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UNIVALENCE OF TWO INTEGRAL OPERATORS

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Abstract. In this work we derive sufficient conditions for the univalence of two integral operators $H_{\alpha,\beta}$ and G_α.

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1. Introduction

We consider the unit open disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{A}$ the class of the functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk $\mathcal{U}$.

Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$, consisting of the functions $f \in \mathcal{A}$, which are univalent in $\mathcal{U}$.

We consider the integral operators

$$H_{\alpha,\beta}(z) = \left[\beta \int_0^z u^{\beta-1} \left(\frac{g(u)}{u}\right)^{\frac{1}{\alpha}} du\right]^{\frac{1}{\beta}}, \tag{1}$$

and

$$G_\alpha(z) = \left[\alpha \int_0^z (h(u))^{\alpha-1} du\right]^{\frac{1}{\alpha}}, \tag{2}$$

for $g,\ h \in \mathcal{A}$ and $\alpha,\ \beta$ be complex numbers, $\alpha \neq 0$, $\beta \neq 0$.

In [3] Pescar has studied the univalence of these integral operators.

In the present paper, we obtain new univalence conditions for the integral operators $H_{\alpha,\beta}$, G_α to be in the class $\mathcal{S}$.

2. Preliminary results

We need the following lemmas.

Lemma 1 (2). . *Let α be a complex number, $Re\ \alpha > 0$ and $f \in \mathcal{A}$.*
If

$$\frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1, \tag{3}$$

for all $z \in \mathcal{U}$, then for any complex number β, $Re\ \beta \geq Re\ \alpha$, the integral operator F_β defined by

$$F_\beta(z) = \left[\beta \int_0^z u^{\beta-1} f'(u)\right]^{\frac{1}{\beta}} \tag{4}$$

is in the class $\mathcal{S}$.

Lemma 2. *(Schwarz [1]). Let f be the function regular in the disk $\mathcal{U}_R = \{z \in \mathbb{C} : |z| < R\}$ with $|f(z)| < M$, M fixed. If $f(z)$ has in $z = 0$ one zero with multiply $\geq m$, then*

$$|f(z)| \leq \frac{M}{R^m}|z|^m,\ (z \in \mathcal{U}_R)\,, \tag{5}$$

the equality (in the inequality (5) for $z \neq 0$) can hold only if

$$f(z) = e^{i\theta}\frac{M}{R^m}z^m,$$

where θ is constant.

3. Main results

Theorem 1. *Let α be a complex number, $Re\ \alpha > 0$, M, L positive real numbers and the function $g \in \mathcal{A}$, $g(z) = z + a_2z^2 + \ldots$.*
If

$$\left|\frac{z^2g'(z)}{g^2(z)} - 1\right| \leq M, \quad (z \in \mathcal{U}), \tag{6}$$

$$|g(z)| \leq L, \quad (z \in \mathcal{U}) \tag{7}$$

and

$$\frac{ML}{(Re\ \alpha+1)^{\frac{Re\ \alpha+1}{Re\ \alpha}}}+\frac{L+1}{Re\ \alpha}\leq|\alpha|, \tag{8}$$

then for any complex number β*,* $Re\ \beta\geq Re\ \alpha$*, the integral operator* $H_{\alpha,\beta}$ *defined by (1) is in the class* $\mathcal{S}$*.*

Proof. Let us consider the function

$$f(z)=\int_0^z\left(\frac{g(u)}{u}\right)^{\frac{1}{\alpha}}du,\quad (z\in\mathcal{U}). \tag{9}$$

The function f is regular in $\mathcal{U}$ and $f(0)=f'(0)-1=0$.
We have

$$f'(z)=\left(\frac{g(z)}{z}\right)^{\frac{1}{\alpha}},$$

$$f''(z)=\frac{1}{\alpha}\left(\frac{g(z)}{z}\right)^{\frac{1}{\alpha}-1}\frac{zg'(z)-g(z)}{z^2},\quad (z\in\mathcal{U}),$$

$$\frac{zf''(z)}{f'(z)}=\frac{1}{\alpha}\left(\frac{zg'(z)}{g(z)}-1\right),$$

hence, we get

$$\frac{zf''(z)}{f'(z)}=\frac{1}{\alpha}\left[\left(\frac{z^2g'(z)}{g^2(z)}-1\right)\frac{g(z)}{z}+\frac{g(z)}{z}-1\right], \tag{10}$$

for all $z\in\mathcal{U}$. From (10) we obtain

$$\frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}\left|\frac{zf''(z)}{f'(z)}\right|\leq\frac{1-|z|^{2Re\ \alpha}}{|\alpha|Re\ \alpha}\left(\left|\frac{z^2g'(z)}{g^2(z)}-1\right|\left|\frac{g(z)}{z}\right|+\left|\frac{g(z)}{z}\right|+1\right) \tag{11}$$

The function $p(z)=\frac{z^2g'(z)}{g^2(z)}-1$ has in $z=0$, one zero with multiply $m=2$ and for the function $g(z)$ we have one zero with multiply $m=1$.

So, from (6), (7) and Lemma 2 we get

$$\left|\frac{z^2g'(z)}{g^2(z)}-1\right|\leq M|z|^2,\quad (z\in\mathcal{U}) \tag{12}$$

and

$$|g(z)| \le L|z|, \quad (z \in \mathcal{U}). \tag{13}$$

From (12), (13) and (11) we obtain

$$\frac{1-|z|^{2Re\,\alpha}}{Re\,\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \le \frac{1-|z|^{2Re\,\alpha}}{|\alpha| Re\,\alpha}|z|^2 ML + \frac{1-|z|^{2Re\,\alpha}}{|\alpha| Re\,\alpha}(L+1), \quad (z \in \mathcal{U}) \tag{14}$$

Since

$$\max_{|z|\le 1}\left[\frac{1-|z|^{2Re\,\alpha}}{Re\,\alpha}|z|^2\right] = \frac{1}{(Re\,\alpha+1)^{\frac{Re\,\alpha+1}{Re\,\alpha}}}, \tag{15}$$

from (8), (15) and (14), we obtain

$$\frac{1-|z|^{2Re\,\alpha}}{Re\,\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \le 1, \quad (z \in \mathcal{U}) \tag{16}$$

and hence, by Lemma 1, it results that the integral operator $H_{\alpha,\beta}$ belongs to the class $\mathcal{S}$. □

Theorem 2. *Let α be a complex number, $Re\,\alpha > 0$, M, L positive real numbers and the function $h \in \mathcal{A}$, $h(z) = z + a_2 z^2 + \ldots$.*

If

$$\left|\frac{z^2 h'(z)}{h^2(z)} - 1\right| \le M, \quad (z \in \mathcal{U}), \tag{17}$$

$$|h(z)| \le L, \quad (z \in \mathcal{U}) \tag{18}$$

and

$$|\alpha - 1|\left[\frac{ML}{(Re\,\alpha+1)^{\frac{Re\,\alpha+1}{Re\,\alpha}}} + \frac{L+1}{Re\,\alpha}\right] \le 1, \tag{19}$$

then the integral operator G_α, defined by (2) belongs to the class $\mathcal{S}$.

Proof. From (2) we have

$$G_\alpha(z) = \left[\alpha \int_0^z u^{\alpha-1}\left(\frac{h(u)}{u}\right)^{\alpha-1} du\right]^{\frac{1}{\alpha}}. \tag{20}$$

We consider the function

$$f(z) = \int_0^z \left(\frac{h(u)}{u}\right)^{\alpha-1} du. \tag{21}$$

The function f is regular in $\mathcal{U}$. From (21) we obtain

$$f'(z) = \left(\frac{h(z)}{z}\right)^{\alpha-1}, \quad f''(z) = (\alpha-1)\left(\frac{h(z)}{z}\right)^{\alpha-2} \frac{zh'(z)-h(z)}{z^2}$$

We get

$$\frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq$$

$$\leq |\alpha-1|\frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}\left(\left|\frac{z^2h'(z)}{h^2(z)}-1\right|\frac{|h(z)|}{|z|}+\frac{|h(z)|}{|z|}+1\right), \tag{22}$$

for all $z \in \mathcal{U}$. By (17), (18), Lemma 2 we obtain

$$\left|\frac{z^2h'(z)}{h^2(z)}-1\right| \leq M|z|^2, \quad (z \in \mathcal{U}),$$

$$|h(z)| \leq L|z|, \quad (z \in \mathcal{U}),$$

hence, by (22) we have

$$\frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq |\alpha-1|\left[\frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}|z|^2ML+\frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}(L+1)\right], \tag{23}$$

for all $z \in \mathcal{U}$.

Since

$$\max_{|z|\leq 1}\left[\frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}|z|^2\right] = \frac{1}{(Re\ \alpha+1)^{\frac{Re\ \alpha+1}{Re\ \alpha}}},$$

from (19) and (23) we get

$$\frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1, \quad (z \in \mathcal{U}). \tag{24}$$

We have $f'(z) = \left(\frac{h(z)}{z}\right)^{\alpha-1}$ and now, by Lemma 1 and (24), for $\beta = \alpha$ we obtain that the integral operator G_α is in the class $\mathcal{S}$.

□

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