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THE FEKETE-SZEGÖ PROBLEM FOR A CLASS DEFINED BY THE HOHLOV OPERATOR

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Abstract. Let $\mathcal{A}$ be the class of analytic functions in the open unit disk $\mathcal{U}$. For complex numbers a, b and c $(c \neq 0, -1, -2, ...)$, let $\mathcal{I}_c^{a,b}$ be the operator defined on $\mathcal{A}$ by

$$(\mathcal{I}_c^{a,b}(f))(z) = z_2F_1(a, b; c; z) * f(z)$$

where ${}_2F_1(a, b; c; z)$ is the *Gaussian hypergeometric function*. The function f in $\mathcal{A}$ is said to be in the class $k - \mathcal{SP}_c^{a,b}$ if $\mathcal{I}_c^{a,b}(f)$ is a *k-parabolic starlike function*. For this class the *Fekete-Szegö problem* is settled in the present paper.

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1. Introduction and definitions

Let $\mathcal{A}$ be the class of functions analytic in the open unit disk

$$\mathcal{U} := \{z : z \in \mathrm{C} \text{ and } |z| < 1\}$$

and let $\mathcal{A}_0$ be the family of functions f in $\mathcal{A}$ satisfying the *normalization* condition (cf.[3]):

$$f(0) = f'(0) - 1 = 0.$$

Thus, the functions in $\mathcal{A}_0$ are given by the power series

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathcal{U}). \tag{1}$$

Let $\mathcal{S}$ denote the class of analytic *univalent* functions in $\mathcal{U}$. For fixed k $(0 \leq k < \infty)$, the function $f \in \mathcal{A}_0$ is said to be in k-$\mathcal{UCV}$, the class of k- *uniformly convex functions* in $\mathcal{U}$, if the the image of every circular arc γ contained in $\mathcal{U}$, with center ξ where $|\xi| \leq k$, is a convex arc. This interesting unification of the concepts of *univalent convex* *functions* (cf.[3]) and *uniformly convex functions* (cf.[5]) is due to Kanas and Wisniowska [8].

The class k- $\mathcal{SP}$, consisting of *k-parabolic starlike functions* is defined from k-$\mathcal{UCV}$ via the *Alexander's transform* (see [9]) i-e

$$f \in k-\mathcal{UCV} \Leftrightarrow g \in k-\mathcal{SP}, \text{ where } g(z) = zf'(z) \quad (z \in \mathcal{U}).$$

The one variable characterization theorem (cf.[8]) of the class k- $\mathcal{UCV}$ gives that $f \in k-\mathcal{UCV}$(respectively $f \in k-\mathcal{SP}$) if and only if the values of

$$p(z) = 1 + \frac{zf''(z)}{f'(z)} \left(\text{respectively} \quad \frac{zf'(z)}{f(z)}\right) \quad (z \in \mathcal{U})$$

lie in the conic region Ω_k in the w- plane, where

$$\Omega_k := \{w = u + iv \in \mathrm{C} : u^2 > k^2(u-1)^2 + k^2v^2,\ u > 0,\ 0 \le k < \infty\}.$$

For details of the geometric description of Ω_k see([8,9]).
If f and g are functions in $\mathcal{A}$ and given by the power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=0}^{\infty} b_n z^n \quad (z \in \mathcal{U}),$$

then the *Hadamard product* (or *convolution*) of f and g denoted by $f * g$ is defined by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n = (g * f)(z) \quad (z \in \mathcal{U}).$$

For complex numbers a, b and c ($c \neq 0, -1, -2, \dots$) the *Gaussian hypergeometric function* ${}_2F_1(z)$ is defined by

$${}_2F_1(z) = {}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} z^n = 1 + \frac{ab}{c}z + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \dots\dots.. \quad (2)$$

where $(\lambda)_n$ is the *Pochhamer symbol* or *shifted factorial*, written in terms of the *gamma function* Γ, by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1, & n = 0 \\ \lambda(\lambda+1)\dots.(\lambda+n-1), & n \in \mathrm{N} := \{1, 2, 3, \dots..\}. \end{cases}$$

Note that ${}_2F_1(z)$ is symmetric in a and b and that the series (2) terminates if at least one of the numerator parameters a and b is zero or a negative integer. Using

Gaussian hypergeometric series Hohlov (cf.[6]) introduced and studied the linear operator $\mathcal{I}_c^{a,b} : \mathcal{A}_0 \to \mathcal{A}_0$ defined by

$$(\mathcal{I}_c^{a,b}(f))(z) = z_2F_1(a,b;c;z) * f(z), \quad (f \in \mathcal{A}_0, z \in \mathcal{U}).$$

Observe that for the function f of the form (1), we have

$$(\mathcal{I}_c^{a,b}(f))(z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} a_n z^n, \qquad (z \in \mathcal{U}). \tag{3}$$

The Hohlov operator $\mathcal{I}_c^{a,b}$ unifies several previously well studied operators. Namely ;

- $\mathcal{I}_1^{2,1}(f) = zf'(z) = \mathcal{A}(f)$ is the Alexander transformation, where as $\mathcal{I}_2^{1,1}(f) = \int_0^z \frac{f(t)}{t} dt$ is its inverse transform (see[3]);
- $\mathcal{I}_3^{1,2}(f) = \mathcal{L}(f)$ is the Libera integral operator (see[24]);
- $\mathcal{I}_{\gamma+2}^{1,\gamma+1}(f) = \mathcal{B}(f)$ is the Bernardi integral operator (see[24]);
- $\mathcal{I}_{n+1}^{2,1}(f) = \mathcal{I}_n(f)$ is the Noor integral operator of order n (see[14-16]);
- $\mathcal{I}_1^{1,n+1}(f) = \mathcal{D}^n(f)$ $(n > -1)$ is the Ruscheweyh derivative of f of order n (see[19,20]);
- $\mathcal{I}_c^{a,1}(f) = \mathcal{L}(a,c)(f)$ is the Carlson -Shaffer operator (see[24]);
- $\mathcal{I}_{2-\lambda}^{2,1}(f) = \Omega^\lambda(f)$ is the Owa- Srivastava operator (see[17]).

In this sequel to earlier work on the classes k- $\mathcal{UCV}$ and k- $\mathcal{SP}$, we now define a new subclass of analytic functions by using the Hohlov operator $\mathcal{I}_c^{a,b}$.
Definition 1: The function $f \in \mathcal{A}_0$ is said to be in the class $k - \mathcal{SP}_c^{a,b}$ $(0 \le k < \infty,\ a,\ b,\ c \in \mathrm{R},\ c \ne\ 0, -1, -2, ...)$ if $\mathcal{I}_c^{a,b}(f) \in k - \mathcal{SP}$ or equivalently

$$\Re\left(\frac{z(\mathcal{I}_c^{a,b}(f))'(z)}{\mathcal{I}_c^{a,b}(f)(z)}\right) > k\left|\frac{z(\mathcal{I}_c^{a,b}(f))'(z)}{\mathcal{I}_c^{a,b}(f)(z)} - 1\right|, \qquad (z \in \mathcal{U}). \tag{4}$$

In the particular case $k = 1$, we denote by $\mathcal{SP}_c^{a,b}$ the class $1 - \mathcal{SP}_c^{a,b}$. We note that, by specializing the parameters k, a, b and c we obtain the following subclasses studied by various authors.

- for $k = 1,\ a = 2,\ b = 1,\ c = 1,\ 1 - \mathcal{SP}_1^{2,1} := \mathcal{UCV}$, the class of uniformly convex functions has been studied by Goodman [5] and Ma and Minda [11].

- for $k = 1,\ a = 1,\ b = 1,\ c = 2,\ 1 - \mathcal{SP}_2^{1,1} := \mathcal{SP}$, the class of parabolic starlike functions has been studied by Rønning [18];
- for $k = 1,\ a = 2,\ b = 1,\ c = 2 - \lambda\ (0 \leq \lambda \leq 1)$, the class $1 - \mathcal{SP}_{2-\lambda}^{2,1} := \mathcal{SP}_\lambda$ has been studied by Srivastava and Mishra [21];
- for $a = 2,\ b = 1,\ c = 2 - \lambda\ (0 \leq \lambda \leq 1)$, the class $k - \mathcal{SP}_{2-\lambda}^{2,1} := k - \mathcal{SP}_\lambda$ has been studied by Mishra and Gochhayat [12];
- for $a = 2,\ b = 1,\ c = n+1$, the class $k - \mathcal{SP}_{n+1}^{2,1} := k - \mathcal{UCV}_n$ has been studied by Mishra and Gochhayat [13].

In the particular cases $k = 0,\ a = 2,\ b = 1,\ c = 1$, we get $0 - \mathcal{SP}_1^{2,1} := \mathcal{CV}$, the class of univalent convex functions [3]. Similarly, taking $k = 0,\ a = 1,\ b = 1,\ c = 2$, we have $0 - \mathcal{SP}_2^{1,1} := \mathcal{S}^*$, the class of univalent starlike functions [3].

It is well known (cf.[3]) that for $f \in \mathcal{S}$ and given by (1), the sharp inequality $|a_3 - a_2^2| \leq 1$ holds. Fekete and Szegö [4] obtained sharp upper bounds for $|\mu a_2^2 - a_3|$ for $f \in \mathcal{S}$ when μ is real. Thus the determination of sharp upper bounds for the nonlinear functional $|\mu a_2^2 - a_3|$ for any compact family $\mathcal{F}$ of functions in $\mathcal{A}_0$ is popularly known as the Fekete-Szegö problem for $\mathcal{F}$. For different subclasses of S , the Fekete-Szegö problem has been investigated by many authors including [4,11-13,21-23] etc. For a brief history of the Fekete-Szegö problem see([23]).
In the present paper the Fekete-Szegö problem for the class $k - \mathcal{SP}_c^{a,b}\quad (0 \leq k < \infty,\ a,\ b,\ c \in \mathrm{R},\ c \neq 0, -1, -2, ...,\ a,\ b \neq 0, -1)$ is settled completely. For particular values of $a,\ b,\ c$ and k, our result include the results found in [11-13,21]. The following definitions, notations and results shall be useful for the presentation of our results.
The *Jacobi elliptic integral* (or *normal elliptic integral*) *of first kind* (cf.[1], [2], also see [24,p.50]) is defined by

$$\mathcal{F}(\omega, t) = \int_0^\omega \frac{dx}{\sqrt{(1 - x^2)(1 - t^2x^2)}} \quad (0 < t < 1). \tag{5}$$

The function $\mathcal{F}(1, t) := \mathcal{K}(t)$ is called the *complete elliptic integral of the first kind.* Changing to the variable $t' = \sqrt{1 - t^2},\ t \in (0, 1)$, we write $\mathcal{K}'(t) := \mathcal{K}(t')$. It should be emphasized here that the symbol $\prime$(prime) does not stand for derivative. The following properties of $\mathcal{K}(t)$ and $\mathcal{K}'(t)$ are well known (cf.[7]).

$$\lim_{t \to 0^+} \mathcal{K}(t) = \frac{\pi}{2} \qquad \lim_{t \to 1^-} \mathcal{K}(t) = \infty.$$

Moreover the function

$$\nu(t) = \frac{\pi}{2} \frac{\mathcal{K}'(t)}{\mathcal{K}(t)}, \quad (t \in (0, 1))$$

strictly decreases from ∞ to 0 as t moves from 0 to 1. Therefore every positive number k can be expressed as

$$k = \cosh(\nu(t)) \tag{6}$$

for some unique $t \in (0,1)$. Finally we introduce the following functions which will be used in the discussion of sharpness of our results. Define the function $\mathcal{G}$ on $\mathcal{U}$ by

$$\mathcal{G}(z) = [z_2F_1(c,b;a;z)] * \left\{ z \ \exp\left(\int_0^z \frac{q_k(\zeta)-1}{\zeta} d\zeta \right) \right\}, \quad (z \in \mathcal{U}), \tag{7}$$

where q_k is the Riemann map of $\mathcal{U}$ onto Ω_k satisfying $q_k(0) = 1$ and $q_k'(0) > 0$. Finally define the the function $\psi(z,\theta,\eta)$ in $k - \mathcal{SP}_c^{a,b}$ by

$$\psi(z,\theta,\eta) \quad = [z_2F_1(c,b;a;z)] * z \exp\left(\int_0^z \left[q_k\left(\frac{e^{i\theta}\zeta(\zeta+\eta)}{1+\eta\zeta} \right) - 1 \right] \frac{d\zeta}{\zeta} \right) \tag{8}$$
$$(0 \leq \theta \leq 2\pi \ ; \ 0 \leq \eta \leq 1).$$

Note that $\psi(z,0,1) = \mathcal{G}(z)$ defined by (1.7) and $\psi(z,\theta,0)$ is an odd function.

2. Preliminary Lemmas

We need the following results in our investigation.

Lemma 1. [7] *Let $k \in [0,\infty)$ be fixed and q_k be the Riemann map of $\mathcal{U}$ onto Ω_k, satisfying $q_k(0) = 1$ and $q_k'(0) > 0$. If*

$$q_k(z) = 1 + Q_1(k)z + Q_2(k)z^2 + \dots, \qquad (z \in \mathcal{U}), \tag{9}$$

then

$$Q_1(k) := \begin{cases} \frac{2A^2}{1-k^2}; & 0 < k < 1, \\ \frac{8}{\pi^2}; & k = 1, \\ \frac{\pi^2}{4(k^2-1)\mathcal{K}^2(t)\sqrt{t}(1+t)}; & k > 1, \end{cases}$$

and

$$Q_2(k) := D(k)Q_1(k)$$

where

$$D(k) = \begin{cases} \frac{(A^2+2)}{3}; & 0 < k < 1, \\ \frac{2}{3}; & k = 1, \\ \frac{(4\mathcal{K}^2(t)(t^2+6t+1)-\pi^2)}{24\mathcal{K}^2(t)\sqrt{t}(1+t)} & k > 1, \end{cases}$$

$$A = \frac{2}{\pi} \arccos k \tag{10}$$

and $\mathcal{K}(t)$ is the complete elliptic integral of first kind.

Lemma 2. [10] *Let the Schwarz function $\omega(z)$ be given by*

$$\omega(z) = d_1 z + d_2 z^2 + ..., \qquad (z \in \mathcal{U}). \tag{11}$$

Then for any complex number s,

$$|d_2 - s d_1^2| \leq 1 + (|s| - 1)|d_1|^2. \tag{12}$$

3. The Fekete-Szegö Inequalities

The following calculations shall be used in each of the proofs of Theorems 1 ,2 3 and 4 (below).

By Definition 1 there exists a function $\omega \in \mathcal{A}$ satisfying the conditions of the Schwarz lemma such that

$$\frac{z(\mathcal{I}_c^{a,b} f(z))'}{\mathcal{I}_c^{a,b} f(z)} = q_k(\omega(z)) \qquad (z \in \mathcal{U}), \tag{13}$$

where q_k is the function defined as in Lemma 1. Suppose

$$\omega(z) = d_1 z + d_2 z^2 + ..., (z \in \mathcal{U}).$$

Substituting this in the series (9) we get

$$q_k(w(z)) = 1 + Q_1(k) d_1 z + \{Q_1(k) d_2 + Q_2(k) d_1^2\} z^2 + ... \tag{14}$$

For brevity of notation, throughout, we shall write $Q_1 := Q_1(k), Q_2 := Q_2(k)$ and $D := D(k)$. Using the expansion (3) and (14) in (13) and equating coefficients we find that

$$a_2 = \frac{c}{ab} Q_1 d_1 \tag{15}$$

and

$$a_3 = \frac{c(c+1)}{ab(a+1)(b+1)} \left[Q_2 \ d_1^2 + Q_1 \ d_2 + \frac{ab}{c} \ Q_1 \ d_1 \ a_2 \right]. \tag{16}$$

We have the following:

Theorem 1. *Let the function f given by (1) be in the class* $k-\mathcal{SP}_c^{a,b}$ $(0 \le k < \infty,\ a,\ b,\ c \in \mathrm{R},\ a,\ b,\ c > 0)$.
Then

$$|\mu a_2^2 - a_3| \le \begin{cases} \frac{c(c+1)}{ab(a+1)(b+1)} Q_1 \left[\frac{(a+1)(b+1)c}{ab(c+1)}\ Q_1\ \mu - Q_1 - D\right], & \mu \ge \alpha_1 \\ \frac{c(c+1)}{ab(a+1)(b+1)} Q_1, & \alpha_2 \le \mu \le \alpha_1 \\ \frac{c(c+1)}{ab(a+1)(b+1)} Q_1 \left[Q_1 + D - \frac{(a+1)(b+1)c}{ab(c+1)}\ Q_1\ \mu\right], & \mu \le \alpha_2 \end{cases} \quad (17)$$

where $Q_1 := Q_1(k)$ and $D := D(k)$ are defined as in Lemma 1 ;

$$\alpha_1 := \alpha_1(k) = \frac{ab(c+1)}{(a+1)(b+1)c} \frac{\left[1 + Q_1 + D\right]}{Q_1} \quad (18)$$

and

$$\alpha_2 := \alpha_2(k) = \frac{ab(c+1)}{(a+1)(b+1)c} \frac{[Q_1 + D - 1]}{Q_1}. \quad (19)$$

Each of the estimates in (17) is sharp.

Proof. Putting the values of a_2 and $Q_2 := DQ_1$ in (16), we have

$$a_3 = \frac{c(c+1)}{ab(a+1)(b+1)}\ Q_1 \left[Q_1 d_1^2 + d_2 + D d_1^2\right]$$

where $D := D(k)$ is as in Lemma 1. Therefore

$$|\mu a_2^2 - a_3| = \left| \frac{c^2}{a^2 b^2} Q_1^2 d_1^2 \mu - \frac{c(c+1)}{ab(a+1)(b+1)}\ Q_1 \left[Q_1 d_1^2 + d_2 + D d_1^2\right] \right|$$

$$= \frac{c(c+1)}{ab(a+1)(b+1)}\ Q_1 \left| \left\{ \frac{(a+1)(b+1)c}{ab(c+1)}\ Q_1\ \mu - Q_1 - D - 1 \right\} d_1^2 + (d_1^2 - d_2) \right| \quad (20)$$

$$\le \frac{c(c+1)}{ab(a+1)(b+1)}\ Q_1 \left\{ \left| \frac{(a+1)(b+1)c}{ab(c+1)}\ Q_1\ \mu - Q_1 - D - 1 \right| |d_1^2| + |d_1^2 - d_2| \right\}. \quad (21)$$

If $\mu \ge \alpha_1$, the expression inside the first modulus symbol on the right hand side of (21) is non negative. An application of Lemma 2 gives

$$|\mu a_2^2 - a_3| \le \frac{c(c+1)}{ab(a+1)(b+1)}\ Q_1 \left\{ \left(\frac{(a+1)(b+1)c}{ab(c+1)}\ Q_1\ \mu - Q_1 - D - 1 \right) 1 + 1 \right\} \quad (22)$$

$$= \frac{c(c+1)}{ab(a+1)(b+1)}\ Q_1 \Big\{ \frac{(a+1)(b+1)c}{ab(c+1)}\ Q_1\ \mu - Q_1 - D \Big\}.$$

This is precisely the first part of the assertion(17).
Next, suppose $\mu \leq \alpha_2$ where α_2 is given by (19). We rewrite (20) as

$$|\mu a_2^2 - a_3| = \frac{c(c+1)}{ab(a+1)(b+1)}\ Q_1 \left| d_2 + \left(Q_1 + D - \frac{(a+1)(b+1)c}{ab(c+1)}\ Q_1\ \mu \right) d_1^2 \right|. \quad (23)$$

An application of the inequality $|d_2| \leq 1 - |d_1|^2$ of Lemma 2 gives

$$|\mu a_2^2 - a_3| \leq \frac{c(c+1)}{ab(a+1)(b+1)}\ Q_1 \Big\{ \left(Q_1 + D - \frac{(a+1)(b+1)c}{ab(c+1)}\ Q_1\ \mu \right) |d_1^2| + 1 - |d_1^2| \Big\} \quad (24)$$

$$= \frac{c(c+1)}{ab(a+1)(b+1)}\ Q_1 \Big\{ \left(Q_1 + D - \frac{(a+1)(b+1)c}{ab(c+1)}\ Q_1\ \mu - 1 \right) |d_1|^2 + 1 \Big\}.$$

Applying Lemma 2 again we get

$$|\mu a_2^2 - a_3| \leq \frac{c(c+1)}{ab(a+1)(b+1)}\ Q_1 \left(Q_1 + D - \frac{(a+1)(b+1)c}{ab(c+1)}\ Q_1\ \mu \right) \quad (25)$$

which is the third part of the inequality in (17).
Observe that if $\alpha_2 \leq \mu \leq \alpha_1$ then

$$\left| Q_1 + D - \frac{(a+1)(b+1)c}{ab(c+1)}\ Q_1\ \mu \right| \leq 1. \quad (26)$$

Therefore (24) gives

$$|\mu a_2^2 - a_3| \leq \frac{c(c+1)}{ab(a+1)(b+1)}\ Q_1 \Big\{ 1 - |d_1^2| + |d_1^2| \Big\} = \frac{c(c+1)}{ab(a+1)(b+1)}\ Q_1. \quad (27)$$

We get the second part of the estimate in (17).
Next we discuss the sharpness of the estimates in (17).
If $\mu > \alpha_1$ then equality holds in (17) if and if equality holds in (22). This happens if and only if $|d_1| = 1$ and $|d_1^2 - d_2| = 1$. Thus $\omega(z) = z$. Equivalently, the extremal function is $\mathcal{G}(z)$ defined by (7) or one of its rotations. However, $\mu = \alpha_1$, is equivalent to

$$\frac{(a+1)(b+1)c}{(c+1)}\ Q_1\ \mu - Q_1 - D - 1 = 0.$$

Therefore equality holds true in (22) if and only if $|d_1^2 - d_2| = 1$ in (21) . Thus

$$\omega(z) = \frac{e^{i\theta} z(z + d_1)}{1 + \overline{d_1} z}, \qquad (0 \leq |d_1| \leq 1, z \in \mathcal{U})$$

for suitable values of θ (e.g $\theta = \pi - 2arg d_1$) and the extremal functions are $\psi(z,\theta,d_1)$ defined by (8) and d_1 is any complex number with $0 \le |d_1| \le 1$. Next, if $\mu < \alpha_2$ then equality holds in (25) if and only if $d_1^2 = -1$ and $d_2 = 0$ in (20) if and only if $d_1 = e^{i\frac{\pi}{2}}$ or $d_1 = e^{i\frac{3\pi}{2}}$ which also gives $d_2 = 0$. Thus $\omega(z) = e^{i\theta}z$ where $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$ and the extremal functions are $\psi(z,\theta,1)$ or one of the rotation. Also, $\mu = \alpha_2$ is equivalent to

$$Q_1 + D - \frac{(a+1)(b+1)c}{ab(c+1)}\, Q_1\ \mu = 1.$$

Therefore, equality holds in (25) if and only if $arg d_2 = 2arg d_1$ and $|d_2| = 1 - |d_1|^2$. Thus the extremal function is $\psi(e^{i\theta}z, 0, \eta)$, $(0 \le \theta \le 2\pi, 0 \le \eta \le 1)$. Lastly if $\alpha_2 < \mu < \alpha_1$, then equality holds true if $|d_1| = 0$ and $|d_2| = 1$. Therefore $\omega(z) = e^{i\theta}z^2$ and the extremal function is $\psi(e^{i\theta}z, 0, 0)$. The Proof of Theorem 1 is complete.
Putting the values of $Q_1 := Q_1(k)$ and $D := D(k)$ from Lemma 1 in Theorem 1 for $0 \le k < 1,\ k = 1$ and $k > 1$ respectively we get the following results:

Theorem 2. *Let the function f given by (1) be in the class $k - \mathcal{SP}_c^{a,b}$ $(0 \le k < 1,\ a,\ b,\ c > 0)$. Then*

$$|\mu a_2^2 - a_3| \le \begin{cases} \frac{2c(c+1)}{ab(a+1)(b+1)}\frac{A^2}{(1-k^2)}\left(\frac{2(a+1)(b+1)c}{ab(c+1)}\frac{A^2}{(1-k^2)}\,\mu - \frac{2A^2}{1-k^2} - \frac{A^2+2}{3}\right), & \mu \ge \rho_1 \\ \frac{2c(c+1)}{ab(a+1)(b+1)}\frac{A^2}{(1-k^2)}, & \rho_2 \le \mu \le \rho_1 \\ \frac{2c(c+1)}{ab(a+1)(b+1)}\frac{A^2}{(1-k^2)}\left(\frac{2A^2}{1-k^2} + \frac{A^2+2}{3} - \frac{2(a+1)(b+1)c}{ab(c+1)}\frac{A^2}{(1-k^2)}\,\mu\right), & \mu < \rho_2 \end{cases} \tag{28}$$

where

$$\rho_1 := \rho_1(k) = \frac{ab(c+1)}{2(a+1)(b+1)c}\frac{(1-k^2)}{A^2}\left(1 + \frac{2A^2}{1-k^2} + \frac{A^2+2}{3}\right) \tag{29}$$

$$\rho_2 = \rho_2(k) = \frac{ab(c+1)}{2(a+1)(b+1)c}\frac{(1-k^2)}{A^2}\left(\frac{2A^2}{1-k^2} + \frac{A^2+2}{3} - 1\right) \tag{30}$$

and the constant A is given by (10). Each of the estimates in (28) is sharp.

Theorem 3. *Let the function f given by (1) be in the class $\mathcal{SP}_c^{a,b}$ $(a,\ b,\ c \in$ R$,\ a,\ b,\ c > 0)$. Then*

$$|\mu a_2^2 - a_3| \le \begin{cases} \frac{16c(c+1)}{ab(a+1)(b+1)\pi^2}\left(\frac{4(a+1)(b+1)c}{ab(c+1)\pi^2}\,\mu - \frac{4}{\pi^2} - \frac{1}{3}\right), & (\mu \ge \beta_1) \\ \frac{8c(c+1)}{ab(a+1)(b+1)\pi^2}, & (\beta_2 \le \mu \le \beta_1) \\ \frac{16c(c+1)}{ab(a+1)(b+1)\pi^2}\left(\frac{4}{\pi^2} + \frac{1}{3} - \frac{4(a+1)(b+1)c}{ab(c+1)\pi^2}\,\mu\right), & (\mu \le \beta_2) \end{cases} \tag{31}$$

where

$$\beta_1 = \frac{ab(c+1)}{(a+1)(b+1)c}\left(\frac{5\pi^2}{24}+1\right) \tag{32}$$

and

$$\beta_2 = \frac{ab(c+1)}{(a+1)(b+1)c}\left(1-\frac{\pi^2}{24}\right). \tag{33}$$

Each of the estimates in (31) is sharp.

Theorem 4. *Let the function f given by (1) be in the class k- $\mathcal{SP}_c^{a,b}$ $(k > 1, a, b, c > 0)$. Then*

$$|\mu a_2^2 - a_3| \leq \begin{cases} \frac{c(c+1)}{ab(a+1)(b+1)}\ Q_1\left(\frac{(a+1)(b+1)c}{ab(c+1)}\ Q_1\ \mu - Q_1 - B(t)\right), & \mu \geq \delta_1 \\ \frac{c(c+1)}{ab(a+1)(b+1)}\ Q_1, & \delta_2 \leq \mu \leq \delta_1 \\ \frac{c(c+1)}{ab(a+1)(b+1)}\ Q_1\left(Q_1 + B(t) - \frac{(a+1)(b+1)c}{ab(c+1)}\ Q_1\ \mu\right), & \mu \leq \delta_2 \end{cases} \tag{34}$$

where $\mathcal{K}(t)$ is the complete elliptic integral of the first kind , $Q_1 := Q_1(k)$ is given in (9),

$$B(t) = \frac{4\mathcal{K}^2(t)(t^2+6t+1)-\pi^2}{24\mathcal{K}^2(t)\sqrt{t}(1+t)},$$

$$\delta_1 = \frac{ab(c+1)}{(a+1)(b+1)c}\frac{(1+Q_1+B(t))}{Q_1} \tag{35}$$

and

$$\delta_2 = \frac{ab(c+1)}{(a+1)(b+1)c}\frac{(Q_1-1+B(t))}{Q_1}. \tag{36}$$

Each of the estimates in (34) is sharp.

Remark 1. Our Theorems 2,3 and 4 include several previous results for special values of $k,\ a,\ b$ and c. For example, taking $a = 2,\ b = 1$ and $c = 2-\lambda$ $(0 \leq \lambda \leq 1)$ in Theorems 2 and 4 we get the Fekete- Szegö inequalities for the class $k - \mathcal{SP}_\lambda$ respectively for $0 < k < 1$ and $k > 1$ [12]. The Fekete- Szegö inequalities for the classes of k- parabolic starlike functions and k- uniformly convex functions $(0 < k < 1,\ k > 1)$ correspond to the special cases $\lambda = 0$ and $\lambda = 1$ of the above. Similarly, the choice $a = 2,\ b = 1$ and $c = 2-\lambda$ in Theorem 3 gives results of Srivastava and Mishra [21] for the class $\mathcal{SP}_\lambda$. By taking $a = 2,\ b = 1$ and $c = n+1$ we get results for a class $k - \mathcal{UCV}_n$ studied recently in [13]. The Fekete-Szegö inequalities for the class of uniformly convex functions are also included in our Theorem 3 for the particular values $a = 2,\ b = 1$ and $c = 1$. The classical results of Keogh and Merkes

[10] on the Fekete- Szegö inequalities for the classes of univalent starlike functions and univalent convex functions are included in our Theorem 2 in the particular cases $k = 0,\ a = 2,\ b = 1,\ c = 2$ and $k = 0,\ a = 2, b = 1,\ c = 1$ respectively.

4. Improvements of the main results

In this section we discuss some improvements of the second part of assertion in (17).

Remark 2. The second inequality in (17) can be improved as follows:

$$|\mu a_2^2 - a_3| + (\mu - \alpha_2)|a_2|^2 \leq \frac{c(c+1)}{ab(a+1)(b+1)}\ Q_1, \qquad \alpha_2 \leq \mu \leq \alpha_3 \tag{37}$$

and

$$|\mu a_2^2 - a_3| + (\alpha_1 - \mu)|a_2|^2 \leq \frac{c(c+1)}{ab(a+1)(b+1)}\ Q_1, \qquad \alpha_3 \leq \mu \leq \alpha_1, \tag{38}$$

where α_3 is given by

$$\alpha_3 = \alpha_3(k) = \frac{ab(c+1)}{(a+1)(b+1)c}\frac{(Q_1 + D)}{Q_1}.$$

Proof. Suppose $0 \leq k < \infty$ and $\alpha_2 \leq \mu \leq \alpha_3$. Using (23) for $|\mu a_2^2 - a_3|$ and putting the value of α_2 we have

$$\begin{aligned}
|\mu a_2^2 - a_3| \quad &+ \quad (\mu - \alpha_2)|a_2|^2 = |\mu a_2^2 - a_3| + \left\{\mu - \frac{ab(c+1)}{(a+1)(b+1)c}\frac{(Q_1 + D - 1)}{Q_1}\right\}|a_2|^2 \\
&\leq \quad \frac{c(c+1)}{ab(a+1)(b+1)}\ Q_1\left\{\left|Q_1 + D - \frac{(a+1)(b+1)c}{ab(c+1)}\ Q_1\ \mu\right| |d_1|^2 + |d_2|\right\} \\
&+ \quad \left\{\mu - \frac{ab(c+1)}{(a+1)(b+1)c}\frac{(Q_1 + D - 1)}{Q_1}\right\}\frac{c^2}{a^2b^2}\ Q_1^2|d_1^2| \\
&= \quad \frac{c(c+1)}{ab(a+1)(b+1)}\ Q_1\left\{|d_2| + \left|Q_1 + D - \frac{(a+1)(b+1)c}{ab(c+1)}\ Q_1\ \mu\right|\ |d_1|^2\right. \\
&+ \quad \left.\left(\frac{(a+1)(b+1)c}{ab(c+1)}\ Q_1\ \mu - Q_1 - D + 1\right)|d_1|^2\right\}.
\end{aligned} \tag{39}$$

Observe that , since $\mu \leq \alpha_3$

$$Q_1 + D - \frac{(a+1)(b+1)c}{ab(c+1)}\ Q_1\ \mu \geq 0.$$

Therefore applying Lemma 2 in (39) we get

$$\begin{aligned} |\mu a_2^2 - a_3| + (\mu - \alpha_2)|a_2|^2 &\leq \frac{c(c+1)}{ab(a+1)(b+1)}\, Q_1[1 - |d_1|^2 + |d_1|^2] \\ &= \frac{c(c+1)}{ab(a+1)(b+1)}\, Q_1, \qquad \alpha_2 \leq \mu \leq \alpha_3. \qquad (40) \end{aligned}$$

This establishes (37).
Similarly the estimate in (38) can be established.

Remark 3. By putting the values of $Q_1(k)$ and $D(k)$ for $0 \leq k < 1$ in (37) and (38) the second part of the estimate in (28) can be improved as follows:

$$|\mu a_2^2 - a_3| + (\mu - \rho_2)|a_2|^2 \leq \frac{2c(c+1)}{ab(a+1)(b+1)} \frac{A^2}{(1-k^2)}, \qquad \rho_2 \leq \mu \leq \rho_3$$

and

$$|\mu a_2^2 - a_3| + (\rho_1 - \mu)|a_2|^2 \leq \frac{2c(c+1)}{ab(a+1)(b+1)} \frac{A^2}{(1-k^2)}, \qquad \rho_3 \leq \mu \leq \rho_1,$$

where ρ_3 is given by

$$\rho_3 = \frac{ab(c+1)}{(a+1)(b+1)c} \left(1 + (A^2+2)\frac{(1-k^2)}{6A^2} \right).$$

Remark 4. By putting the values of $Q_1(k)$ and $D(k)$ for $k = 1$ in (37) and (38) the second part of the assertion in (31) can be improved as follows:

$$|\mu a_2^2 - a_3| + (\mu - \beta_2)|a_2|^2 \leq \frac{8c(c+1)}{ab(a+1)(b+1)\pi^2}, \quad \beta_2 \leq \mu \leq \beta_3$$

and

$$|\mu a_2^2 - a_3| + (\rho_1 - \mu)|a_2|^2 \leq \frac{8c(c+1)}{ab(a+1)(b+1)\pi^2}, \qquad \beta_3 \leq \mu \leq \beta_1,$$

where β_3 is given by

$$\beta_3 = \frac{ab(c+1)}{(a+1)(b+1)c} \left(\frac{8}{\pi^2} + \frac{2}{3} \right).$$

Remark 5. By putting the values of $Q_1(k)$ and $D(k)$ for $k > 1$ in (37) and (38) the second part of the assertion in (34) can be improved as follows:

$$|\mu a_2^2 - a_3| + (\mu - \delta_2)|a_2|^2 \leq \frac{c(c+1)\pi^2}{4ab(a+1)(b+1)(k^2-1)\mathcal{K}^2(t)\sqrt{t}(1+t)}$$

and

$$|\mu a_2^2 - a_3| + (\delta_1 - \mu)|a_2|^2 \leq \frac{c(c+1)\pi^2}{4ab(a+1)(b+1)(k^2-1)\mathcal{K}^2(t)\sqrt{t}(1+t)},$$

where δ_3 is given by

$$\delta_3 = \frac{ab(c+1)}{(a+1)(b+1)c}\left(1 + \frac{2\left(\mathcal{K}^2(t)(t^2+6t+1) - \pi^2\right)(k^2-1)}{3\pi^2}\right).$$

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