Acta Universitatis Apulensis No. 29/2012
ISSN: 1582-5329 pp. 99-103

A SUBORDINATION RESULT FOR A CLASS OF ANALYTIC FUNCTIONS

B.A. Frasin

Abstract.In this paper, we derive a subordination result for the class $\mathcal{A}^{\star}(\beta_1, \beta_2, \beta_3; \lambda)$ of analytic functions, where β_1, β_2 and β_3 are complex numbers and $\lambda > 0$.

2000 *Mathematics Subject Classification*: 30C45.

1.Introduction and definitions

Let $\mathcal{A}$ denote the class of functions of the form :

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disk $\Delta = \{z : |z| < 1\}$. For two functions $f(z)$ given by (1) and $g(z)$ given by

$$g(z) = z + \sum_{n=2}^{\infty} c_n z^n \tag{2}$$

their Hadamard product (or convolution) is defined by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n c_n z^n. \tag{3}$$

Let $\mathcal{A}(\beta_1, \beta_2, \beta_3; \lambda)$ denote the subclass of $\mathcal{A}$ consisting of functions $f(z)$ satisfying

$$\left|\beta_1 z \left(\frac{f(z)}{z}\right)' + \beta_2 z^2 \left(\frac{f(z)}{z}\right)'' + \beta_3 z^3 \left(\frac{f(z)}{z}\right)'''\right| \leq \lambda \qquad (z \in \Delta) \tag{4}$$

for some complex numbers β_1, β_2 and β_3, and for some real $\lambda > 0$. The class $\mathcal{A}(\beta_1, \beta_2, \beta_3; \lambda)$ was considered by Uyanik et al. [6].

Uyanik et al. [6], proved that, if $f(z) \in \mathcal{A}$ satisfies

$$\sum_{n=2}^{\infty}(n-1)(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|)|a_n| \leq \lambda \tag{5}$$

for some complex numbers β_1, β_2 and β_3, and for some real $\lambda > 0$, then $f(z) \in \mathcal{A}(\beta_1, \beta_2, \beta_3; \lambda)$.

Let us denote by $\mathcal{A}^\star(\beta_1, \beta_2, \beta_3; \lambda)$, the class of functions $f(z) \in \mathcal{A}$ whose coefficients satisfy the condition (5). We note that $\mathcal{A}^\star(\beta_1, \beta_2, \beta_3; \lambda) \subseteq \mathcal{A}(\beta_1, \beta_2, \beta_3; \lambda)$.

In this paper, we prove a subordination result for the class $\mathcal{A}^\star(\beta_1, \beta_2, \beta_3; \lambda)$.

Before we state and prove our main result we need the following definitions and lemma.

Definition 1. (Subordination Principle). *Let $g(z)$ be analytic and univalent in Δ. If $f(z)$ is analytic in Δ, $f(0) = g(0)$, and $f(\Delta) \subset g(\Delta)$, then we see that the function $f(z)$ is subordinate to $g(z)$ in Δ, and we write $f(z) \prec g(z)$.*

Definition 2. (Subordinating Factor Sequence). *A sequence $\{b_n\}_{n=1}^{\infty}$ of complex numbers is called a subordinating factor sequence if, whenever $f(z)$ is analytic, univalent and convex in Δ, we have the subordination given by*

$$\sum_{n=2}^{\infty} b_n a_n z^n \prec f(z) \qquad (z \in \Delta,\ a_1 = 1). \tag{6}$$

Lemma 1. ([7]). *The sequence $\{b_n\}_{n=1}^{\infty}$ is a subordinating factor sequence if and only if*

$$\operatorname{Re}\left\{1 + 2\sum_{n=1}^{\infty} b_n z^n\right\} > 0 \qquad (z \in \Delta). \tag{7}$$

2. Main Theorem

Employing the techniques used by Singh [4], Srivastava and Attiya [5] and Attiya [2] (see also [1], [3]), we state and prove the following theorem.

Theorem 1. *Let the function $f(z)$ be defined by (1) be in the class $\mathcal{A}^\star(\beta_1, \beta_2, \beta_3; \lambda)$. Also let $\mathcal{K}$ denote the familiar class of functions $f(z) \in \mathcal{A}$ which are also univalent and convex in Δ. Then*

$$\frac{|\beta_1|}{2(\lambda + |\beta_1|)}(f * g)(z) \prec g(z) \qquad (\ z \in \Delta;\ g \in \mathcal{K}), \tag{8}$$

and

$$\operatorname{Re}(f(z)) > -\frac{\lambda + |\beta_1|}{|\beta_1|}, \qquad (\beta_1 \in \mathbb{C}-\{0\},\ z \in \Delta). \tag{9}$$

The constant $\frac{|\beta_1|}{2(\lambda+|\beta_1|)}$ *is the best estimate.*

Proof . Let $f(z) \in \mathcal{A}^\star(\beta_1, \beta_2, \beta_3; \lambda)$ and let $g(z) = z + \sum_{n=2}^{\infty} c_n z^n \in \mathcal{K}$. Then

$$\begin{aligned} & \frac{|\beta_1|}{2(\lambda + |\beta_1|)}(f * g)(z) \\ = \ & \frac{|\beta_1|}{2(\lambda + |\beta_1|)} \left(z + \sum_{n=2}^{\infty} a_n c_n z^n \right). \end{aligned}$$

Thus, by Definition 2, the assertion of our theorem will hold if the sequence

$$\left\{ \frac{|\beta_1|}{2(\lambda + |\beta_1|)} a_n \right\}_{n=1}^{\infty}$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 1 , this will be the case if and only if

$$\operatorname{Re}\left\{ 1 + \sum_{n=1}^{\infty} \frac{|\beta_1|}{\lambda + |\beta_1|} a_n z^n \right\} > 0 \ (z \in \Delta). \tag{10}$$

Now, since

$$\sigma(n) = (n-1)(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|)$$

is an increasing function of $n (n \geq 2)$, we have

$$\begin{aligned} & \operatorname{Re}\left\{ 1 + \frac{|\beta_1|}{\lambda + |\beta_1|} \sum_{n=1}^{\infty} a_n z^n \right\} \\ = \ & \operatorname{Re}\left\{ 1 + \frac{|\beta_1|}{\lambda + |\beta_1|} z + \frac{1}{\lambda + |\beta_1|} \sum_{n=1}^{\infty} |\beta_1| a_n z^n \right\} \\ \geq \ & 1 - \frac{2|\beta_1|}{\lambda + 2|\beta_1|} r - \frac{1}{\lambda + |\beta_1|} \sum_{n=1}^{\infty} (n-1)(|\beta_1| + (n-2)|\beta_2| \\ + \ & (n-2)(n-3)|\beta_3|)|a_n| r^n \\ \geq \ & 1 - \frac{|\beta_1|}{\lambda + |\beta_1|} r - \frac{\lambda}{\lambda + |\beta_1|} r \\ > \ & 0 \quad (|z| = r < 1), \end{aligned}$$

Thus (10) holds true in Δ. This proves the inequality (8). The inequality (9) follows by taking the convex function $g(z) = \frac{z}{1-z} = z + \sum\limits_{n=2}^{\infty} z^n$ in (8). To prove the sharpness of the constant $\frac{|\beta_1|}{\lambda+2|\beta_1|}$, we consider the function $f_0(z) \in \mathcal{A}^\star(\beta_1, \beta_2, \beta_3; \lambda)$ given by

$$f_0(z) = z - \frac{\lambda}{|\beta_1|} z^2.$$

Thus from (8), we have

$$\frac{|\beta_1|}{2(\lambda + |\beta_1|)} f_0(z) \prec \frac{z}{1-z}.$$

It can easily verified that

$$\min\left\{ \mathrm{Re}\left(\frac{|\beta_1|}{2(\lambda + |\beta_1|)} f_0(z) \right) \right\} = -\frac{1}{2} \qquad (z \in \Delta).$$

This shows that the constant $\frac{|\beta_1|}{2(\lambda+|\beta_1|)}$ is best possible.

Putting $\beta_1 = \lambda = 1$ and $\beta_2 = \beta_3 = 0$ in Theorem 1, we immediately obtain the following corollary:

Corollary 1. *Let the function $f(z)$ be defined by (1) and satisfies the condition*

$$\sum_{n=2}^{\infty} (n-1)\,|a_n| \leq 1. \tag{11}$$

Also let $\mathcal{K}$ denote the familiar class of functions $f(z) \in \mathcal{A}$ which are also univalent and convex in Δ. Then

$$\frac{1}{4}(f * g)(z) \prec g(z) \qquad (\ z \in \Delta;\ g \in \mathcal{K}), \tag{12}$$

and

$$\mathrm{Re}(f(z)) > -2, \qquad (\ z \in \Delta). \tag{13}$$

The constant $\frac{1}{4}$ is the best estimate.

References

[1] M. K. Aouf , A. Shamandy , A. O. Mostafa and S. M. Madian, *Subordination properties for analytic functions defined by convolution*, Proc. Pakistan Acad. Sci. 46(4) (2009), 227-232.

[2] A.A. Attiya, *On some application of a subordination theorems*, J. Math. Anal. Appl. 311 (2005), 489-494.

[3] B. A. Frasin, *Subordination results for a class of analytic functions defined by a linear operator*, Journal of Inequalities in Pure and Applied Mathematics, vol. 7, no. 4, article 134, pp. 1–7, 2006.

[4] S. Singh, *A subordination theorems for spirallike functions*, IJMMS, 24(7) (2000), 433-435.

[5] H.M. Srivastava and A.A. Attiya, *Some subordination results associated with certain subclasses of analytic functions,* J. Inequal. Pure Appl. Math., 5(4) (2004), Article 82, 1-6.

[6] N. Uyanik and S. Owa, *New extensions for classes of analytic functions associated with close-to-convex and starlike of order* α, Mathematical and Computer Modelling, Volume 54, Issues 1-2 (2011), 359-366.

[7] H.S. Wilf, *Subordinating factor sequence for convex maps of the unit circle*, Proc. Amer. Math. Soc. 12 (1961), 689-693.

B.A. Frasin
Faculty of Science
Department of Mathematics
Al al-Bayt University
P.O. Box: 130095 Mafraq, Jordan
email: *bafrasin@yahoo.com*