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HARMONIC MULTIVALENT FUNCTIONS DEFINED BY AN INTEGRAL OPERATOR

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Abstract. In this paper, we introduce a new class of p-valent harmonic functions defined by an integral operator. Coefficient inequalities, distortion bounds for the functions belonging to this class are obtained. The invariance of the class $\overline{H}_p(n, \alpha)$ under convolution, convex linear combination and also under generalized Bernardi-Libera Livingston integral operator are discussed.

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1. Introduction

A continuous complex valued function $f = u + iv$ is said to be harmonic in a simply connected domain D if both u and v are real harmonic in D. In any simply connected domain we can write $f = h + \overline{g}$ where h and g are analytic in D. We call h, the analytic part and g, the co-analytic part of f. In this case, the Jacobian of f is given by $J_f(z) = |h'(z)|^2 - |g'(z)|^2$.

A necessary and sufficient condition for f to be locally univalent and sense preserving in D is that $|h'(z)| > |g'(z)|$, $z \in D$.

Denote by H the class of functions $f = h + \overline{g}$ that are harmonic univalent and sense preserving in the unit disc $\mathcal{U} = \{z : |z| < 1\}$ so that $f = h + \overline{g}$ is normalized by $f(0) = f_z(0) - 1 = 0$.

Ahuja and Jahangiri [2] defined the class $H_p(n)$ $(p, n \in \mathbb{N})$ consisting of all p-valent harmonic functions $f = h + \overline{g}$ that are sense preserving in $\mathcal{U}$ and h and g are of the form

$$h(z) = z^p + \sum_{k=2}^{\infty} a_{k+p-1} z^{k+p-1}, \quad g(z) = \sum_{k=1}^{\infty} b_{k+p-1} z^{k+p-1}, \quad |b_p| < 1 \qquad (1)$$

The integral operator I^n was introduced by Salagean [7], given by

$$I^0 f(z) = f(z)$$
$$I^1 f(z) = If(z) = \int_0^z f(t)t^{-1}dt$$
$$I^n f(z) = I(I^{n-1}f(z)), \quad n \in \mathbb{N} \ \ and \ \ f(z) = z + a_2 z^2 + a_3 z^3 + \ldots,$$

$z \in \mathcal{U}$, which is holomorphic in $\mathcal{U}$.

For $f = h + \overline{g}$ given by (1), the integral operator I^n of f is defined as

$$I^n f(z) = I^n h(z) + (-1)^n \overline{I^n g(z)} \tag{2}$$

where

$$I^n h(z) = z^p + \sum_{k=2}^{\infty} \left(\frac{k+p-1}{p} \right)^{-n} a_{k+p-1} z^{k+p-1}$$

and

$$I^n g(z) = \sum_{k=1}^{\infty} \left(\frac{k+p-1}{p} \right)^{-n} b_{k+p-1} z^{k+p-1}$$

For harmonic functions

$$f(z) = z^p + \sum_{k=2}^{\infty} a_{k+p-1} z^{k+p-1} + \sum_{k=1}^{\infty} \overline{b}_{k+p-1} \overline{z}^{k+p-1}$$

and

$$F(z) = z^p + \sum_{k=2}^{\infty} A_{k+p-1} z^{k+p-1} + \sum_{k=1}^{\infty} \overline{B}_{k+p-1} \overline{z}^{k+p-1}$$

of $H_p(n)$, we define the convolution of f and F as

$$(f * F)(z) = z^p + \sum_{k=2}^{\infty} (a_{k+p-1})(A_{k+p-1}) z^{k+p-1} + \sum_{k=1}^{\infty} (\overline{b}_{k+p-1})(\overline{B}_{k+p-1}) \overline{z}^{k+p-1} \tag{3}$$

which belongs to $H_p(n)$.
For $0 \leq \alpha < 1$, $p, n \in \mathbb{N}$ and $z \in \mathcal{U}$, let $H_p(n, \alpha)$ denote the subclass of $H_p(n)$ consisting of harmonic functions of the form (3) such that

$$Re \left\{ \frac{I^n (f * F)(z)}{I^{n+1} (f * F)(z)} \right\} > \alpha \tag{4}$$

where I^n is defined by (2).

We further denote by $\overline{H}_p(n,\alpha)$ the subclass of $H_p(n,\alpha)$ such that the functions h and g_n in $f_n = h + \overline{g}_n$ are of the form

$$h(z) = z^p - \sum_{k=2}^{\infty} a_{k+p-1} z^{k+p-1}, \quad g_n(z) = (-1)^{n-1} \sum_{k=1}^{\infty} b_{k+p-1} z^{k+p-1} \tag{5}$$

where $a_{k+p-1}, b_{k+p-1} \geq 0$ and $|b_p| < 1$.

The convolution of the harmonic functions f_n and F_n of the form (5) is defined as

$$(f_n * F_n)(z) = z^p - \sum_{k=2}^{\infty} a_{k+p-1} A_{k+p-1} z^{k+p-1} + (-1)^{n-1} \sum_{k=1}^{\infty} \overline{b}_{k+p-1} \overline{B}_{k+p-1} \overline{z}^{k+p-1} \tag{6}$$

which belongs to $\overline{H}_p(n,\alpha)$.

The families $H_p(n+1,n,\alpha)$ and $\overline{H}_p(n+1,n,\alpha)$ include a variety of well known classes of harmonic functions as well as many new ones. For example $\overline{H}_1(1,0,\alpha) = HS(\alpha)$ is the class of sense-preserving, harmonic, univalent functions which are starlike of order α in $\mathcal{U}$ and $\overline{H}_1(2,1,\alpha) = HK(\alpha)$ is the class of sense preserving, harmonic univalent functions f which are convex of order α in $\mathcal{U}$ and $\overline{H}_1(n+1,n,\alpha) = \overline{H}(n,\alpha)$ is the class of Salagean type harmonic univalent functions.

For harmonic functions f of the form (1) with $b_1 = 0$, Avei and Zlotkiewicz [1] showed that if $\sum_{k=2}^{\infty} k(|a_k|+|b_k|) \leq 1$ then $f \in HS(0) = \overline{H}_1(1,0,0)$ and if $\sum_{k=2}^{\infty} k^2(|a_k|+|b_k|) \leq 1$ then $f \in HK(0) = \overline{H}_1(2,1,0)$.

For harmonic functions f of the form (5) with $n = 1$, Jahangiri [4] showed that $f \in HS(\alpha)$ if and only if $\sum_{k=2}^{\infty}(k-\alpha)|a_k| + \sum_{k=2}^{\infty}(k+\alpha)|b_k| \leq 1-\alpha$ and $f \in \overline{H}_1(2,1,\alpha) = HK(\alpha)$ if and only if

$$\sum_{k=2}^{\infty} k(k-\alpha)|a_k| + \sum_{k=2}^{\infty} k(k+\alpha)|b_k| \leq 1-\alpha.$$

2. Main Results

In this section we prove a characterization theorem for the class $H_p(n,\alpha)$. The results obtained here generalizes the results of Luminita Ioana Cotîrlă [6].

Theorem 2.1. *Let $f * F$ by given by (3). If*

$$\sum_{k=1}^{\infty} \{\psi(n,k,p,\alpha)|a_{k+p-1}||A_{k+p-1}| + \eta(n,k,p,\alpha)|b_{k+p-1}||B_{k+p-1}|\} \leq 2 \tag{7}$$

where

$$\psi(n,k,p,\alpha)=\frac{\left(\frac{k+p-1}{p}\right)^{-n}-\alpha\left(\frac{k+p-1}{p}\right)^{-(n+1)}}{1-\alpha}$$
$$\eta(n,k,p,\alpha)=\frac{\left(\frac{k+p-1}{p}\right)^{-n}+\alpha\left(\frac{k+p-1}{p}\right)^{-(n+1)}}{1-\alpha}$$

$a_p=1=A_p$, α $(0\le\alpha<1)$, $p,n\in\mathbb{N}$ *then* $f*F$ *is sense preserving in* $\mathcal{U}$ *and* $f*F\in H_p(n,\alpha)$.

Proof. Suppose that (7) holds. According to (2), (3) and (4) we only need to show that

$$Re\left\{\frac{I^n(f*F)(z)-\alpha I^{n+1}(f*F)(z)}{I^{n+1}(f*F)(z)}\right\}\ge 0.$$

The case $r=0$ is obvious.
For $0<r<1$, it follows that

$$Re\left\{\frac{I^n(f*F)(z)-\alpha I^{n+1}(f*F)(z)}{I^{n+1}(f*F)(z)}\right\}$$
$$=Re\left\{\frac{z^p(1-\alpha)+\sum_{k=2}^{\infty}\left[\left(\frac{k+p-1}{p}\right)^{-n}-\alpha\left(\frac{k+p-1}{p}\right)^{-(n+1)}\right]a_{k+p-1}A_{k+p-1}z^{k+p-1}}{z^p+\sum_{k=2}^{\infty}\left(\frac{k+p-1}{p}\right)^{-(n+1)}a_{k+p-1}A_{k+p-1}z^{k+p-1}+(-1)^{n+1}\sum_{k=1}^{\infty}\left(\frac{k+p-1}{p}\right)^{-(n+1)}\bar{b}_{k+p-1}\overline{B}_{k+p-1}\overline{z}^{k+p-1}}\right.$$
$$\left.+\frac{(-1)^n\sum_{k=1}^{\infty}\left[\left(\frac{k+p-1}{p}\right)^{-n}+\alpha\left(\frac{k+p-1}{p}\right)^{-(n+1)}\right]\bar{b}_{k+p-1}\overline{B}_{k+p-1}\overline{z}^{k+p-1}}{z^p+\sum_{k=2}^{\infty}\left(\frac{k+p-1}{p}\right)^{-(n+1)}a_{k+p-1}A_{k+p-1}z^{k+p-1}+(-1)^{n+1}\sum_{k=1}^{\infty}\left(\frac{k+p-1}{p}\right)^{-(n+1)}\bar{b}_{k+p-1}\overline{B}_{k+p-1}\overline{z}^{k+p-1}}\right\}$$
$$=Re\left\{\frac{(1-\alpha)+\sum_{k=2}^{\infty}\left[\left(\frac{k+p-1}{p}\right)^{-n}-\alpha\left(\frac{k+p-1}{p}\right)^{-(n+1)}\right]a_{k+p-1}A_{k+p-1}z^{k-1}}{1+\sum_{k=2}^{\infty}\left(\frac{k+p-1}{p}\right)^{-(n+1)}a_{k+p-1}A_{k+p-1}z^{k-1}+(-1)^{n+1}\sum_{k=1}^{\infty}\left(\frac{k+p-1}{p}\right)^{-(n+1)}\bar{b}_{k+p-1}\overline{B}_{k+p-1}\overline{z}^{k+p-1}z^{-p}}\right.$$
$$\left.+\frac{(-1)^n\sum_{k=1}^{\infty}\left[\left(\frac{k+p-1}{p}\right)^{-n}+\alpha\left(\frac{k+p-1}{p}\right)^{-(n+1)}\right]\bar{b}_{k+p-1}\overline{B}_{k+p-1}\overline{z}^{k+p-1}z^{-p}}{1+\sum_{k=2}^{\infty}\left(\frac{k+p-1}{p}\right)^{-(n+1)}a_{k+p-1}A_{k+p-1}z^{k-1}+(-1)^{n+1}\sum_{k=1}^{\infty}\left(\frac{k+p-1}{p}\right)^{-(n+1)}\bar{b}_{k+p-1}\overline{B}_{k+p-1}\overline{z}^{k+p-1}z^{-p}}\right\}$$
$$=Re\left[\frac{(1-\alpha)+C(z)}{1+D(z)}\right].$$

For $z = re^{i\theta}$,

$$C(re^{i\theta}) = \sum_{k=2}^{\infty}\left[\left(\frac{k+p-1}{p}\right)^{-n} - \alpha\left(\frac{k+p-1}{p}\right)^{-(n+1)}\right] a_{k+p-1}A_{k+p-1}r^{k-1}e^{i(k-1)\theta}$$
$$+(-1)^n\sum_{k=1}^{\infty}\left[\left(\frac{k+p-1}{p}\right)^{-n} + \alpha\left(\frac{k+p-1}{p}\right)^{-(n+1)}\right] \bar{b}_{k+p-1}\overline{B}_{k+p-1}r^{k-1}e^{-i(k+2p-1)\theta}$$

and

$$D(re^{i\theta}) = \sum_{k=2}^{\infty}\left(\frac{k+p-1}{p}\right)^{-(n+1)} a_{k+p-1}A_{k+p-1}r^{k-1}e^{i(k-1)\theta}$$
$$+(-1)^{n+1}\sum_{k=1}^{\infty}\left(\frac{k+p-1}{p}\right)^{-(n+1)} \bar{b}_{k+p-1}\overline{B}_{k+p-1}r^{k-1}.e^{-i(k+2p-1)\theta}$$

By setting

$$\frac{(1-\alpha)+C(z)}{1+D(z)} = (1-\alpha)\frac{1+w(z)}{1-w(z)}$$

we can complete the proof by showing that $|w(z)| \leq r < 1$.

$$|w(z)| = \left|\frac{C(z)-(1-\alpha)D(z)}{C(z)+(1-\alpha)D(z)+2(1-\alpha)}\right|$$

$$= \left|\frac{\sum_{k=2}^{\infty}\left[\left(\frac{k+p-1}{p}\right)^{-n} - \left(\frac{k+p-1}{p}\right)^{-(n+1)}\right] a_{k+p-1}A_{k+p-1}r^{k-1}e^{i(k-1)\theta}}{2(1-\alpha)+\sum_{k=2}^{\infty}R(n,k,p,\alpha)a_{k+p-1}A_{k+p-1}r^{k-1}e^{i(k-1)\theta}+(-1)^n\sum_{k=1}^{\infty}M(n,k,p,\alpha)\bar{b}_{k+p-1}\overline{B}_{k+p-1}r^{k-1}e^{-i(k+2p-1)\theta}}\right.$$
$$\left.+\frac{(-1)^n\sum_{k=1}^{\infty}\left[\left(\frac{k+p-1}{p}\right)^{-n} + \left(\frac{k+p-1}{p}\right)^{-(n+1)}\right] \bar{b}_{k+p-1}\overline{B}_{k+p-1}r^{k-1}e^{-i(k+2p-1)\theta}}{2(1-\alpha)+\sum_{k=2}^{\infty}R(n,k,p,\alpha)a_{k+p-1}A_{k+p-1}r^{k-1}e^{i(k-1)\theta}+(-1)^n\sum_{k=1}^{\infty}M(n,k,p,\alpha)\bar{b}_{k+p-1}\overline{B}_{k+p-1}r^{k-1}e^{-i(k+2p-1)\theta}}\right|$$

where

$$R(n,k,p,\alpha) = \left(\frac{k+p-1}{p}\right)^{-n} + (1-2\alpha)\left(\frac{k+p-1}{p}\right)^{-(n+1)}$$

and

$$M(n,k,p,\alpha) = \left(\frac{k+p-1}{p}\right)^{-n} + (2\alpha-1)\left(\frac{k+p-1}{p}\right)^{-(n+1)}.$$

$$
\begin{aligned}
&|w(z)| \\
&\leq \frac{\sum_{k=2}^{\infty}\left[\left(\frac{k+p-1}{p}\right)^{-n}-\left(\frac{k+p-1}{p}\right)^{-(n+1)}\right]|a_{k+p-1}||A_{k+p-1}|r^{k-1}}{2(1-\alpha)-\sum_{k=2}^{\infty}R(n,k,p,\alpha)|a_{k+p-1}||A_{k+p-1}|r^{k-1}-\sum_{k=1}^{\infty}M(n,k,p,\alpha)|b_{k+p-1}||B_{k+p-1}|r^{k-1}} \\
&+\frac{\sum_{k=1}^{\infty}\left[\left(\frac{k+p-1}{p}\right)^{-n}+\left(\frac{k+p-1}{p}\right)^{-(n+1)}\right]|b_{k+p-1}||B_{k+p-1}|r^{k-1}}{2(1-\alpha)-\sum_{k=2}^{\infty}R(n,k,p,\alpha)|a_{k+p-1}||A_{k+p-1}|r^{k-1}-\sum_{k=1}^{\infty}M(n,k,p,\alpha)|b_{k+p-1}||B_{k+p-1}|r^{k-1}} \\
&=\frac{\sum_{k=1}^{\infty}\left[\left(\frac{k+p-1}{p}\right)^{-n}-\left(\frac{k+p-1}{p}\right)^{-(n+1)}\right]|a_{k+p-1}||A_{k+p-1}|r^{k-1}}{4(1-\alpha)-\sum_{k=1}^{\infty}\left\{R(n,k,p,\alpha)|a_{k+p-1}||A_{k+p-1}|+M(n,k,p,\alpha)|b_{k+p-1}||B_{k+p-1}|\right\}r^{k-1}} \\
&+\frac{\sum_{k=1}^{\infty}\left[\left(\frac{k+p-1}{p}\right)^{-n}+\left(\frac{k+p-1}{p}\right)^{-(n+1)}\right]|b_{k+p-1}||B_{k+p-1}|r^{k-1}}{4(1-\alpha)-\sum_{k=2}^{\infty}\left\{R(n,k,p,\alpha)|a_{k+p-1}||A_{k+p-1}|+M(n,k,p,\alpha)|b_{k+p-1}||B_{k+p-1}|\right\}r^{k-1}}
\end{aligned}
$$

$$
\begin{aligned}
&<\frac{\sum_{k=1}^{\infty}\left[\left(\frac{k+p-1}{p}\right)^{-n}-\left(\frac{k+p-1}{p}\right)^{-(n+1)}\right]|a_{k+p-1}||A_{k+p-1}|}{4(1-\alpha)-\sum_{k=1}^{\infty}\left\{R(n,k,p,\alpha)|a_{k+p-1}||A_{k+p-1}|+M(n,k,p,\alpha)|b_{k+p-1}||B_{k+p-1}|\right\}} \\
&+\frac{\sum_{k=1}^{\infty}\left[\left(\frac{k+p-1}{p}\right)^{-n}+\left(\frac{k+p-1}{p}\right)^{-(n+1)}\right]|b_{k+p-1}||B_{k+p-1}|}{4(1-\alpha)-\sum_{k=2}^{\infty}\left\{R(n,k,p,\alpha)|a_{k+p-1}||A_{k+p-1}|+M(n,k,p,\alpha)|b_{k+p-1}||B_{k+p-1}|\right\}} \\
&\leq 1.
\end{aligned}
$$

The harmonic multivalent functions

$$f(z)=z^p+\sum_{k=2}^{\infty}\frac{1}{\psi(n,k,p,\alpha)}x_k z^{k+p-1}+\sum_{k=1}^{\infty}\frac{1}{\eta(n,k,p,\alpha)}\overline{y_k z^{k+p-1}} \tag{8}$$

where $p,n\in\mathbb{N}$ and $\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=1$, show that the coefficient bound given by

(7) is sharp. The functions of the form (8) are in $H_p(n,\alpha)$ because

$$\sum_{k=1}^{\infty}\{\psi(n,k,p,\alpha)|a_{k+p-1}||A_{k+p-1}|+\eta(n,k,p,\alpha)|b_{k+p-1}||B_{k+p-1}|\}$$
$$=1+\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=2.$$

Theorem 2.2. *Let $f_n * F_n$ be given by (6). Then $f_n * F_n \in \overline{H}_p(n,\alpha)$ if and only if*

$$\sum_{k=1}^{\infty}\{\psi(n,k,p,\alpha)|a_{k+p-1}||A_{k+p-1}|+\eta(n,k,p,\alpha)|b_{k+p-1}||B_{k+p-1}|\}\leq 2 \tag{9}$$

with $a_p = 1 = A_p$, $0 \leq \alpha < 1$, $n,p \in \mathbb{N}$.

Proof. Since $\overline{H}_p(n,\alpha) \subseteq H_p(n,\alpha)$, the if part follows from Theorem 2.1.
For the only if part, we show $f_n * F_n \notin \overline{H}_p(n,\alpha)$ if the condition (9) does not hold.
For functions $f_n * F_n$ of the form (6), we note that the condition

$$Re\left\{\frac{I^n(f_n * F_n)(z)}{I^{n+1}(f_n * F_n)(z)}\right\} > \alpha$$

is equivalent to

$$Re\left\{\frac{I^n(f_n * F_n)(z)-\alpha I^{n+1}(f_n * F_n)(z)}{I^{n+1}(f_n * F_n)(z)}\right\} \geq 0$$

which is

$$Re\left\{\frac{(1-\alpha)z^p-\sum_{k=2}^{\infty}\left[\left(\frac{k+p-1}{p}\right)^{-n}-\alpha\left(\frac{k+p-1}{p}\right)^{-(n+1)}\right]a_{k+p-1}A_{k+p-1}z^{k+p-1}}{z^p-\sum_{k=2}^{\infty}\left(\frac{k+p-1}{p}\right)^{-(n+1)}a_{k+p-1}A_{k+p-1}z^{k+p-1}+(-1)^{2n}\sum_{k=1}^{\infty}\left(\frac{k+p-1}{p}\right)^{-(n+1)}\overline{b}_{k+p-1}\overline{B}_{k+p-1}\overline{z}^{k+p-1}}\right.$$
$$\left.+\frac{(-1)^{2n-1}\sum_{k=1}^{\infty}\left[\left(\frac{k+p-1}{p}\right)^{-n}+\alpha\left(\frac{k+p-1}{p}\right)^{-(n+1)}\right]b_{k+p-1}B_{k+p-1}\overline{z}^{k+p-1}}{z^p-\sum_{k=2}^{\infty}\left(\frac{k+p-1}{p}\right)^{-(n+1)}a_{k+p-1}A_{k+p-1}z^{k+p-1}+(-1)^{2n}\sum_{k=1}^{\infty}\left(\frac{k+p-1}{p}\right)^{-(n+1)}b_{k+p-1}B_{k+p-1}\overline{z}^{k+p-1}}\right\}$$
$$\geq 0. \tag{10}$$

The above required condition (10) must hold for all values of z in $\mathcal{U}$. Upon choosing the values of z on the positive real axis where $0 \leq z = r < 1$ we must have

$$\frac{(1-\alpha) - \sum_{k=2}^{\infty}\left[\left(\frac{k+p-1}{p}\right)^{-n} - \alpha\left(\frac{k+p-1}{p}\right)^{-(n+1)}\right] a_{k+p-1}A_{k+p-1}r^{k-1}}{1 - \sum_{k=2}^{\infty}\left(\frac{k+p-1}{p}\right)^{-(n+1)} a_{k+p-1}A_{k+p-1}r^{k-1} + \sum_{k=1}^{\infty}\left(\frac{k+p-1}{p}\right)^{-(n+1)} b_{k+p-1}B_{k+p-1}r^{k-1}}$$

$$- \frac{\sum_{k=1}^{\infty}\left[\left(\frac{k+p-1}{p}\right)^{-n} + \alpha\left(\frac{k+p-1}{p}\right)^{-(n+1)}\right] b_{k+p-1}B_{k+p-1}r^{k-1}}{1 - \sum_{k=2}^{\infty}\left(\frac{k+p-1}{p}\right)^{-(n+1)} a_{k+p-1}A_{k+p-1}r^{k-1} + \sum_{k=1}^{\infty}\left(\frac{k+p-1}{p}\right)^{-(n+1)} b_{k+p-1}B_{k+p-1}r^{k-1}} \geq 0 \tag{11}$$

If the condition (9) does not hold, then the expressions in (11) is negative for r sufficiently close to 1. Hence there exist $z_0 = r_0$ in $(0,1)$ for which the quotient in (11) is negative. This contradicts the required condition for $f_n * F_n \in \overline{H}_p(n,\alpha)$.

3. Distortion Bounds

In this section, we obtain distortion bounds for functions in $\overline{H}_p(n,\alpha)$ which yields the covering results for this class.

Theorem 3.1. *Let $f_n * F_n \in \overline{H}_p(n,\alpha)$. Then for $|z| = r < 1$,*

$$|(f_n * F_n)(z)| \leq (1 + b_pB_p)r^p + \left\{\Phi(n,p,\alpha) - \Omega(n,p,\alpha)b_pB_p\right\} r^{n+p}$$

and

$$|(f_n * F_n)(z)| \geq (1 - b_pB_p)r^p - \left\{\Phi(n,p,\alpha) - \Omega(n,p,\alpha)b_pB_p\right\} r^{n+p}$$

where

$$\Phi(n,p,\alpha) = \frac{1-\alpha}{\left(\frac{p+1}{p}\right)^{-n} - \alpha\left(\frac{p+1}{p}\right)^{-(n+1)}}$$

and

$$\Omega(n,p,\alpha) = \frac{1+\alpha}{\left(\frac{p+1}{p}\right)^{-n} - \alpha\left(\frac{p+1}{p}\right)^{-(n+1)}}.$$

Proof. Let $f_n * F_n \in \overline{H}_p(n,\alpha)$. Taking the absolute value of $f_n * F_n$ we obtain

$$\begin{aligned}
|(f_n * F_n)(z)| &= \left| z^p - \sum_{k=2}^{\infty} a_{k+p-1}A_{k+p-1}z^{k+p-1} + (-1)^{n-1}\sum_{k=1}^{\infty} b_{k+p-1}B_{k+p-1}\overline{z}^{k+p-1} \right| \\
&\leq r^p + \sum_{k=2}^{\infty} a_{k+p-1}A_{k+p-1}r^{k+p-1} + \sum_{k=1}^{\infty} b_{k+p-1}B_{k+p-1}r^{k+p-1} \\
&= r^p + b_pB_pr^p + \sum_{k=2}^{\infty}(a_{k+p-1}A_{k+p-1} + b_{k+p-1}B_{k+p-1})r^{k+p-1} \\
&\leq r^p + b_pB_pr^p + \sum_{k=2}^{\infty}(a_{k+p-1}A_{k+p-1} + b_{k+p-1}B_{k+p-1})r^{p+1} \\
&= (1+b_pB_p)r^p + \Phi(n,p,\alpha)\sum_{k=2}^{\infty}\frac{1}{\Phi(n,p,\alpha)}(a_{k+p-1}A_{k+p-1} + b_{k+p-1}B_{k+p-1})r^{p+1} \\
&\leq (1+b_pB_p)r^p + \Phi(n,p,\alpha)r^{n+p}\times \\
&\quad \left[\sum_{k=2}^{\infty}\{\psi(n,k,p,\alpha)a_{k+p-1}A_{k+p-1} + \eta(n,k,p,\alpha)b_{k+p-1}B_{k+p-1}\}\right]
\end{aligned}$$

$$\leq (1+b_pB_p)r^p + \{\Phi(n,p,\alpha) - \Omega(n,p,\alpha)b_pB_p\}\, r^{n+p} \qquad \text{in view of (9)}$$

and

$$\begin{aligned}
|(f_n * F_n)(z)| &\geq (1+b_pB_p)r^p + \sum_{k=2}^{\infty}(a_{k+p-1}A_{k+p-1} + b_{k+p-1}B_{k+p-1})r^{k+p-1} \\
&\geq (1-b_pB_p)r^p - \sum_{k=2}^{\infty}(a_{k+p-1}A_{k+p-1} + b_{k+p-1}B_{k+p-1})r^{p+1} \\
&= (1-b_pB_p)r^p - \Phi(n,p,\alpha)\sum_{k=2}^{\infty}\frac{1}{\Phi(n,p,\alpha)}(a_{k+p-1}A_{k+p-1} + b_{k+p-1}B_{k+p-1})r^{p+1} \\
&\geq (1-b_pB_p)r^p - \Phi(n,p,\alpha)r^{n+p}\times \\
&\quad \left[\sum_{k=2}^{\infty}\{\psi(n,k,p,\alpha)a_{k+p-1}A_{k+p-1} + \eta(n,k,p,\alpha)b_{k+p-1}B_{k+p-1}\}\right] \\
&\geq (1-b_pB_p)r^p - \{\Phi(n,p,\alpha) - \Omega(n,p,\alpha)b_pB_p\}\, r^{n+p}
\end{aligned}$$

by (9)

The bounds given in Theorem 3.1 for the functions of the form (6) also hold for functions of the form (3) if the coefficient condition (7) is satisfied.

The following covering result follows from the left hand inequality in Theorem 3.1.

Corollary 3.1. *Let $f_n * F_n \in \overline{H}_p(n,\alpha)$, then for $|z| = r < 1$ we have*

$$\{w : |w| < 1 - b_p B_p - [\phi(n,p,\alpha) - \Omega(n,p,\alpha) b_p B_p]\} \subset (f_n * F_n)(\mathcal{U}).$$

4. Closure Property of the Class $\overline{H}_p(n,\alpha)$

First we show that $\overline{H}_p(n,\alpha)$ is closed under convolution.

Theorem 4.1. *For $0 \leq \beta \leq \alpha < 1$, let $f_n(z) \in \overline{H}_p(n,\alpha)$ and $F_n(z) \in \overline{H}_p(n,\beta)$. Then*

$$f_n * F_n \in \overline{H}_p(n,\alpha) \subset \overline{H}_p(n,\beta).$$

Proof. Suppose f_n and F_n are so that $f_n * F_n$ is given by the convolution (6). Since $f_n(z) \in \overline{H}_p(n,\alpha)$ and $F_n(z) \in \overline{H}_p(n,\beta)$, the coefficients of f_n and F_n must be satisfying the conditions given by Theorem 2.2. Hence for the coefficients of $f_n * F_n$ we find that

$$\sum_{k=1}^{\infty} \{\psi(n,k,p,\alpha)|a_{k+p-1}||A_{k+p-1}| + \eta(n,k,p,\alpha)|b_{k+p-1}||B_{k+p-1}|\}$$

where $\psi(n,k,p,\alpha)$ and $\eta(n,k,p,\alpha)$ as given in (7), is bounded by 2. Thus $f_n * F_n \in \overline{H}_p(n,\alpha) \subset \overline{H}_p(n,\beta)$.

Now we show that $\overline{H}_p(n,\alpha)$ is closed under convex linear combinations of its members.

Theorem 4.2. *The family $\overline{H}_p(n,\alpha)$ is closed under convex combination.*

Proof. For $i = 1,2,3,\ldots$, let $f_{n_i} * F_{n_i} \in \overline{H}_p(n,\alpha)$ where

$$(f_{n_i} * F_{n_i})(z) = z^p - \sum_{k=2}^{\infty} a_{i,k+p-1} A_{i,k+p-1} z^{k+p-1} + (-1)^{n-1} \sum_{k=1}^{\infty} b_{i,k+p-1} B_{i,k+p-1} \overline{z}^{k+p-1}.$$

Then

$$\sum_{k=1}^{\infty} \{\psi(n,k,p,\alpha) a_{i,k+p-1} A_{i,k+p-1} + \eta(n,k,p,\alpha) b_{i,k+p-1} B_{i,k+p-1}\} \leq 2\ . \tag{12}$$

For $\sum_{i=1}^{\infty} t_i = 1$, $0 \leq t_i \leq 1$, the convex linear combination of $f_{n_i} * F_{n_i}$ may be written as

$$\sum_{i=1}^{\infty} t_i (f_{n_i} * F_{n_i})(z) = z^p - \sum_{k=2}^{\infty} \left[\sum_{i=1}^{\infty} t_i\ a_{i,k+p-1} A_{i,k+p-1} \right] z^{k+p-1}$$
$$+ (-1)^{n-1} \sum_{k=1}^{\infty} \left[\sum_{i=1}^{\infty} t_i\ b_{i,k+p-1} B_{i,k+p-1} \right] \overline{z}^{k+p-1}.$$

Then by (12)

$$\sum_{k=1}^{\infty} \left\{ \psi(n,k,p,\alpha) \sum_{i=1}^{\infty} t_i\ a_{i,k+p-1} A_{i,k+p-1} + \eta(n,k,p,\alpha) \sum_{i=1}^{\infty} t_i\ b_{i,k+p-1} B_{i,k+p-1} \right\}$$
$$= \sum_{i=1}^{\infty} t_i \left\{ \sum_{k=1}^{\infty} \psi(n,k,p,\alpha) a_{i,k+p-1} A_{i,k+p-1} + \eta(n,k,p,\alpha) b_{i,k+p-1} B_{i,k+p-1} \right\}$$
$$\leq 2 \sum_{i=1}^{\infty} t_i = 2$$

which implies that $\sum_{i=1}^{\infty} t_i (f_{n_i} * F_{n_i})(z) \in \overline{H}_p(n,\alpha)$ in view of Theorem 2.2.

Next we show that $\overline{H}_p(n,\alpha)$ is closed under the generalized Bernardi-Libera Livingston integral operator $L_c(f)$ which is defined by

$$L_c(f(z)) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t) dt, \quad c > -1. \tag{13}$$

Theorem 4.3. *Let $f_n * F_n \in \overline{H}_p(n,\alpha)$. Then $L_c[(f_n * F_n)(z)]$ belongs to the class $\overline{H}_p(n,\alpha)$.*

Proof. From the representation (13), it follows that

$$
\begin{aligned}
&L_c[(f_n * F_n)(z)] \\
&= \frac{c+p}{z^c}\int_0^z t^{c-1}\left\{t^p - \sum_{k=2}^{\infty} a_{k+p-1}A_{k+p-1}t^{k+p-1} + (-1)^{n-1}\sum_{k=1}^{\infty}\overline{b_{k+p-1}B_{k+p-1}t^{k+p-1}}\right\}dt \\
&= z^p - \sum_{k=2}^{\infty}\frac{c+p}{c+p+k-1}a_{k+p-1}A_{k+p-1}z^{k+p-1} + (-1)^{n-1}\sum_{k=1}^{\infty}\frac{c+p}{c+p+k-1}b_{k+p-1}B_{k+p-1}z^{k+p-1} \\
&= z^p - \sum_{k=2}^{\infty}X_{k+p-1}z^{k+p-1} + (-1)^{n-1}\sum_{k=1}^{\infty}Y_{k+p-1}z^{k+p-1}
\end{aligned}
$$

where

$$
\begin{aligned}
X_{k+p-1} &= \frac{c+p}{c+p+k-1}a_{k+p-1}A_{k+p-1} \quad \text{and} \\
Y_{k+p-1} &= \frac{c+p}{c+p+k-1}b_{k+p-1}B_{k+p-1}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\sum_{k=1}^{\infty}\left\{\psi(n,k,p,\alpha)\frac{c+p}{c+p+k-1}a_{k+p-1}A_{k+p-1} + \eta(n,k,p,\alpha)\frac{c+p}{c+p+k-1}b_{k+p-1}B_{k+p-1}\right\} \\
&\le \sum_{k=1}^{\infty}\{\psi(n,k,p,\alpha)a_{k+p-1}A_{k+p-1} + \eta(n,k,p,\alpha)b_{k+p-1}B_{k+p-1}\} \le 2 \text{ by (9)}
\end{aligned}
$$

which implies $L_c\{(f_n * F_n)(z)\} \in \overline{H}_p(n,\alpha)$.

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