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SOLUTIONS TO THE SET-THEORETICAL YANG BAXTER EQUATION DERIVED FROM RELATIONS

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Abstract. Given a binary relation R on the set X, we define $S\colon X\times X \to X\times X$ by setting $S((u,v))$ to be (u,v) if uRv, and (v,u) otherwise. This construction represent a way to associate a function to an arbitrary relation. We give simple conditions that completely characterize the relations R for which S satisfies the set-theoretical Yang-Baxter equation.

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1. Introduction and preliminaries

The *Yang-Baxter equation* (YBE) first appeared in C.N. Yang's paper [9]. (Prof. Yang was a physics professor at SUNY Stony Brook and a Nobel laureate.) Independently, This equation was also discovered by Baxter, in [1, 2] (in Statistical Mechanics). It then turned out that this equation was also important in mathematical physics, the theory of quantum groups, knot theory, non-commutative geometry, quantum computing, etc.

Finding all solutions of the Yang-Baxter equation is a difficult task that is far from being completed. Nevertheless many solutions of these equations have been found during the last 30 years, and the related algebraic structures (Hopf algebras, Yetter Drinfeld structures and categories, etc) have been studied (for example, see [7]). At present, the study of solutions of the Yang-Baxter equation attracts the attention of a broad circle of scientists.

In this paper we study solutions of the set-theoretical Yang-Baxter equation derived from relations on X (which are not necessarily invertible). These solutions generalize the twist map. Comparing our solutions with those from [4, 5], we observe that there is just a trivial overlap. The only non-degenerate function derived from a relation is the one that always takes (x, y) to (y, x), often written as (the twist map). Other applications of our construction are also presented in our paper.

The main results are presented in a convenient way in next section. We conclude with examples of relations which lead to set-theoretical solutions for the Yang-Baxter equation.

The notations and terminology related to the Yang-Baxter equation are the following (see, for example, [6]). Let V be a vector space over a field k. A linear automorphism R of $V \otimes V$ is a solution of the Yang-Baxter equation, if the equality

$$(R \otimes id)(id \otimes R)(R \otimes id) = (id \otimes R)(R \otimes id)(id \otimes R) \tag{1}$$

holds in the automorphism group of $V \otimes V \otimes V$.

R is a solution of the *quantum Yang-Baxter equation* (QYBE) if

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{2}$$

where R_{ij} means R acting on the i-th and j-th components.

Let T be the twist map, $T(v \otimes w) = w \otimes v$. Then R satisfies equation (1) if and only if $R \circ T$ satisfies equation (2) if and only if $T \circ R$ satisfies equation (2).

V. Drinfeld ([3]) posed the problem of studying set-theoretical solutions of the Yang-Baxter equation. Specifically, we consider a set X and an invertible map S. We think of the equation (1) as an equality of maps from $X \times X$ to $X \times X$:

$$(S \times id)(id \times S)(S \times id) = (id \times S)(S \times id)(id \times S). \tag{3}$$

2. Main results

For an arbitrary relation on a set X, we associate a function from $X \times X$ into itself in a natural way.

Given a binary relation R on X (i.e., a subset of $X \times X$), we define $S_R = S\colon X \times X \to X \times X$ by

$$S(u, v) = \begin{cases} (u, v) & \text{if } uRv; \\ (v, u) & \text{otherwise.} \end{cases} \tag{4}$$

So, S switches u and v if and only if u is not related to v. If $u = v$, the output is the same. In other words, S is the same regardless of what $R \cap \{(x, x) : x \in X\}$ is. Therefore, we can assume without loss of generality that R is reflexive, and will do so throughout this paper.

We often deal with the complement of R, which we will denote by N (i.e., xNy if and only if x is not related to y by R).

We proceed to investigate the relations R for which S_R is a solution of the set-theoretical Yang-Baxter equation.

For three arbitrary elements $a, b, c \in X$, we consider all possible patterns of relation/non-relation between them. There are 6 ordered pairs which can be formed from the set $\{a, b, c\}$. There are 2^6 patterns of relation/non-relation, but we can simplify this step as follows. We consider the non-diagonal pairs from $\{a, b, c\}$ in the following order: (a, b) first, (b, c) second, (a, c) third, (b, a) fourth, (c, b) fifth, and (c, a) sixth. Then we write the symbols for these six pairs in order as a string. For example, the string RN---- represents the situation where aRb, bNc, and the other pairs can be anything.

From now on, LHS $= S_{12} \circ S_{23} \circ S_{12}(a, b, c)$, and RHS $= S_{23} \circ S_{12} \circ S_{23}(a, b, c)$. Thus the YBE is satisfied for (a, b, c) if and only if LHS $=$ RHS for (a, b, c).

Next, we consider a collection of cases which exhausts all possible patterns.

1. RR---- Here we have aRb and bRc, so both S_{12} and S_{23} leave (a, b, c) fixed. Thus LHS $= (a, b, c) =$ RHS (the YBE is satisfied).

2. RNR-R- In this case, aRb, bNc, aRc and cRb. LHS $= S_{12} \circ S_{23} \circ S_{12}(a, b, c) = S_{12} \circ S_{23}(a, b, c) = S_{12}(a, c, b) = (a, c, b)$, and RHS $= S_{23} \circ S_{12} \circ S_{23}(a, b, c) = S_{23} \circ S_{12}(a, c, b) = S_{23}(a, c, b) = (a, c, b)$. The YBE is satisfied.

   We leave it to the reader to verify the results in the remaining cases.

3. RNR-N- We have LHS $= (a, c, b)$, RHS $= (a, b, c)$; the YBE is not satisfied. (Since R is reflexive and cNb, $c \neq b$, and $(a, c, b) \neq (a, b, c)$.)

4. RNN--- Here, LHS $= (c, a, b) =$ RHS and the YBE is satisfied.

5. NRRR-- Here, LHS $= (b, a, c) =$ RHS and the YBE is satisfied.

6. NRRN-- We have LHS $= (a, b, c)$, RHS $= (b, a, c)$ and the YBE is not satisfied. (Similarly to (iii), aRb is false, $a \neq b$ and $(a, b, c) \neq (b, a, c)$.)

7. NRN--- Here, LHS $= (b, c, a) =$ RHS and the YBE is satisfied.

8. NNRR-- We have LHS $= (b, a, c)$, and left the status of the fifth pair (c,b) undecided. If cRb, then RHS $= (a, c, b)$. And if cNb, then RHS $= (a, b, c)$. In any event, LHS $\neq$ RHS and the YBE is not satisfied. (Similarly to (iii), we have aNb, $a \neq b$, and so on.)

9. NNRNR- We have LHS $= (a, b, c)$, RHS $= (a, c, b)$, and the YBE is not satisfied. (Similarly to (iii), bNc, $b \neq c$, and $(a, b, c) \neq (a, c, b)$.)

10. NNRNN- Here, LHS $= (a, b, c) =$ RHS and the YBE is satisfied.

11. NNN--- Here, LHS $= (c, b, a) =$ RHS and the YBE is satisfied.

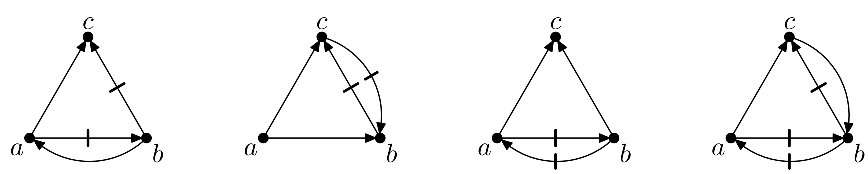
Thus S will satisfy the YBE for all triples from X if and only if the patterns in cases (iii), (vi), (viii), and (ix) never occur.

The rest of the analysis will be easier if we use a graphical representation. We represent xRy by an arrow from x to y, and xNy by a crossed arrow from x to y.

While binary relations are often represented by digraphs, our scheme is slightly unusual. The typical method is to have an arrow from x to y if and only if xRy. We augment this by including crossed arrows for all the pairs where xNy.

We obtain the four *forbidden diagrams* drawn in Figure 1. For example, diagram (α) corresponds to case (viii) in our earlier analysis (which showed that the pattern NNRR– could not occur if S satisfies the YBE).

Diagrams (β), (γ) and (δ) are in a similar situation. They can not appear as subgraphs of the labeled digraph for R, since they are ruled out by cases (iii), (vi) and (ix), respectively.



α (case viii) β (case iii) γ (case vi) δ (case ix)

Figure 1: Forbidden diagrams

Since a, b, and c were arbitrary, we have that when S satisfies the YBE, our digraph for R can have no labeled subgraphs isomorphic to any of the four forbidden diagrams.

We also do not have to worry about the possibility that a, b and c are not distinct in a forbidden diagram. We chose R to be reflexive, so the existence of a crossed arrow between x and y implies $x \neq y$. We have that every arrow has a unique labeling, so two elements with differently labeled arrows to (or from) a common element must be distinct. For instance in (α), we have $a \neq b$ and $b \neq c$ since there are crossed arrows from a to b and from b to c. And $a \neq c$ since b has a plain arrow to a and a crossed arrow to c. The arguments for (β), (γ) and (δ) are similar.

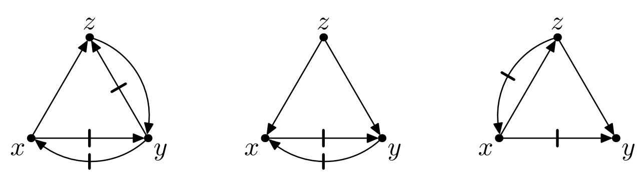
We already have that if S satisfies the YBE, the digraph for R does not contain any of the four forbidden diagrams. But the converse is also true. For suppose S does not satisfy the YBE. Then some particular triple (a, b, c) must have LHS $\neq$ RHS, and fall into a case corresponding to one of the forbidden diagrams. This implies that a, b and c are distinct, so the digraph for R has one of these diagrams as a labeled subgraph. Thus we have the following result.

Theorem 1 *If S is derived from the binary relation R as in (4), then S satisfies the YBE if and only if the labeled digraph for R does not have any labeled subgraphs isomorphic to any of the four forbidden diagrams in Figure 1.*

Our next task is to produce a nicer description of relations R with digraphs that omit the forbidden diagrams. Looking at these diagrams, we see that diagram (δ) is redundant.

Lemma 2 *If the labeled digraph of R does not have any subgraphs isomorphic to (α), (β) or (γ), then it also has none isomorphic to (δ).*

Proof. Suppose that there is a subgraph isomorphic to (δ). We call its elements x, y and z, as in (a) of Figure 2. Now consider the pair (z, x). If there is a plain arrow from z to x, then we have a subgraph isomorphic to forbidden diagram (β), as shown in (b). The other possibility is that there is a crossed arrow from z to x, but this yields a subgraph isomorphic to (α), as shown in (c).



(a) (b) (c)

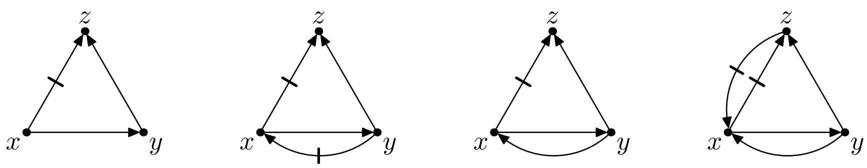
Figure 2: Proof of Lemma 2

We will assume for the moment that the digraph for R does not have any subgraphs isomorphic to (α), (β) or (γ).

We use R^{-1} to denote the inverse relation of R, so $(x, y) \in R^{-1}$ if and only if $(y, x) \in R$. Now consider $R \cup R^{-1}$, the "related at least one way" relation. It is reflexive, since R is. By construction, it is symmetric. We claim that it is also transitive.

For suppose (x, y) and (y, z) are both in $R \cup R^{-1}$. We consider four cases, as follows.

1. xRy and yRz. (Refer to Figure 3.) If xRz, the claim is proved. So assume that xNz, as in (a). Now if yNx, we have a subgraph isomorphic to diagram (α), as shown in (b). So we have yRx, which gives the situation shown in (c). If zNx, we get a subgraph isomorphic to diagram (β), as in (d). Thus zRx, and (x, z) is in $R \cup R^{-1}$.



(a) (b) (gives (α)) (c) (d) (gives (β))

Figure 3:

2. $xR^{-1}y$ and $yR^{-1}z$. Then z R y and y R x, and we proceed as in Case 1.

3. xRy and $yR^{-1}z$. We are done unless both xRz and $xR^{-1}z$ are false. But if they are both false, we have an instance of diagram (β), which is not allowed.

4. $xR^{-1}y$ and yRz. Similarly to Case 3, we are done unless both xRz and $xR^{-1}z$ are false. But this can not happen, since it gives diagram (γ).

This argument shows that $R \cup R^{-1}$ is transitive, and we have established the following.

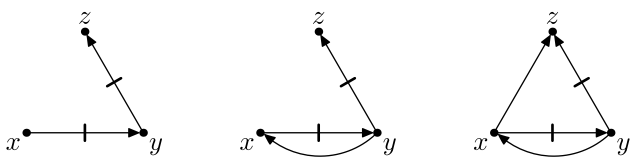
Theorem 3 *If the digraph for R does not have any subgraphs isomorphic to (α), (β) or (γ), then $R \cup R^{-1}$ is an equivalence relation.*

Still assuming that the digraph for R does not have (α), (β) or (γ), we have that X is partitioned into equivalence classes of $R \cup R^{-1}$. All three of (α), (β) and (γ) are connected by $R \cup R^{-1}$, so any subgraph isomorphic to one of them must be contained in an equivalence class.

We will now focus on a particular equivalence class Y of $R \cup R^{-1}$. Then every two elements of Y are related by $R \cup R^{-1}$, and there are never elements x and y where both xNy and yNx.

We want to focus on the relation N, the complement of R. We claim that N is a strict partial order on Y. In other words, N is irreflexive, antisymmetric and transitive. N is irreflexive, since yRy for all $y \in Y$, making yNy always false. And N is antisymmetric on Y, for if xNy and yNx, we have $(x, y) \notin R \cup R^{-1}$, which is impossible since Y is a class of $R \cup R^{-1}$.

To see that N is transitive on Y, suppose xNy and yNz as in (a) of Figure 4. Since Y is a class of $R \cup R^{-1}$, we must have yRx as in (b). Then if xRz as in (c), there is a subgraph isomorphic to (α). Thus xNz, showing N is transitive.



(a) (b) (c) (yields (α))

Figure 4:

Thus N must be a strict partial order on Y. It turns out that the converse of this is also true. Let Y be any class of $R \cup R^{-1}$, and assume that N, the complement of R, is a strict partial order on Y. Looking at the three diagrams (α), (β) and (γ) in Figure 1, we see that none can be isomorphic to subgraphs of the labeled digraph on Y. For (α) has aNb and bNc, but not aNc, contradicting transitivity. The forbidden diagrams (β) and (γ) both contradict the antisymmetry of N. Thus we have the following.

Theorem 4 *Let Y be an equivalence class of $R \cup R^{-1}$. Then Y has no subgraphs isomorphic to (α), (β) or (γ) if and only if N is a strict partial order on Y.*

Putting this theorem together with our previous results, we obtain our main result.

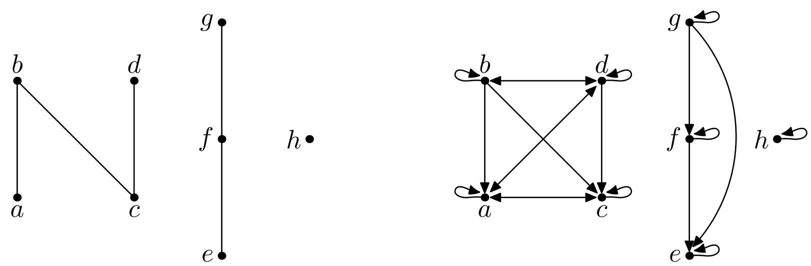
Theorem 5 *The function S derived from the relation R satisfies the YBE if and only if $R \cup R^{-1}$ is an equivalence relation on X and N is a strict partial order on each class of $R \cup R^{-1}$.*

Remark. The result of Theorem 5 was partially anticipated by the second author's work in [8]. It was shown there that when $(A, \vee, \wedge, 0, 1, ')$ is a Boolean algebra, then the function $Q(x, y) = (x \vee y, x \wedge y)$ is a solution to the YBE. The proof is a direct calculation involving $\vee$ and $\wedge$ which works in any distributive lattice, so $Q(x, y)$ as above is a solution to the YBE on any distributive lattice.

An ordered chain is a distributive lattice, so [8] established a special case of our Theorem 5. Given the linearly ordered chain $a_1 < a_2 < a_3 \cdots a_n$, the operations $\vee$ and $\wedge$ are defined in terms of the order. For any $i < j$, we have $a_i < a_j$, $Q(a_i, a_j) = (a_i \vee a_j, a_i \wedge a_j) = (a_j, a_i)$ and $Q(a_j, a_i) = (a_j \vee a_i, a_j \wedge a_i) = (a_j, a_i)$.

Taking our relation R to be $\geq$, we have that Q switches x and y iff xNy. Thus Q is identical with S in this case, where the entire chain is an equivalence class of $R \cup R^{-1}$ and N is the strict partial order $<$.

Here is an example showing a typical relation R corresponding to a solution of the YBE. Let X be the set $\{a, b, c, d, e, f, g, h\}$. We let the classes of $R \cup R^{-1}$ be $\{a, b, c, d\}$, $\{e, f, g\}$ and $\{h\}$. The partial order N will be $\{(a, b), (c, b), (c, d)\}$ on $\{a, b, c, d\}$, $\{(e, f), (e, g), (f, g)\}$ on $\{e, f, g\}$, and $\emptyset$ on $\{h\}$. This gives the Hasse diagrams shown in (a) of Figure 5. Now R is completely determined, and is shown in (b).



(a) Hasse diagrams for N (b) R (crossed arrows not shown)

Figure 5:

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