

ON A NEW SUBFAMILIES OF BAZILEVIC FUNCTIONS

A.T. OLADIPO

ABSTRACT. In this paper, the author make use of the newly introduced concept of analytic functions by Kanas and Ronning [7] and apply Aouf et al derivative operator with the technique of Hayami, Owa and Srivastava [17] to define a more larger and generalized class of Bazilevic functions. Characterization and coefficient bounds of this new class of Bazilevic functions are considered.

2000 Mathematics Subject Classification: Primary 30C45

Keywords and phrases: analytic functions, univalent functions, characterization, coefficient bounds, Aouf et al derivative operator.

1. INTRODUCTION

Several authors have discussed various subfamilies of Bazilevic functions of type α from various perspective. They discussed it from the perspective of convexity, inclusion theorem, radii of starlikeness, and convexity boundary rotational problem, subordination just to mention few. The most amazing thing is that, it is difficult to see any of this authors discussing the coefficient inequalities, and coefficient bounds of these subfamilies of Bazilevic function most especially when the parameter α is greater than one (α is real).

The main aim of the present paper is to define a class of subfamilies of Bazilevic functions using the newly introduced concept of analytic functions by kanas and Ronning [7] and by using the method of Cauchy and Holder inequalities [17], we comprehensively study these classes in the areas of characterization and coefficient bounds. The classes to be defined here in this work serves as a larger and a new generalization of some of the existing subfamilies of Bazilevic functions.

However, before we go on to our discussion we would like to make mention of various authors that have studied one aspect or the other of Bazilevic functions. The likes of Macregor [8], Yamaguchi [20], Opoola [13], Oladipo[10,11,12] Thomas [16], Bazilevic [6], Tuan [18], Abduhalim [1] etc.

Let $\Gamma(\omega)$ denote the class of analytic functions $f(z)$ of the form

$$f(z) = (z - \omega) + \sum_{k=2}^{\infty} a_k(z - \omega)^k \tag{1}$$

which are regular in the unit disk $U = \{z : |z| < 1\}$ and normalized with $f(\omega) = f'(\omega) - 1 = 0$ and ω is a fixed point in U [7]. Kanas and Ronning [1] used this concept to define the classes of ω -starlike and ω -convex functions respectively as follows

$$ST(\omega) = S^*(\omega) = \left\{ f(z) \in S(\omega) : \operatorname{Re} \frac{(z - \omega)f'(z)}{f(z)} > 0, z \in U \right\}$$

and

$$CV(\omega) = S^c(\omega) = \left\{ f(z) \in S(\omega) : 1 + \operatorname{Re} \frac{(z - \omega)f'(z)}{f(z)} > 0, z \in U \right\}$$

where $S(\omega) \subset \Gamma(\omega)$ denote the class of Univalent functions and ω is a fixed point U .

Several other authors the likes of Acu and Owa [9], Aouf etal [3], Oladipo [11] have made various studies on these classes with some various extensions.

Let ω be a fixed point in U , $\lambda \geq 0$, $l \geq 0$, $m \in N_0$ we define according to [3] the derivative operator $I_{\omega}^m(\lambda, l):A(\omega) \rightarrow A(\omega)$ as follows

$$I_{\omega}^0(\lambda, l)f(z) = f(z)$$

$$I_{\omega}^1(\lambda, l)f(z) = I_{\omega}(\lambda, l)f(z) = I_{\omega}^0(\lambda, l)f(z) \frac{1 - \lambda + l}{1 + l} + (I_{\omega}^0(\lambda, l)f(z))' \frac{\lambda(z - \omega)}{1 + l}$$

$$= (z - \omega) + \sum_{k=2}^{\infty} \left(\frac{1 + \lambda(k - 1) + l}{1 + l} \right) a_k(z - \omega)^k$$

$$I_{\omega}^2(\lambda, l)f(z) = I_{\omega}(\lambda, l)f(z) \frac{1 - \lambda + l}{1 + l} + (I_{\omega}(\lambda, l)f(z))' \frac{\lambda(z - \omega)}{1 + l}$$

$$= (z - \omega) + \sum_{k=2}^{\infty} \left(\frac{1 + \lambda(k - 1) + l}{1 + l} \right)^2 a_k(z - \omega)^k$$

and in general

$$I_{\omega}^m(\lambda, l)f(z) = I_{\omega}(\lambda, l)(I_{\omega}^{m-1}(\lambda, l)f(z))$$

$$= (z - \omega) + \sum_{k=2}^{\infty} \left(\frac{1 + \lambda(k-1) + l}{1+l} \right)^m a_k (z - \omega)^k \quad (2)$$

Using the operator above, we give the definition of a more larger and generalized classes of family of Bazilevic functions of type α as follows:

Definition A. Let $T_m^\alpha(\omega, \lambda, \beta, \gamma, l)$ denote the subclass of $\Gamma(\omega)$ consisting of functions $f(z)$ which satisfy the inequality

$$Re \left\{ \frac{I_\omega^m(\lambda, l)(f(z))^\alpha}{\left(\frac{1+\lambda(\alpha-1)+l}{1+l} \right)^m (z - \omega)^\alpha} \right\} > \gamma \left| \frac{I_\omega^m(\lambda, l)(f(z))^\alpha}{\left(\frac{1+\lambda(\alpha-1)+l}{1+l} \right)^m (z - \omega)^\alpha} - 1 \right| + \beta \quad (3)$$

for some $\lambda \geq 0, l \geq 0, 0 \leq \beta < 1, \gamma \geq 0, m \in N_0 = 0, 1, 2, \dots, \alpha > 0$ (α is real), ω is a fixed point in U and that all index are meant principal determination only.

From Definition A, we have the following remarks to make

Remark A. (i) For $\omega = 0$ in (3), we have

$$Re \left\{ \frac{I_0^m(\lambda, l)(f(z))^\alpha}{\left(\frac{1+\lambda(\alpha-1)+l}{1+l} \right)^m z^\alpha} \right\} > \gamma \left| \frac{I_0^m(\lambda, l)(f(z))^\alpha}{\left(\frac{1+\lambda(\alpha-1)+l}{1+l} \right)^m z^\alpha} - 1 \right| + \beta$$

that is, $f \in T_m^\alpha(0, \lambda, \beta, \gamma, l)$ which is the class of functions studied by Oladipo and Breaz in [12]

(ii) $\gamma = 0$ in (3), we have

$$Re \left\{ \frac{I_\omega^m(\lambda, l)(f(z))^\alpha}{\left(\frac{1+\lambda(\alpha-1)+l}{1+l} \right)^m (z - \omega)^\alpha} \right\} > \beta$$

that is, $f \in T_m^\alpha(\omega, \lambda, \beta, 0, l)$ which is a presumed new class of Bazilevic functions and it shall be included as a Corollary in the result presented in this paper.

(iii) if we further put $\omega = 0$ in Remark A (ii) we have another presumed new class of Bazilevic functions. That is, we have

$$Re \left\{ \frac{I_0^m(\lambda, l)(f(z))^\alpha}{\left(\frac{1+\lambda(\alpha-1)+l}{1+l} \right)^m z^\alpha} \right\} > \beta$$

then $f \in T_m^\alpha(0, \lambda, \beta, 0, l)$

(iv) for $\gamma = 0, l = 0$ in (3) we have

$$Re \frac{I_{\omega}^m(\lambda, 0)f(z)^{\alpha}}{(1 + \lambda(\alpha - 1))^m (z - \omega)^{\alpha}} > \beta \Leftrightarrow Re \frac{D_{\lambda, \omega}^m f(z)^{\alpha}}{(1 + \lambda(\alpha - 1))^m (z - \omega)^{\alpha}} > \beta$$

where $D_{\lambda, \omega}^m$ is the ω -modified AL-oboudi derivative operator and $f \in T_m^{\alpha}(0, \lambda, \beta, 0, 0)$ which is also new and it shall be seen as a Corollary in the present work.

(v) If we put $\omega = 0$ in (iv) then we have

$$Re \frac{I_0^m(\lambda, 0)f(z)^{\alpha}}{(1 + \lambda(\alpha - 1))^m z^{\alpha}} > \beta \Leftrightarrow Re \frac{D_{0, \lambda}^m f(z)^{\alpha}}{(1 + \lambda(\alpha - 1))^m z^{\alpha}} > \beta$$

where $D_{0, \lambda}^m$ is the well known AL-oboudi derivative operator and $f \in T_m^{\alpha}(0, \lambda, \beta, 0, 0)$

(vi) for $\gamma = 0, l = 0, \lambda = 1$ in (3) we have

$$Re \frac{I_{\omega}^m(1, 0)f(z)^{\alpha}}{\alpha^m (z - \omega)^{\alpha}} > \beta \Leftrightarrow Re \frac{D_{\omega}^m f(z)^{\alpha}}{\alpha^m (z - \omega)^{\alpha}} > \beta$$

which is the class of functions studied by Oladipo [10].

(vii) For $\omega = 0, \gamma = 0, l = 0, \lambda = 1$ in (3) we have

$$Re \frac{I_0^m(1, 0)f(z)^{\alpha}}{\alpha^m z^{\alpha}} > \beta \Leftrightarrow Re \frac{D_0^m f(z)^{\alpha}}{\alpha^m z^{\alpha}} > \beta$$

which is the class of function studied by Opoola in [13] and Babalola and Opoola in [5] and Babalola [4].

(viii) For $\omega = 0, \gamma = 0, l = 0, \lambda = 1, \beta = 0$ in (3) we have

$$Re \frac{I_0^m(1, 0)f(z)^{\alpha}}{z^{\alpha}} > 0 \Leftrightarrow Re \frac{D_0^m f(z)^{\alpha}}{z^{\alpha}} > 0$$

which is the class of function studied by Abduhalim in [1] and D^m is the well know Salagean derivative operator [15]. That is $f \in T_m^{\alpha}(0, 1, 0, 0, 0) = T_m^{\alpha}(0) \equiv B_n(\alpha)$.

(ix) For $\omega = 0, \gamma = 0, l = 0, \lambda = 1, \beta = 0, m = 0, \alpha = 1$ in (3) we have

$$Re \frac{I_0^0(1, 0)f(z)}{z} > 0 \Leftrightarrow Re \frac{f(z)}{z} > 0$$

which is the class of function studied by Yamaguchi in [20]. That is

$$f \in T_m^1(0, 1, 0, 0, 0) = T_0^1(0).$$

(x) For $\omega = 0, l = 0, \lambda = 1, \beta = 0, m = 1$ in (3), we have

$$Re \frac{I_0^1(1, 0)f(z)^{\alpha}}{z^{\alpha}} > 0 \Rightarrow Re \frac{D^1 f(z)^{\alpha}}{z^{\alpha}} > 0 \Leftrightarrow Re \left\{ \frac{\alpha f' f^{\alpha-1}}{z^{\alpha}} \right\} > 0$$

The functions with this properties belongs in the class $B_1(\alpha)$; which is the class of Bazilevic functions that was studied by Singh in [14].

(xi) For $\omega = 0, l = 0, \lambda = 1, 0 \leq \beta < 1, m = 0, \alpha = 1$ in (3), we have

$$\operatorname{Re} \frac{I_0^0(1,0)f(z)}{z} > \beta \Leftrightarrow \operatorname{Re} \frac{f(z)}{z} > \beta$$

which is the class of functions studied in [18] by Tuan and Anh.

(xii) For $\omega = 0, \gamma = 0, l = 0, \lambda = 1, \beta = 0, m = 1, \alpha = 1$ in (3), we have

$$\operatorname{Re} \frac{I_0^1(1,0)f(z)}{z} > 0$$

which is the class of functions studied in [8] by MacGregor.

As earlier mentioned, this paper is designed to address the problem of coefficient inequalities and coefficient bounds of subfamilies of Bazilevic functions given in Definition A. The technique employed in this work follows that of Hayami, Owa and Srivastava [17].

For the sake of clarity we wish to state the following, that is, from (1) we can write that

$$(f(z))^\alpha = \left((z - \omega) + \sum_{k=2}^{\infty} a_k(z - \omega) \right)^\alpha \quad (4)$$

Using binomial expansion on (4)[12], we have

$$(f(z))^\alpha = (z - \omega)^\alpha + \sum_{k=2}^{\infty} a_k(\alpha)(z - \omega)^{\alpha+k-1} \quad (5)$$

We know from Definition A that all the index are meant principal determination only. Therefore, the coefficient a_k shall depend so much on the parameter α . On applying (2) on (5) we obtain

$$\begin{aligned} I_\omega^m(\lambda, l)(f(z))^\alpha &= \left(\frac{1 + \lambda(\alpha - 1) + l}{1 + l} \right)^m (z - \omega)^\alpha \\ &+ \sum_{k=2}^{\infty} \left(\frac{1 + \lambda(\alpha + k - 2) + l}{1 + l} \right)^m a_k(\alpha)(z - \omega)^{\alpha+k-1} \end{aligned} \quad (6)$$

where all the parameters are as earlier defined and ω is a fixed point in U .

For the purpose of the present investigation the following Lemma shall be necessary.

Lemma A. A function $p_\omega(z) \in \Omega$

$$Rep_\omega(z) > 0, \quad z \in U$$

if and only if

$$p_\omega(z) \neq \frac{\psi - 1}{\psi + 1}, \quad (z \in U, \psi \in C; |\psi| = 1).$$

and our $p_\omega(z)$ is given as

$$P_\omega(z) = 1 + \sum_{k=1}^{\infty} B_k(z - \omega)^k$$

where

$$|B_k| \leq \frac{2}{(1+d)(1-d)^k}, \quad |\omega| = d$$

and ω is a fixed point in U [3].

Proof. For the sake of completeness, we shall give the proof of Lemma as appeared in [17], even though it is a bit obvious that the following bilinear (or Mobius) transformation

$$h = \frac{z - 1}{z + 1}$$

maps the unit circle ∂u on to imaginary axis $R(h)=0$. Indeed, for all ψ such that $|\psi| = 1$ ($\psi \in C$), we set

$$h = \frac{\psi - 1}{\psi + 1} \quad (\psi \in C; |\psi| = 1).$$

Then

$$|\psi| = \left| \frac{1+h}{1-h} \right| = 1,$$

which shows that

$$Re(h) = R\left(\frac{\psi - 1}{\psi + 1}\right) = 0 \quad (\psi \in C; |\psi| = 1).$$

Moreover, by noting that $p_\omega(\omega) = 1$ for $p_\omega(z) \in \Omega$, we know that

$$p_\omega(z) \neq \frac{\psi - 1}{\psi + 1}, \quad (z \in U, \psi \in C; |\psi| = 1).$$

and ω is a fixed point in U . This complete the proof of Lemma A.

2. COEFFICIENT INEQUALITIES

For the purpose of our investigation we shall first derive the following lemma.

Lemma 2.1. *A function $f(z) \in \Gamma(\omega)$ is in the class $T_m^\alpha(\omega, \lambda, \beta, \gamma, l)$ if and only if*

$$1 + \sum_{k=2}^{\infty} A_k(z - \omega)^{k-1} \neq 0 \quad (7)$$

where

$$A_k = \frac{(\psi + 1)(1 - \gamma)}{2(1 - \beta)} \left(\frac{1 + \lambda(\alpha + k - 2) + l}{1 + l} \right)^m a_k(\alpha)$$

and ω is a fixed point in U and all the parameters remain as earlier defined.

Proof. Let us set

$$p_\omega(z) = \frac{\frac{(1-\gamma)I_\omega^m(\lambda, l)(f(z))^\alpha}{\left(\frac{1+\lambda(\alpha-1)+l}{1+l}\right)^m (z-\omega)^\alpha} - (\beta - \gamma)}{1 - \beta}, \quad (f \in T_m^\alpha(\omega, \lambda, \beta, \gamma, l))$$

we find that $p_\omega(z) \in \Omega$ and $Rep_\omega(z) > 0$ ($z \in U$) and ω is a fixed point in U . By using Lemma A, we have

$$\frac{\frac{(1-\gamma)I_\omega^m(\lambda, l)(f(z))^\alpha}{\left(\frac{1+\lambda(\alpha-1)+l}{1+l}\right)^m (z-\omega)^\alpha} - (\beta - \gamma)}{1 - \beta} \neq \frac{\psi - 1}{\psi + 1}, \quad (z \in U, \psi \in C, |\psi| = 1)$$

which readily yields

$$\begin{aligned} & (1 - \gamma)(\psi + 1)I_\omega^m(\lambda, l)f(z)^\alpha + [1 - 2\beta + \gamma - (1 - \gamma)\psi] \cdot \\ & \cdot \left(\frac{1 + \lambda(\alpha - 1) + l}{1 + l} \right)^m (z - \omega)^\alpha \neq 0 \end{aligned}$$

Thus we find that

$$\begin{aligned} & 2(1 - \beta) \left(\frac{1 + \lambda(\alpha - 1) + l}{1 + l} \right)^m (z - \omega)^\alpha + \\ & + \sum_{k=2}^{\infty} (1 - \gamma)(\psi + 1) \left(\frac{1 + \lambda(\alpha + k - 2) + l}{1 + l} \right)^m a_k(\alpha)(z - \omega)^{\alpha+k-1} \neq 0 \end{aligned}$$

that is,

$$2(1 - \beta) \left(\frac{1 + \lambda(\alpha - 1) + l}{1 + l} \right)^m (z - \omega)^\alpha. \tag{8}$$

$$\cdot \left[1 + \sum_{k=2}^{\infty} \frac{(1 - \gamma)(\psi + 1)}{2(1 - \beta)} \left(\frac{1 + \lambda(\alpha + k - 2) + l}{1 + \lambda(\alpha - 1) + l} \right)^m a_k(\alpha)(z - \omega)^{k-1} \right] \neq 0 \tag{9}$$

On dividing both sides of (10) by $2(1 - \beta) \left(\frac{1 + \lambda(\alpha - 1) + l}{1 + l} \right)^m (z - \omega)^\alpha (z \neq \omega)$, we obtain

$$1 + \sum_{k=2}^{\infty} \frac{(1 - \gamma)(\psi + 1)}{2(1 - \beta)} \left(\frac{1 + \lambda(\alpha + k - 2) + l}{1 + \lambda(\alpha - 1) + l} \right)^m a_k(\alpha)(z - \omega)^{k-1} \neq 0$$

which completes the proof of Lemma 2.1.

For $\omega = 0$ in Lemma 2.1 we have

Corollary A. *A function $f(z) \in \Gamma(0)$ is in the class $T_m^\alpha(0, \lambda, \beta, \gamma, l)$ if and only if*

$$1 + \sum_{k=2}^{\infty} A_k z^{k-1} \neq 0$$

where

$$A_k = \frac{(\psi + 1)(1 - \gamma)}{2(1 - \beta)} \left(\frac{1 + \lambda(\alpha + k - 2) + l}{1 + \lambda(\alpha - 1) + l} \right)^m a_k(\alpha),$$

this holds for the class of functions studied by Oladipo and Breez in [12].

For $\gamma = 0$ in Lemma 2.1 we have

Corollary B. *A function $f(z) \in \Gamma(\omega)$ is in the class $T_m^\alpha(\omega, \lambda, \beta, 0, l)$ if and only if*

$$1 + \sum_{k=2}^{\infty} A_k (z - \omega)^{k-1} \neq 0$$

where

$$A_k = \frac{(\psi + 1)}{2(1 - \beta)} \left(\frac{1 + \lambda(\alpha + k - 2) + l}{1 + \lambda(\alpha - 1) + l} \right)^m a_k(\alpha),$$

this is a presumed new subfamilies of class of Bazilevic function.

For $\gamma = 0, l = 0$ in Lemma 2.1 we have

Corollary C. *A function $f(z) \in \Gamma(\omega)$ is in the class $T_m^\alpha(\omega, \lambda, \beta, 0, 0)$ if and only if*

$$1 + \sum_{k=2}^{\infty} A_k(z - \omega)^{k-1} \neq 0$$

where

$$A_k = \frac{(\psi + 1)}{2(1 - \beta)} \left(\frac{1 + \lambda(\alpha + k - 2)}{1 + \lambda(\alpha - 1)} \right)^m a_k(\alpha),$$

which is also a presumed new class of Bazilevic functions.

For $\gamma = 0$, $l = 0$, $\lambda = 1$ in Lemma 2.1 we have:

Corollary D. *A function $f(z) \in \Gamma(\omega)$ is in the class $T_m^\alpha(\omega, 1, \beta, 0, 0)$ if and only if*

$$1 + \sum_{k=2}^{\infty} A_k(z - \omega)^{k-1} \neq 0$$

where

$$A_k = \frac{(\psi + 1)}{2(1 - \beta)} \left(\frac{\alpha + k - 1}{\alpha} \right)^m a_k(\alpha),$$

and ω is a fixed point in U , holds for the class of functions studied by Oladipo in [10].

Setting $\omega = 0$ in Corollary D, we have:

Corollary E. *A function $f(z) \in \Gamma(0)$ is in the class $T_m^\alpha(0, \beta,) \equiv T_m^\alpha(\beta,)$ if and only if*

$$1 + \sum_{k=2}^{\infty} A_k z^{k-1} \neq 0$$

where

$$A_k = \frac{(\psi + 1)}{2(1 - \beta)} \left(\frac{(\alpha + k - 1)}{\alpha} \right)^m a_k(\alpha),$$

This holds true for the class of functions studied by Opoola [13] and Babalola [9,10].

For $\beta = 0$ in Corollary E, we have:

Corollary F. A function $f(z) \in \Gamma(\omega)$ is in the class $T_m^\alpha(0) \equiv B_n(\alpha,)$ if and only if

$$1 + \sum_{k=2}^{\infty} A_k z^{k-1} \neq 0$$

where

$$A_k = \frac{\psi + 1}{2} \left(\frac{(\alpha + k - 1)}{\alpha} \right)^m a_k(\alpha),$$

This holds true for the class of functions studied by Abduhalim in [1].

For $\alpha = 1$ in Corollary F we have:

Corollary G. A function $f(z) \in \Gamma(0)$ is in the class $T_m^1(0) \equiv B_n$ if and only if

$$1 + \sum_{k=2}^{\infty} A_k z^{k-1} \neq 0$$

where

$$A_k = \frac{\psi + 1}{2} k^m a_k(1),$$

For $\alpha = 1, m = 1$ in Corollary F, we have:

Corollary H. A function $f(z) \in \Gamma(0)$ is in the class $T_1^1(0) \equiv$ if and only if

$$1 + \sum_{k=2}^{\infty} A_k z^{k-1} \neq 0$$

where

$$A_k = \frac{\psi + 1}{2} k a_k(1),$$

This holds true for the class of functions studied by Abduhalim in [1].

For $m = 1$ in Corollary F, we have:

Corollary I. A function $f(z) \in \Gamma(0)$ is in the class $T_1^\alpha(0) \equiv B_1(\alpha,)$ if and only if

$$1 + \sum_{k=2}^{\infty} A_k z^{k-1} \neq 0$$

where

$$A_k = \frac{\psi + 1}{2} \left(\frac{\alpha + k - 1}{\alpha} \right) a_k(\alpha),$$

This holds true for the class of functions studied by Singh in [14].

With the aid of Lemma 2.1 we state and proof the following:

Theorem 2.1. *If $f(z) \in \Gamma(\omega)$ satisfies the following conditon*

$$\begin{aligned} \sum_{k=2}^{\infty} (r+d)^{k-1} & \left(\left| \sum_{\sigma=1}^k \left[\sum_{q=1}^{\sigma} (-1)^{\sigma-q} (1-\gamma) \left(\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l} \right)^m {}^{(\tau)}_{\sigma-q} a_q(\alpha) \right] \binom{\rho}{k-\sigma} \right| \right. \\ & \left. + \left| \sum_{\sigma=1}^k \left[\sum_{q=1}^{\sigma} (-1)^{\sigma-q} (1-\gamma) \left(\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l} \right)^m {}^{(\tau)}_{\sigma-q} a_q(\alpha) \right] \binom{\rho}{k-\sigma} \right| \right) \\ & \leq 2(1-\beta) \end{aligned} \tag{10}$$

$m \in N_0, \gamma \geq 0, l \geq 0, \lambda \geq 0, 0 \leq \beta < 1, \alpha > 0, (\alpha \text{ is real}), \rho, \tau \in R$ then $f \in T_m^\alpha(\omega, \lambda, \beta, \gamma, l)$.

Proof. We first note that

$(1 - (z - \omega))^\tau \neq 0$ and $(1 - (z - \omega))^\rho \neq 0$ ($z \in U, \tau \in R, \rho \in R$).

Hence the following inequality

$$\left(1 + \sum_{k=2}^{\infty} A_k (z - \omega)^{k-1} \right) (1 - (z - \omega))^\tau (1 + (z - \omega))^\rho \neq 0$$

holds true, then we have

$$1 + \sum_{k=2}^{\infty} A_k (z - \omega)^{k-1} \neq 0$$

which is the relation (7) of Lemma 2.1. It is easy to see that (9) is equivalent to

$$\left(1 + \sum_{k=2}^{\infty} A_k (z - \omega)^{k-1} \right) \left(\sum_{k=0}^{\infty} (-1)^k b_k (z - \omega)^k \right) \left(\sum_{k=0}^{\infty} c_k (z - \omega)^k \right) \neq 0 \tag{11}$$

where for convinience

$$b_k = {}^{(\tau)}_k \text{ and } c_k = \binom{\rho}{k}.$$

Considering the Cauchy product of the first two factors, (10) can be re-written as follows

$$\left(1 + \sum_{k=2}^{\infty} B_k(z - \omega)^{k-1}\right) \left(\sum_{k=0}^{\infty} c_k(z - \omega)^k\right) \neq 0 \tag{12}$$

where

$$B_k = \sum_{q=1}^k (-1)^{k-q} A_q b_{k-q}.$$

Further more, by applying the same method for the Cauchy product in (11), we find that

$$1 + \sum_{k=2}^{\infty} \left(\sum_{\sigma=1}^k B_k c_{k-\sigma}\right) (z - \omega)^{k-1} \neq 0 \quad (z \in U)$$

or equivalently as

$$1 + \sum_{k=2}^{\infty} \left[\sum_{\sigma=1}^k \left(\sum_{q=1}^{\sigma} (-1)^{\sigma-q} A_q b_{\sigma-q} \right) c_{k-\sigma} \right] (z - \omega)^{k-1} \neq 0 \quad (z \in U)$$

and ω is a fixed point in U .

Thus, if $f(z) \in \Gamma(\omega)$ satisfies the following inequality

$$\sum_{k=2}^{\infty} (r + d)^{k-1} \left| \sum_{\sigma=1}^k \left(\sum_{q=1}^{\sigma} (-1)^{\sigma-q} A_q b_{\sigma-q} \right) c_{k-\sigma} \right| \leq 1,$$

that is, if

$$\begin{aligned} & \frac{1}{2(1 - \beta)} \sum_{k=2}^{\infty} (r + d)^{k-1} \left| \sum_{\sigma=1}^k \left(\sum_{q=1}^{\sigma} (-1)^{\sigma-q} [(1 - \gamma) \left(\frac{1 + \lambda(\alpha + k - 2) + l}{1 + \lambda(\alpha - 1) + l} \right)^m \right. \right. \\ & \quad \left. \left. + \psi(1 - \gamma) \left(\frac{1 + \lambda(\alpha + k - 2) + l}{1 + \lambda(\alpha - 1) + l} \right)^m \right] a_q(\alpha) b_{\sigma-q} \right| c_{k-\sigma} \\ & \leq \frac{1}{2(1 - \beta)}. \end{aligned}$$

$$\sum_{k=2}^{\infty} (r + d)^{k-1} \left| \sum_{\sigma=1}^k \left[\sum_{q=1}^{\sigma} (-1)^{\sigma-q} (1 - \gamma) \left(\frac{1 + \lambda(\alpha + k - 2) + l}{1 + \lambda(\alpha - 1) + l} \right)^m a_q(\alpha) b_{\sigma-q} \right] c_{k-\sigma} \right|$$

$$+|\psi| \left| \sum_{\sigma=1}^k \left[\sum_{q=1}^{\sigma} (-1)^{\sigma-q} (1-\gamma) \left(\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l} \right)^m a_q(\alpha) b_{\sigma-q} \right] C_{k-\sigma} \right| \leq 1,$$

all the parameters are as earlier defined, then $f(z) \in T_m(\omega, \lambda, \beta, \gamma, l)$. This complete the proof of Theorem 2.1.

Remark B. If we decide to be varying the parametrics as we did in the earlier Corollaries, Coefficient inequalities for various classes of functions studied in the cited literatures and many other new ones will be obtained.

In our next result we consider the subclass $T_m^\alpha(\omega, \theta, \lambda, \beta, \gamma, l)$ of $\Gamma(\omega)$, which consists of functions $f(z) \in \Gamma(\omega)$ if and only if the following inequality holds true

$$Re \left[e^{i\theta} \frac{I_\omega^m(\lambda, l) f(z)^\alpha}{\left(\frac{1+\lambda(\alpha-1)+l}{1+l} \right)^m (z-\omega)^\alpha} - (\beta-\gamma) \right] > 0,$$

for $\lambda \geq 0, l \geq 0, \gamma \geq 0, m \in N_0, \alpha > 0$ (is real), $0 \leq \beta < 1, z \in U$ and ω is a fixed point in $U, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

For function $f(z) \in T_m^\alpha(\omega, \theta, \lambda, \beta, \gamma, l)$ we first derive Lemma 2.2 below.

Lemma 2.2. *A function $f(z) \in \Gamma(\omega)$ is in the class $T_m^\alpha(\omega, \theta, \lambda, \beta, \gamma, l)$ if and only if*

$$1 + \sum_{k=2}^{\infty} Y_k (z-\omega)^{k-1} \neq 0 \tag{13}$$

where

$$Y = \frac{e^{i\theta}(1-\gamma)(\psi+1)}{2(1-\beta)\text{Cos}\theta} \left(\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l} \right)^m a_k(\alpha)$$

and ω is a fixed point in $U, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

Proof. The proof follows the same method as in the proof of Lemma 2.1 therefore, it is omitted. Various choices of parameters involved will hold for most of the classes in the cited literatures and many new ones could be obtained.

Theorem 2.2. *If $f(z) \in \Gamma(\omega)$ satisfies the following condition*

$$\sum_{k=2}^{\infty} (r+d)^{k-1} \left| \sum_{\sigma=1}^k \left[\sum_{q=1}^{\sigma} (-1)^{\sigma-q} e^{i\theta} (1-\gamma) \left(\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l} \right)^m (\tau_{\sigma-q}) a_q(\alpha) \right] \right| \binom{\rho}{k-\sigma}$$

$$\begin{aligned}
 & + |\psi| \left| \sum_{\sigma=1}^k \left[\sum_{q=1}^{\sigma} (-1)^{\sigma-q} e^{i\theta} (1-\gamma) \left(\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l} \right)^m (\tau_{\sigma-q}) a_q(\alpha) \right] \binom{\rho}{k-\sigma} \right| \\
 & \leq 2(1-\beta) \cos\theta. \tag{14}
 \end{aligned}$$

Then $f(z) \in T_m^\alpha(\omega, \lambda, \beta, \theta, \gamma, l)$.

Proof. The proof is similar to that of Theorem 2.2 and therefore omitted.

3. COEFFICIENT BOUNDS

In this section we consider the coefficient bound for the functions in the class $T_m^\alpha(\lambda, \beta, \gamma, l)$ as we proof the following:

Theorem 3.1. *If $f(z) \in T_m^\alpha(\omega, \lambda, \beta, \gamma, l)$, then*

$$|a_2| \leq \frac{2(1-\beta)}{\alpha|1-\gamma|(1-d^2)\Psi_1^m} \tag{15}$$

$$|a_3| \leq \left\{ \begin{array}{l} \frac{2(1-\beta)}{\alpha|1-\gamma|(1-d^2)(1-d)\Psi_2^m} - \frac{2(\alpha-1)(1-\beta)^2}{\alpha^2|1-\gamma|^2(1-d^2)^2\Psi_1^{2m}}, 0 < \alpha < 1 \\ \frac{2(1-\beta)}{\alpha|1-\gamma|(1-d^2)(1-d)\Psi_2^m}, \alpha \geq 1 \end{array} \right\}$$

$$|a_4| \leq \left\{ \begin{array}{l} \Omega_1 + \Omega_2 + \Omega_3, \alpha \in (0, 1) \\ \Omega_1 + \Omega_3 + \Omega_4, \alpha \in [1, 2) \\ \Omega_1 + \Omega_3, \alpha \in [2, \infty) \end{array} \right\}.$$

where

$$\Omega_1 = \frac{2(1-\beta)}{\alpha|1-\gamma|(1-d^2)(1-d)^2\Psi_3^m}$$

$$\Omega_2 = \frac{4(\alpha-1)(1-\beta)^2}{\alpha^2|1-\gamma|^2(1-d^2)^2(1-d)\Psi_2^m\Psi_1^m}$$

$$\Omega_3 = \frac{4(\alpha-1)^2(1-\beta)^2}{\alpha^3|1-\gamma|^3(1-d^2)^3\Psi_1^{3m}}$$

$$\Omega_4 = \frac{4(1-\beta)^3(\alpha-1)(2-\alpha)}{3\alpha^3|1-\gamma|(1-d^2)^3\Psi_1^{3m}}$$

$$|a_5| \leq \begin{cases} \Omega_1 + \Omega_2 + \Omega_4 + \Omega_5 + \Omega_6 + \Omega_8, \alpha \in (0, 1) \\ \Omega_1 + \Omega_2 + \Omega_7, \alpha \in [1, 2) \\ \Omega_1 + \Omega_2 + \Omega_3 + \Omega_8, \alpha \in [2, 3) \\ \Omega_1 + \Omega_2 + \Omega_8, \alpha \in [3, \infty) \end{cases}$$

where

$$\Omega_1 = \frac{2(1-\beta)}{\alpha|1-\gamma|(1-d^2)(1-d)^3\Psi_4^m}, \quad \Omega_2 = \frac{12(\alpha-1)^2(1-\beta)^3}{\alpha^3|1-\gamma|^3(1-d^2)^3(1-d)\Psi_2^m\Psi_1^{2m}}$$

$$\Omega_3 = \frac{20(\alpha-1)^2(2-\alpha)(1-\beta)^4}{3\alpha^4|1-\gamma|^4(1-d^2)^4\Psi_1^{4m}}, \quad \Omega_4 = \frac{10(1-\alpha)^3(1-\beta)^4}{\alpha^4|1-\gamma|^4(1-d^2)^4\Psi_1^{4m}}$$

$$\Omega_5 = \frac{4(1-\alpha)(1-\beta)^2}{\alpha^2|1-\gamma|^2(1-d^2)^2(1-d)^2\Psi_1^m\Psi_3^m}, \quad \Omega_6 = \frac{2(1-\alpha)(1-\beta)^2}{\alpha^2|1-\gamma|(1-d^2)^2(1-d)^2\Psi_2^{2m}}$$

$$\Omega_7 = \frac{4(\alpha-1)(2-\alpha)(1-\beta)^3}{\alpha^3|1-\gamma|^3(1-d^2)^3(1-d)\Psi_1^{2m}\Psi_2^m}, \quad \Omega_8 = \frac{2(\alpha-1)(\alpha-2)(3-\alpha)(1-\beta)^4}{3\alpha^4|1-\gamma|^2(1-d^2)^4\Psi_1^{4m}}$$

and

$$\Psi_1^m = \left(\frac{1 + \lambda\alpha + l}{1 + \lambda(\alpha - 1) + l} \right)^m$$

$$\Psi_2^m = \left(\frac{1 + \lambda(\alpha + 1) + l}{1 + \lambda(\alpha - 1) + l} \right)^m$$

$$\Psi_3^m = \left(\frac{1 + \lambda(\alpha + 2) + l}{1 + \lambda(\alpha - 1) + l} \right)^m$$

$$\Psi_4^m = \left(\frac{1 + \lambda(\alpha + 3) + l}{1 + \lambda(\alpha - 1) + l} \right)^m$$

Proof. Note that, for $f(z) \in T_m^\alpha(\omega, \lambda, \beta, \gamma, l)$,

$$Re \frac{I^m(\lambda, l)f(z)^\alpha}{\left(\frac{1+\lambda(\alpha-1)+l}{1+l}\right)^m (z-\omega)^\alpha} > \frac{\beta-\gamma}{1-\gamma}, \quad (z \in U)$$

and all the parameters are as earlier defined and ω is a fixed point in U .

If we define the function $p_\omega(z)$ by

$$p_\omega(z) = \frac{(1-\gamma) \frac{I^m(\lambda, l)f(z)^\alpha}{\left(\frac{1+\lambda(\alpha-1)+l}{1+l}\right)^m (z-\omega)^\alpha} - (\beta-\gamma)}{1-\beta} = 1 + B_1(z-\omega) + B_2(z-\omega)^2 + \dots (16)$$

Then $p_\omega(z)$ is analytic in U with $p_\omega(\omega) = 1$ and $Rep_\omega(z) > 0$, $z \in U$ and ω is a fixed point in U .

For the sake of clarity we let

$$f(z)^\alpha = (z - \omega)^\alpha [1 + \sum_{j=1}^{\infty} \alpha_j (a_2(z - \omega) + a_3(z - \omega)^2 + \dots)^j] \quad (17)$$

where for convinience in the above we let

$$\alpha_j = \binom{\alpha}{j}, \quad j = 1, 2, 3, \dots \quad (18)$$

hence from (16) and (17) we have

$$\begin{aligned} p_\omega(z) = & 1 + \alpha_1 \frac{\Psi_1^m}{B} a_2(z - \omega) + (\alpha_1 a_3 + \alpha_2 a_2^2) \frac{\Psi_2^m}{B} (z - \omega)^2 \\ & + (\alpha_1 a_4 + 2\alpha_2 a_2 a_3 + \alpha_3 a_2^3) \frac{\Psi_3^m}{B} (z - \omega)^3 \\ & + (\alpha_1 a_5 + \alpha_2(2a_2 a_4 + a_3^2) + 3\alpha_3 a_2^2 a_3 + \alpha_4 a_2^4) \frac{\Psi_4^m}{B} (z - \omega)^4 \end{aligned} \quad (19)$$

where $\alpha_j (j = 1, 2, 3, \dots)$ is as earlier defined in (17),

$$\Psi_j^m = \left(\frac{1 + \lambda(\alpha + j - 1) + l}{1 + \lambda(\alpha - 1) + l} \right)^m, \quad j = 1, 2, 3, \dots$$

$$B = \frac{1 - \beta}{|1 - \gamma|}$$

On comparing coefficients in (18) and using the fact the $|B_k| \leq \frac{2}{(1+d)(1-d)^k}$, $k \geq 1$ [3] the results follow and the proof is complete. On setting $\lambda = 1, l = 0$ in Theorem 3.1 we have

Corollary 3.1. *If $f(z) \in T_m^\alpha(\omega, 1, \gamma, \beta, 0) \equiv T_m^\alpha(\omega, \gamma, \beta)$, for $\alpha > 0, 0 \leq \beta < 1, 0 \leq \gamma \leq \beta$, or $\gamma > \frac{1+\beta}{2}, n = 0, 1, 2, \dots$ then*

$$|a_2| \leq \frac{2(1 - \beta)\alpha^{n-1}}{(\alpha + 1)^n |1 - \gamma|(1 - d^2)}$$

$$|a_3| \leq \left\{ \begin{array}{l} \frac{2(1-\beta)\alpha^{n-1}}{(\alpha+2)^n |1-\gamma|(1-d^2)(1-d)} - \frac{2(\alpha-1)(1-\beta)^2 \alpha^{2n-2}}{(\alpha+1)^{2n} |1-\gamma|^2 (1-d^2)^2}, 0 < \alpha < 1 \\ \frac{2(1-\beta)\alpha^{n-1}}{(\alpha+2)^n |1-\gamma|(1-d^2)(1-d)}, \alpha \geq 1 \end{array} \right\}$$

$$|a_4| \leq \begin{cases} \Omega_1 + \Omega_2 + \Omega_3, \alpha \in (0, 1) \\ \Omega_1 + \Omega_3 + \Omega_4, \alpha \in [1, 2) \\ \Omega_1 + \Omega_3, \alpha \in [2, \infty) \end{cases}$$

where

$$\Omega_1 = \frac{2(1 - \beta)\alpha^{m-1}}{(\alpha + 3)^m |1 - \gamma|(1 - d^2)(1 - d)^2}$$

$$\Omega_2 = \frac{4(\alpha - 1)(1 - \beta)^2 \alpha^{2m-2}}{(\alpha - 1)^m (\alpha + 2)^m |1 - \gamma|^2 (1 - d^2)^2 (1 - d)}$$

$$\Omega_3 = \frac{4(\alpha - 1)^2 (1 - \beta)^3 \alpha^{3m-3}}{(\alpha + 1)^{3m} |1 - \gamma|^3 (1 - d^2)^3}$$

$$\Omega_4 = \frac{4(\alpha - 1)(2 - \alpha)(1 - \beta)^3 \alpha^{3m-3}}{3(\alpha + 1)^{3m} |1 - \gamma|^3 (1 - d^2)^3}$$

$$|a_5| \leq \begin{cases} \Omega_1 + \Omega_2 + \Omega_4 + \Omega_5 + \Omega_6 + \Omega_8, \alpha \in (0, 1) \\ \Omega_1 + \Omega_2 + \Omega_7, \alpha \in [1, 2) \\ \Omega_1 + \Omega_2 + \Omega_3 + \Omega_8, \alpha \in [2, 3) \\ \Omega_1 + \Omega_2 + \Omega_8, \alpha \in [3, \infty) \end{cases}$$

where

$$\Omega_1 = \frac{2(1 - \beta)\alpha^{m-1}}{(\alpha + 4)^m |1 - \gamma|(1 - d^2)(1 - d)^3},$$

$$\Omega_2 = \frac{12(\alpha - 1)^2 (1 - \beta)^3 \alpha^{3m-3}}{(\alpha + 1)^{2m} (\alpha + 2)^m |1 - \gamma|^3 (1 - d^2)^3 (1 - d)}$$

$$\Omega_3 = \frac{20(\alpha - 1)^2 (2 - \alpha)(1 - \beta)^4 \alpha^{4m-4}}{3(\alpha + 1)^{4m} |1 - \gamma|^4 (1 - d^2)^4},$$

$$\Omega_4 = \frac{10(1 - \alpha)^3 (1 - \beta)^4 \alpha^{4m-4}}{(\alpha + 1)^{4m} |1 - \gamma|^4 (1 - d^2)^4}$$

$$\Omega_5 = \frac{4(1 - \alpha)(1 - \beta)^2 \alpha^{2m-2}}{(\alpha + 1)^m (\alpha + 3)^m |1 - \gamma|^2 (1 - d^2)^2 (1 - d)^2},$$

$$\Omega_6 = \frac{2(1 - \alpha)(1 - \beta)^2 \alpha^{2m-2}}{(\alpha + 2)^{2m} |1 - \gamma|^2 (1 - d^2)^2 (1 - d)^2}$$

$$\Omega_7 = \frac{4(\alpha - 1)(2 - \alpha)(1 - \beta)^3 \alpha^{3m-3}}{(\alpha + 1)^{2m} (\alpha + 2)^m |1 - \gamma|^3 (1 - d^2)^3 (1 - d)},$$

$$\Omega_8 = \frac{2(\alpha - 1)(\alpha - 2)(3 - \alpha)(1 - \beta)^4 \alpha^{4m-4}}{3(\alpha + 1)^{4m} |1 - \gamma|^4 (1 - d^2)^4},$$

On setting $\omega = 0$ in Corollary 3.1, we have:

Corollary 3.2. *If $f(z) \in T_m^\alpha(\lambda, \beta, \gamma, l)$, then*

$$|a_2| \leq \frac{2(1-\beta)}{\alpha|1-\gamma|\Psi_1^m}$$

$$|a_3| \leq \left\{ \begin{array}{l} \frac{2(1-\beta)}{\alpha|1-\gamma|\Psi_2^m} - \frac{2(\alpha-1)(1-\beta)^2}{\alpha^2|1-\gamma|^2\Psi_1^{2m}}, 0 < \alpha < 1 \\ \frac{2(1-\beta)}{\alpha|1-\gamma|\Psi_2^m}, \alpha \geq 1 \end{array} \right\}$$

$$|a_4| \leq \left\{ \begin{array}{l} \Omega_1 + \Omega_2 + \Omega_3, \alpha \in (0, 1) \\ \Omega_1 + \Omega_3 + \Omega_4, \alpha \in [1, 2) \\ \Omega_1 + \Omega_3, \alpha \in [2, \infty) \end{array} \right.$$

where

$$\Omega_1 = \frac{2(1-\beta)}{\alpha|1-\gamma|\Psi_3^m}; \quad \Omega_2 = \frac{4(\alpha-1)(1-\beta)^2}{\alpha^2|1-\gamma|^2\Psi_2^m\Psi_1^m}$$

$$\Omega_3 = \frac{a(\alpha-1)^2(1-\beta)^2}{\alpha^3|1-\gamma|^3\Psi_1^{3m}}; \quad \Omega_4 = \frac{4(1-\beta)^3(\alpha-1)(2-\alpha)}{3\alpha^3|1-\gamma|\Psi_1^{3m}}$$

$$|a_5| \leq \left\{ \begin{array}{l} \Omega_1 + \Omega_2 + \Omega_4 + \Omega_5 + \Omega_6 + \Omega_8, \alpha \in (0, 1) \\ \Omega_1 + \Omega_2 + \Omega_7, \alpha \in [1, 2) \\ \Omega_1 + \Omega_2 + \Omega_3 + \Omega_8, \alpha \in [2, 3) \\ \Omega_1 + \Omega_2 + \Omega_8, \alpha \in [3, \infty) \end{array} \right.$$

where

$$\Omega_1 = \frac{2(1-\beta)}{\alpha|1-\gamma|\Psi_4^m}, \quad \Omega_2 = \frac{12(\alpha-1)^2(1-\beta)^3}{\alpha^3|1-\gamma|^3\Psi_2^m\Psi_1^{2m}}$$

$$\Omega_3 = \frac{20(\alpha-1)^2(2-\alpha)(1-\beta)^4}{3\alpha^4|1-\gamma|^4\Psi_1^{4m}}, \quad \Omega_4 = \frac{10(1-\alpha)^3(1-\beta)^4}{\alpha^4|1-\gamma|^4\Psi_1^{4m}}$$

$$\Omega_5 = \frac{4(1-\alpha)(1-\beta)^2}{\alpha^2|1-\gamma|^2\Psi_1^m\Psi_3^m}, \quad \Omega_6 = \frac{2(1-\alpha)(1-\beta)^2}{\alpha^2|1-\gamma|\Psi_2^{2m}}$$

$$\Omega_7 = \frac{4(\alpha-1)(2-\alpha)(1-\beta)^3}{\alpha^3|1-\gamma|^3\Psi_1^{2m}\Psi_2^m}, \quad \Omega_8 = \frac{2(\alpha-1)(\alpha-2)(3-\alpha)(1-\beta)^4}{3\alpha^4|1-\gamma|^2\Psi_1^{4m}}$$

and

$$\begin{aligned}\Psi_1^m &= \left(\frac{1 + \lambda\alpha + l}{1 + \lambda(\alpha - 1) + l} \right)^m \\ \Psi_2^m &= \left(\frac{1 + \lambda(\alpha + 1) + l}{1 + \lambda(\alpha - 1) + l} \right)^m \\ \Psi_3^m &= \left(\frac{1 + \lambda(\alpha + 2) + l}{1 + \lambda(\alpha - 1) + l} \right)^m \\ \Psi_4^m &= \left(\frac{1 + \lambda(\alpha + 3) + l}{1 + \lambda(\alpha - 1) + l} \right)^m\end{aligned}$$

On setting $\gamma = 0$ in Corollary 3.2, we have:

Corollary 3.3. *If $f(z) \in T_m^\alpha(1, 0, \beta, 0) \equiv T_m^\alpha(\beta)$, for $\alpha > 0$, $0 \leq \beta < 1$, $0 \leq \gamma \leq \beta$, or $\gamma > \frac{1+\beta}{2}$, $n = 0, 1, 2, \dots$ then*

$$|a_2| \leq \frac{2(1 - \beta)\alpha^{n-1}}{(\alpha + 1)^n}$$

$$|a_3| \leq \left\{ \begin{array}{l} \frac{2(1-\beta)\alpha^{n-1}}{(\alpha+2)^n} - \frac{2(\alpha-1)(1-\beta)^2\alpha^{2n-2}}{(\alpha+1)^{2n}}, 0 < \alpha < 1 \\ \frac{2(1-\beta)\alpha^{n-1}}{(\alpha+2)^n}, \alpha \geq 1 \end{array} \right\}$$

$$|a_4| \leq \left\{ \begin{array}{l} \Omega_1 + \Omega_2 + \Omega_3, \alpha \in (0, 1) \\ \Omega_1 + \Omega_3 + \Omega_4, \alpha \in [1, 2) \\ \Omega_1 + \Omega_3, \alpha \in [2, \infty) \end{array} \right.$$

where

$$\Omega_1 = \frac{2(1 - \beta)\alpha^{m-1}}{(\alpha + 3)^m}$$

$$\Omega_2 = \frac{4(\alpha - 1)(1 - \beta)^2\alpha^{2m-2}}{(\alpha - 1)^m(\alpha + 2)^m}$$

$$\Omega_3 = \frac{4(\alpha - 1)^2(1 - \beta)^3\alpha^{3m-3}}{(\alpha + 1)^{3m}}$$

$$\Omega_4 = \frac{4(\alpha - 1)(2 - \alpha)(1 - \beta)^3\alpha^{3m-3}}{3(\alpha + 1)^{3m}}$$

$$|a_5| \leq \begin{cases} \Omega_1 + \Omega_2 + \Omega_4 + \Omega_5 + \Omega_6 + \Omega_8, \alpha \in (0, 1) \\ \Omega_1 + \Omega_2 + \Omega_7, \alpha \in [1, 2) \\ \Omega_1 + \Omega_2 + \Omega_3 + \Omega_8, \alpha \in [2, 3) \\ \Omega_1 + \Omega_2 + \Omega_8, \alpha \in [3, \infty) \end{cases}$$

where

$$\Omega_1 = \frac{2(1-\beta)\alpha^{m-1}}{(\alpha+4)^m}, \quad \Omega_2 = \frac{12(\alpha-1)^2(1-\beta)^3\alpha^{3m-3}}{(\alpha+1)^{2m}(\alpha+2)^m}$$

$$\Omega_3 = \frac{20(\alpha-1)^2(2-\alpha)(1-\beta)^4\alpha^{4m-4}}{3(\alpha+1)^{4m}|1-\gamma|^4}, \quad \Omega_4 = \frac{10(1-\alpha)^3(1-\beta)^4\alpha^{4m-4}}{(\alpha+1)^{4m}|1-\gamma|^4}$$

$$\Omega_5 = \frac{4(1-\alpha)(1-\beta)^2\alpha^{2m-2}}{(\alpha+1)^m(\alpha+3)^m}, \quad \Omega_6 = \frac{2(1-\alpha)(1-\beta)^2\alpha^{2m-2}}{(\alpha+2)^{2m}}$$

$$\Omega_7 = \frac{4(\alpha-1)(2-\alpha)(1-\beta)^3\alpha^{3m-3}}{(\alpha+1)^{2m}(\alpha+2)^m}, \quad \Omega_8 = \frac{2(\alpha-1)(\alpha-2)(3-\alpha)(1-\beta)^4\alpha^{4m-4}}{3(\alpha+1)^{4m}},$$

With different choices of the parameters involved, various coefficient bounds for the classes of functions in the cited literatures could be derived.

REFERENCES

- [1] Abduhalim S., *On a class of analytic functions involving Salagean differential operator*, Tamkang Journal of Mathematics, Vol., 23, No. 1, (1992) 51 - 58.
- [2] Al-Oboudi, F.M. , *On univalent functions defined by a generalized Salagean operator*, Internat. J. Math. Math. Sci. (2004) 1429-1436.
- [3] Aouf, M.K.; Shamandy, A. Mostafa, A.O. and Madian, S.M. , *A subclass of M-W- starlike functions*, Acta Universitatis Apulensis, No. 21, (2010) 135-142.
- [4] Babalola K. O. and Opoola T. O., *Iterated integral transforms of Caratheodory functions and their applications to analytic and univalent functions*, Tamkang Journal of Mathematics 37 (4), (2006) 355 - 366.
- [5] Babalola, K.O. (2009), *Convex null sequence technique for analytic and univalent mappings of the unit disk*, Tamkang Journal of Mathematics 40, No.2 201-209.
- [6] Bazilevic I.E. (1955) *On a case of integrability in quadrature of the Loewner-Kufareu equation*, Mat Sb 37 (79), 471-476 (Russian) MR 17, 356.

- [7] Kanas S. and Ronning F., *Uniformly starlike and convex functions and other related classes of univalent functions*, Ann. Univ. Mariae Curie - Sklodowska Section A, 53, (1999) 95-105.
- [8] Mac Gregor, T.H. ,*Functions whose derivatives have positive real part*, Trans America Mathematical Society, 104, (1962) 532-537, MR 25-797.
- [9] Mugur Acu and Shigeyoshi Owa , *On some subclasses of univalent functions*, Journal of inequalities in Pure and Applied Mathematics Vol, 6 issue 3, Article 70, (2005) 1 - 14
- [10] Oladipo, A.T., *On subclasses of analytic and univalent functions*, Advances in Applied Mathematical Analysis, Vol. 4, No 1 (2009) 87-93.
- [11] Oladipo, A.T. and Daniel Breaz , *On class of family of Bazilevic functions* , Acta Universitatis Apulensis, No 24, (2010) 319-330.
- [12] Oladipo, A.T., *On certain subclass of analytic functions involving convolution operators*, Acta Universitatis Apulensis, No 20, (2009) 163-174.
- [13] Opoola T. O., *On a new subclass of univalent functions*, Mathematicae Cluj (36) 59 (2), (1994) 195 - 200
- [14] Ram Singh ,*On Bazilevic functions*, Proceedings of The America Mathematical Society, Vol. 38, No 2, (1973) 261-271.
- [15] Salagean, G.S. ,*Subclasses of univalent functions*, Lecture Notes in Math 1013, 362-372 Sringer Verlag, Berlin,Heidelberg and New York (1983).
- [16] THomas D.K. , *On Bazilevic functions*, Tran, Amer. Math. Soc. 132, (1968) 353-361 MR 36 5330.
- [17] Toshio Hayami, Shigeyoshi Owa and Srivastava, H.M., *Coefficient inequalities for certain classes of analytic and univalent functions*, J.Ineq. in Pure and Appl. Mathematics. Vol. 8, Issue 4, Art. 95, (2007) 1-21.
- [18] Tuan, P.D. and Anh, V.V. (1978), *Radii of starlikeness and convexity of certain classes of analytic functions*. J.Math. Anal. and Appl. 64, 146-158.
- [19] Wald, J.K., *On starlike functions*, Ph.D Thesis, University of Delaware, New Ark, Delaware (1978).
- [20] Yamaguchi, K., *On functions satisfying $Re \left(\frac{f(z)}{z} \right) > 0$* , Proceedings of The American Mathematical Society, 17, (1966) 588-591 MR 33-268.

A.T. Oladipo
 Department of Pure and Applied Mathematics,
 Ladoke Akintola University of Technology, Ogbomoso
 P. M. B. 4000, Ogbomoso, Nigeria.
 e-mail: *atlab_3@yahoo.com*