. It follows that the equation $K^{p_0}(u) = u$ has a unique solution $u^\beta \in C([0,T]; L^2(I))$. We claim that $K(u^\beta) = u^\beta$. In fact, one has

$$K(K^{P_0}(u^\beta)) = K(u^\beta).$$

Hence

$$K^{P_0}(K(u^\beta)) = K(u^\beta).$$

By the uniqueness of the fixed point of G^{P_0}, one has $G(u^\beta) = u^\beta$, i.e., the equation $G(u) = u$ has a unique solution $u^\beta \in C([0,T]; L^2(I))$. The proof is completed.

Proof of Theorem 2. Let u and v be two solutions of (2) corresponding to the values φ and ω.

From (2) one has

$$\begin{aligned} \|u(.,.,t) - v(.,.,t)\|^2 \;\; \le \;\; & \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(A_\beta(i,j,t)|\varphi_{ij} - \omega_{ij}|)^2 \\ & + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\int_t^T C_\beta(i,j,t,s)|g_{ij}(u)(s) - g_{ij}(v)(s)|ds\right)^2 \end{aligned} \quad (9)$$

It follows from (21) that

$$\begin{aligned} \|u(.,.,t) - v(.,.,t)\|^2 \;\; & \le \;\; 2\beta^{\frac{2t}{T}-2}(M_\beta)^{2-\frac{2t}{T}}\|\varphi - \omega\|^2 + \\ & \le \;\; 2k^2(T-t)\beta^{\frac{2t}{T}}(M_\beta)^{2-\frac{2t}{T}}\int_t^T \beta^{\frac{-2s}{T}}(M_\beta)^{\frac{2s}{T}-2}\|u(.,.,s) - v(.,.,s)\|^2 ds. \end{aligned}$$

Hence

$$\begin{aligned} \beta^{\frac{-2t}{T}}(M_\beta)^{\frac{2t}{T}-2}\|u(.,.,t) - v(.,.,t)\|^2 \;\; & \le \;\; 2\beta^{-2}\|\varphi - \omega\|^2 \\ & + \;\; 2k^2(T-t)\int_t^T \beta^{\frac{-2s}{T}}(M_\beta)^{\frac{2s}{T}-2}\|u(.,.,s) - v(.,.,s)\|^2 ds. \end{aligned}$$

By using Gronwall's inequality, we find that

$$\|u(.,.,t) - v(.,.,t)\| \le 2\beta^{\frac{t}{T}-1}(M_\beta)^{1-\frac{t}{T}}\exp(k^2(T-t)^2)\|\varphi - \omega\|.$$

This completes the proof of the theorem.

Proof of Theorem 3.

We have

$$
\begin{aligned}
& |u_{ij}(t) - u_{ij}^{\beta}(t)| \\
\leq\ & \left|(A(i,j,0) - A_\beta(i,j,0)) \left(e^{-t\lambda_{ij}}\varphi_{ij} - \int_t^T e^{(s-t-T)\lambda_{ij}} g_{ij}(u)(s)ds\right)\right| \\
& \qquad + \left|\int_t^T C_\beta(i,j,s,t)(g_{ij}(u)(s) - g_{ij}(u^\beta)(s))ds)\right| \\
\leq\ & \left|\beta(i^2+j^2)A_\beta(i,j,t)\left(e^{T(i^2+j^2)}g_{ij} - \int_t^T e^{s\lambda_{ij}} g_{ij}(u)(s)ds\right)\right| \\
& \qquad + \int_t^T C_\beta(i,j,s,t)|g_{ij}(u)(s) - g_{ij}(u^\beta)(s)|ds \\
\leq\ & \left|\beta A_\beta(i,j,t)\lambda_{ij}e^{t\lambda_{ij}}u_{ij}(t)\right| \\
& \qquad + \int_t^T C_\beta(i,j,s,t)|g_{ij}(u)(s) - g_{ij}(u^\beta)(s)|ds \\
\leq\ & \beta.\beta^{\frac{t}{T}-1}(M_\beta)^{1-\frac{t}{T}}|\lambda_{ij}e^{t\lambda_{ij}}u_{ij}(t)| + \\
& \qquad + \int_t^T \beta^{t/T-1}(M_\beta)^{1-\frac{t}{T}}|g_{ij}(u)(s) - g_{ij}(u^\beta)(s)|ds.
\end{aligned}
$$

It follows from (10) that

$$
\begin{aligned}
\||u(.,.,t) - u^\beta(.,.,t)\|^2 &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|u_{ij}(t) - u_{ij}^\beta(t)|^2 \\
&\leq 2\beta^{\frac{2t}{T}}(M_\beta)^{2-\frac{2t}{T}}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|\lambda_{ij}e^{t\lambda_{ij}}u_{ij}(t)|^2 + \\
& 2\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\int_t^T \beta^{-\frac{s}{T}}(M_\beta)^{\frac{s}{T}-1}|g_{ij}(u)(s) - g_{ij}(u^\beta)(s)|ds\right)^2.
\end{aligned}
$$

This implies

$$
\begin{aligned}
\||u(.,.,t) - u^\beta(.,.,t)\|^2 &\leq 2\beta^{\frac{2t}{T}}(M_\beta)^{2-\frac{2t}{T}}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\lambda_{ij}^2 e^{2t\lambda_{ij}}u_{ij}^2(t) && (10) \\
&+ 2k^2T\beta^{\frac{2t}{T}}(M_\beta)^{2-\frac{2t}{T}}\int_t^T \beta^{-\frac{2s}{T}}(M_\beta)^{\frac{2s}{T}-2}\||u(.,.,s) - u^\beta(.,.,s)\|^2 ds && (11)
\end{aligned}
$$

By using Gronwall's inequality, we get:

$$
\beta^{\frac{-2t}{T}}\left(\frac{T}{1+\ln(\frac{T}{\beta})}\right)^{\frac{2t}{T}-2}\||u(.,.,t) - u^\beta(.,.,t)\|^2 \leq Pe^{2k^2T(T-t)}.
$$

Proof of Theorem 4.
Let w^β and u^β be the solution of problem (7) corresponding to φ_β and φ. Using Theorems 2 and 3, we get

$$\begin{aligned}\|w^\beta(.,.,t)-u(.,.,t)\| &\le \|w^\beta(.,.,t)-u^\beta(.,.,t)\|+\|u^\beta(.,.,t)-u(.,.,t)\|\\ &\le 2\beta^{\frac{t}{T}-1}(M_\beta)^{1-\frac{t}{T}}\exp(k^2(T-t)^2)\|\|\varphi_\beta-\varphi\\ &\qquad\qquad +\sqrt{P}e^{k^2T(T-t)}\beta^{\frac{t}{T}}\left(\frac{T}{1+\ln(\frac{T}{\beta})}\right)^{1-\frac{t}{T}}\\ &\le (2+\sqrt{P})e^{k^2T(T-t)}\beta^{\frac{t}{T}}\left(\frac{T}{1+\ln(\frac{T}{\beta})}\right)^{1-\frac{t}{T}}.\end{aligned}$$

4. Numerical example

Let us consider the two dimensional Allen-Cahn equation as follows

$$\begin{cases} u_t-u_{xx}-u_{yy}=u-u^3+f(x,y,t), \quad (x,y,t)\in(0,\pi)\times(0,\pi)\times(0,1),\\ u(x,y,t)=0 \quad (x,y,t)\in\partial I\times[0,T]\\ u(x,y,1)=\varphi(x,y),\ \ x,y\in(0,\pi)\times(0,\pi)\end{cases} \tag{12}$$

where

$$f(x,y,t)=2e^t\sin x\sin y+e^{3t}\sin^3 x\sin^3 y,$$

and

$$u(x,y,1)=\varphi_0(x,y)\equiv e\sin x\sin y.$$

The exact solution of the latter equation is

$$u(x,y,t)=e^t\sin x\sin y.$$

Especially

$$u\left(x,y,\frac{999}{1000}\right)\equiv u(x,y)=\exp\left(\frac{999}{1000}\right)\sin x\sin y.$$

Denote the regularization parameter $\beta=\epsilon$. Let $\varphi_\epsilon(x,y)\equiv\varphi(x,y)=(\epsilon+1)e\sin x\sin y$.

We have

$$\|\varphi_\epsilon-\varphi\|_2=\sqrt{\int_0^\pi\int_0^\pi\epsilon^2e^2\sin^2(x)\sin^2 y\,dxdy}=\epsilon e\frac{\pi}{2}.$$

We find the regularized solution $u_\epsilon\left(x,y,\frac{999}{1000}\right)\equiv u_\epsilon(x,y)$ having the following form

$$u_\epsilon(x,y)=v_m(x,y) \;=\; w_{11,m}\sin x\sin y+w_{33,m}\sin 3x\sin 3y,$$

where

$$v_1(x,y) = (\epsilon+1)e\sin x\sin y$$
$$w_{11,1} = (\epsilon+1)e,$$
$$w_{12,1} = w_{13,1} = w_{21,1} = w_{22,1} = w_{23,1} = w_{31,1} = w_{32,1} = w_{33,1} = 0.$$

and

$$\begin{cases} a = \frac{1}{40000} \\ t_m = 1 - am \quad m = 1,2,...,40 \\ w_{ij,m+1} = \frac{e^{-t_{m+1}(i^2+j^2)}}{\epsilon(i^2+j^2)+e^{-t_m(i^2+j^2)}} w_{ij,m} - \\ -\frac{4}{\pi^2} \int\limits_{t_{m+1}}^{t_m} \frac{e^{-t_{m+1}(i^2+j^2)}}{\epsilon(i^2+j^2)+e^{-t_m(i^2+j^2)}} \left(\int\limits_0^\pi \int\limits_0^\pi \big(v_m - v_m^3(x,y) + f(x,y,s) \big) \sin ix \sin jy dx dy \right) ds, \\ i,j = 1,2,3. \end{cases}$$

Let $a_\epsilon = \|u_\epsilon - u\|$ be the error between the regularized solution u_ϵ and the exact solution u.

Let $\epsilon = \epsilon_1 = 10^{-5}, \epsilon = \epsilon_2 = 10^{-7}, \epsilon = \epsilon_3 = 10^{-10}$, we have

ϵ	u_ϵ	a_ϵ
$\epsilon_1 = 10^{-5}$	$2.699490181\sin x\sin y$ $-0.0002082242787\sin 3x\sin 3y$	0.01607476736
$\epsilon_2 = 10^{-7}$	$2.715403794\sin x\sin y$ $-0.0002055494193\sin 3x\sin 3y$	0.0001611532506
$\epsilon_3 = 10^{-10}$	$2.715563078\sin x\sin y$ $-0.001936581654\sin 3x\sin 3y$	0.000005577348503

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