Acta Universitatis Apulensis
ISSN: 1582-5329

No. 30/2012
pp. 49-54

SOME INEQUALITIES IN NUMERICAL SEMIGROUPS

Sedat İlhan

Abstract. In this paper, we give some inequalities in numerical semigroups. We also describe a question of Wilf which will be shown to be equivalent to the statement that $e(S)n(S) \geq g(S)+1$.

2000 *AMS Classification:* 20M14

Keywords: Numerical semigroup, Embedding dimension, Maximal length.

1. Introduction

Let $\mathbb{N}$ be the set of nonnegative integers. A subsemigroup S of $(\mathbb{N},+)$ with $0 \in S$ is called a numerical semigroup. It is well known that any numerical semigroup is finitely generated. Thus, there exist $s_1, s_2, ..., s_p \in S$ such that $S =< s_1, s_2, ..., s_p >= \{s_1k_1 + s_2k_2 + ... + s_pk_p : k_i \in \mathbb{N}, 1 \leq i \leq p\}$ where $s_1 < s_2 < ... < s_p$. In this case, we say that $\{s_1, s_{2,...,}s_p\} \subset S$ is generating set of S. The set $\{s_1, s_2, ..., s_p\}$ is called minimal generating set of S if no proper subset of it is a generating set of S. $e(S) = \sharp(\{s_1, s_{2,...,}s_p\})$ is called emmedding dimension of S. It is observed that the set $\mathbb{N}\backslash S$ is finite if and only if $g.c.d\,\{s_1, s_2, ..., s_p\} = 1$ ($g.c.d$ stands for greatest common divisor)[5].

Another important invariant of S is the largest integer not belonging to S, known as the Frobenius number of S and denoted by $g(S)$, that is, $g(S) = \max\{x \in \mathbb{Z} : x \notin S\}$ [3,6]. For a numerical semigroup S , we can define $n\,(S) = \sharp\,(\{0, 1, ..., g\,(S)\} \cap S)$ and $\mu(S) = \min\{s \in S : s > 0\}$ where $\sharp\,(A)$ denotes the cardinality of A and $\mu(S)$ is called the multiplicity of S. Notice that $e(S) \leq \mu(S)$. We say that S has maximal embedding dimension if $e(S) = \mu(S)$ [1,4].

If S is a numerical semigroup, then we can write $S = \{0, s_1, s_2, ..., s_{n-1}, s_n = g(S)$ $+1, \rightarrow ...\}$ where $\rightarrow$ means that every integer greater than $g(S)+1$ belongs to S, $n = n(S)$ and $s_i < s_{i+1}$ for $i = 1, 2, ..., n$.

Let S be a numerical semigroup and $i \geq 0$. The sets S_i and $S(i)$ are defined by $S_i = \{s \in S : s \geq s_i\}$ and $S(i) = \{x \in \mathbb{N} : x + S_i \subseteq S\}$, respectively. It is obvious that every $S(i)$ is itself a numerical semigroup and we obtain the following chain:

$$S_n \subset S_{n-1} \subset ... \subset S_1 \subset S \subset S(1) \subset ... \subset S(n-1) \subset S(n) = \mathbb{N}$$

The number $\sharp(S(1)\backslash S)$ is called the type of S and denoted by $t(S)$. Likewise, we put $t_i = \sharp(S(i)\backslash S(i-1))$ for $i \geq 1$. Clearly $t_1(S) = t(S)$ but in general case $t_i(S) \neq t(S(i))$. In this way, it is possible to associate every numerical semigroup S with a numerical sequence $\left\{t_1, t_2, ..., t_{n(S)}\right\}$ called the type sequence of S. We have $1 \leq t_i(S) \leq t_1(S)$ by [6]. In this case, since $\sharp\,(\mathbb{N}\backslash S) = g(S) + 1 - n(S)$, and from the definition of the numbers $t_i(S)$, we can write

$$\sum_{i=1}^{n(S)} t_i(S) = g(S) + 1 - n(S). \tag{1.1}$$

Moreover, by [2] it is not difficult to deduce the inequalites

$$2n(S) \leq g(S) + 1 \leq (t(S) + 1)n(S) \tag{1.2}$$

and

$$t(S) + 1 \leq \mu(S) \leq t(S) + n(S) - 1. \tag{1.3}$$

We say that a numerical semigroup S has maximal length and almost maximal length, if $g(S) + 1 - n(S) = t(S)n(S)$ and $g(S) + 2 - n(S) = t(S)n(S)$, respectively [6].

A numerical semigroup S is pseudo-symmetric if $g(S)$ is even and $x = \frac{g(S)}{2}$ is only integer such that $x \in \mathbb{Z}\backslash S$ and $g(S) - x \notin S$. Also S is symmetric if $g(S) - x \in S$ for all $x \in \mathbb{Z}\backslash S$. On the other hand, we note that symmetric and pseudo-symmetric numerical semigroups are of type sequences $\{1, 1, ..., 1\}$ and $\{2, 1, ..., 1\}$, respectively [2].

Furthermore, all symmetric numerical semigroups have maximal length, but need not have maximal embedding dimension and almost maximal length. For example, $S =< 4, 5 >$ is maximal length but has not maximal embedding dimension and almost maximal length, since $\mu(S) \neq e(S)$ and $n(S) \neq \frac{g(S)+2}{t(S)+1}$, respectively. On the other hand, a numerical semigroup S has maximal embedding dimension, but it need not has maximal length. For example, $S =< 3, 5, 7 >$ has maximal embedding dimension but not has maximal length since $n(S) \neq \frac{g(S)+1}{t(S)+1}$.

Notice that a numerical semigroup S of almost maximal length can not be symmetric, and need not has maximal embedding dimension. For example, $S =<$

$4, 5, 11 >$ has almost maximal length but S is not symmetric and has not maximal embedding dimension.

If S is pseudo-symmetric then it need not has maximal length, almost maximal length and maximal embedding dimension. For example, $S =< 4, 5, 7 >$ is pseudo-symmetric but has not maximal length, almost maximal length and maximal embedding dimension. When S has maximal length or almost maximal length or maximal embedding dimension, it need not be pseudo-symmetric[6]. For example, we can consider the semigroup $S =< 4, 5, 6 >$ of maximal length, $S =< 4, 7, 13 >$ of almost maximal length and $S =< 3, 7, 8 >$ of maximal embedding dimension, it is clear that of them is not pseudo-symmetric, respectively.

The contents of this study are organized as follows. In Section 2, we give some inequalities in numerical semigroup. In particular, the main goal of this section is to give some criteria for maximal and almost maximal length as Propositions 2.4 and 2.5. Furthermore, the aim of Section 3 is to give some results about a question of Wilf in [1] which are satisfied inequality $e(S)n(S) \geq g(S) + 1$.

Throughout this paper, we will assume numerical semigroup S as

$$S = \{0, s_1, s_2, ..., s_{n-1}, s_n = g(S) + 1, \rightarrow ...\}$$

where $n = n(S) \geq 2$ and $g.c.d\,\{s_1, s_2, ..., s_p\} = 1$.

2. Some Inequalities in Numerical Semigroups

In this section, we give some inequalities in numerical semigroups and some results concerning the numerical semigroups of maximal and almost maximal length.

Our first lemma gives a bound for multiplicity of S.

Lemma 1 *Let S be a numerical semigroup, $t(S)$ be its type, the integers $g(S)$ be the frobenius number and $\mu(S)$ be multiplicity of S. We have*

i) If S has maximal length, then $\mu(S) \leq \frac{(t(S))^2+g(S)}{t(S)+1}$.

ii) If S has almost maximal length, then $\mu(S) \leq \frac{(t(S))^2+g(S)+1}{t(S)+1}$.

Proof. In order to establish the required results in Lemma 2.1, it is sufficient to take $n(S) = \frac{g(S)+1}{t(S)+1}$ and $n(S) = \frac{g(S)+2}{t(S)+1}$ in formula (1.3), respectively. ∎

Corollary 2 *Let S be a numerical semigroup of almost maximal length. If S is pseudo-symmetric, then $\mu(S) \leq 3$.*

Proof. If S is pseudo-symmetric and almost maximal length then we have $g(S) = 4$ since $n(S) = \frac{g(S)}{2} = \frac{g(S)+2}{t(S)+1} = \frac{g(S)+2}{3}$. Thus we write $\mu(S) \leq \frac{(t(S))^2+g(S)+1}{t(S)+1} = \frac{4+4+1}{3} = 3$. ■

Lemma 3 *Let S be a numerical semigroup and $\{t_i(S)\}$ is type sequence of S, for $i = 1, 2, ..., n(S)$. Then,*

$$\sum_{i=1}^{n(S)} t_i(S) \leq n(S)t(S).$$

Proof. We can write $\sum\limits_{i=1}^{n(S)} t_i(S) + n(S) = g(S) + 1$ from formula (1.1). We find that

$$\sum_{i=1}^{n(S)} t_i(S) + n(S) = g(S) + 1 \leq (t(S) + 1)n(S),$$

i.e. $\sum\limits_{i=1}^{n(S)} t_i(S) \leq n(S)t(S)$, from formula (1.2). ■

Proposition 4 *Let S be a numerical semigroup and $\{t_i(S)\}$ be type sequence of S, for $i = 1, 2, ..., n(S)$. If S has maximal length, then*

$$\sum_{i=1}^{n(S)} t_i(S) + n(S) \leq g(S) + 1.$$

Proof. If S has maximal length, then from Lemma 2.3 we can obtain

$g(S) + 1 - n(S) = t(S)n(S) \geq \sum\limits_{i=1}^{n(S)} t_i(S)$. ■

Proposition 5 *Let S be a numerical semigroup and $\{t_i(S)\}$ is type sequence of S, for $i = 1, 2, ..., n(S)$. If S has almost maximal length, then $\sum\limits_{i=1}^{n(S)} t_i(S) + n(S) \leq g(S) + 2$.*

Proof. If S has almost maximal length, then by formula (1.4) we can obtain

$g(S) + 2 - n(S) = t(S)n(S) \geq \sum\limits_{i=1}^{n(S)} t_i(S)$. ■

The following example show that the inverses of above propositions are not true.

Example 6 *Let us consider the numerical semigroup S given by $S =< 7, 9, 10, 12, 13, 15 >= \{0, 7, 9, 10, 12, \to ...\}$. Then, we can find that the sequence $\{5, 1, 1, 1\}$ is type of S. Thus we write*

$$g(S) + 1 = 12 \geq n(S) + \sum_{i=1}^{4} t_i(S) = 4 + (5 + 1 + 1 + 1) \text{ but } n(S) \neq \tfrac{g(S)+1}{t(S)+1} = \tfrac{12}{6} =$$

2, and $g(S)+2 = 13 \geq n(S)+\sum_{i=1}^{4} t_i(S) = 4+(5+1+1+1)$ but $n(S) \neq \frac{g(S)+2}{t(S)+1} = \frac{13}{6}$.

3. Some Results for a Question of Wilf

Wilf's a question : When the inequality $e(S)n(S) \geq g(S) + 1$. is satisfied? This question has answered affirmatively if S is symmetric, pseudo-symmetric or has maximal embedding dimension. The same time, it has also answered affirmatively for the cases, $e(S) \leq 3$, $g(S) \leq 20, n(S) \leq 4$ and $n(S) \geq \frac{g(S)+1}{4}$ [1].

In this section, we give some results for Wilf's a question.

Proposition 7 *Let S be a numerical semigroup of almost maximal length and $e(S) \geq 4$. If $t(S) + 1 \leq \frac{4g(S)+8}{g(S)+1}$ then $e(S)n(S) \geq g(S) + 1$.*

Proof. If S has almost maximal length then $n(S) = \frac{g(S)+2}{t(S)+1}$. Thus we have

$$e(S)n(S) = e(S)\left(\frac{g(S)+2}{t(S)+1}\right) \geq e(S)\left(\frac{g(S)+2}{\frac{4g(S)+8}{g(S)+1}}\right) = e(S)\left(\frac{g(S)+1}{4}\right) \geq g(S)+1.$$

■

Example 8 *Let us consider the numerical semigroup S given by*

$$S =< 4, 15, 17, 18 >= \{0, 4, 8, 12, 15, \to ...\}.$$

Thus we write $g(S) = 14$, $n(S) = 4$, $e(S) = 4$ and $t(S) = 3$. In this case, we obtain $t(S) + 1 = 4 \leq \frac{4.14+8}{14+1}$ and $e(S)n(S) \geq g(S) + 1$.

Now, we give some results when $e(S)n(S) \geq g(S) + 1$. We note that S need not be maximal or almost maximal length when $e(S)n(S) \geq g(S) + 1$. For example, let us consider a numerical semigroup S given by $S =< 5, 6, 13 >= \{0, 5, 6, 10, 11, 12, 13, 15, \to ...\}$. Thus we write $g(S) = 14$, $n(S) = 7$, $e(S) = 3$ and $t(S) = 2$. It follows $e(S)n(S) \geq g(S) + 1$ but S is not maximal and almost maximal length.

Proposition 9 *Let S be a numerical semigroup. If S has maximal length and $\mu(S) = 3$, then $t(S) + 1 \leq e(S)$.*

Proof. We have $e(S)n(S) \geq g(S) + 1$, since $e(S) \leq \mu(S) = 3$ by [1]. If S has maximal length then $n(S) = \frac{g(S)+1}{t(S)+1}$. In this case, we obtain $e(S) \geq t(S) + 1$ from $e(S)n(S) \geq g(S) + 1$. ■

Corollary 10 *Let S be a numerical semigroup. If S has maximal length and $\mu(S) = 3$, then S is symmetric or pseudo-symmetric.*

Proof. If S has maximal length then $n(S) = \frac{g(S)+1}{t(S)+1}$. From proposition 3.3, we write $e(S) \geq t(S) + 1$. Thus, we can obtain $t(S) + 1 \leq 3$ since $\mu(S) = 3$. Finally, we find that S is symmetric or pseudo-symmetric. ■

References

[1] D.E.Dobbs and G.L.Matthews, *On a question of Wilf Cocerning Numerical Semigroups, manuscript, 16 pages.*

[2] M.D'Anna, *Type Sequences of Numerical Semigroups,* Semigroup Forum, **56,** 1-31, 1998.

[3] J.C. Rosales and P.A. Garcia-Sanchez , *Pseudo-Symmetric Numerical Semigroups with Three Generators,* Journal of Algebra, **291**, 46-54, 2005.

[4] J.C. Rosales and P.A. Garcia-Sanchez, *Finitely Generated Commutative Monoids,* Nova Science Publishers, New York, 1999.

[5] V.Barucci, D.E. Dobbs and M. Fontana, *Maximality Properties in Numerical Semigroups and Applications To One-Dimensional Analyticalle Irreducible Local Domains,* Memoirs of The Amer. Math. Soc. , **598,**13-25, 1997.

[6] W.C. Brown and Frank Curtis, *Numerical Semigroups of Maximal and Almost Maximal length,* Semigroup Forum, **42,** 219-235, 1991.

Sedat İlhan
Dicle University, Faculty of Science, Department of Mathematics,
21280 Diyarbakır, Turkey
email: *sedati@dicle.edu.tr*