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NEW SOLUTION OF DIFFERENTIAL EQUATION FOR DUAL CURVATURES OF DUAL SPACELIKE BIHARMONIC CURVES WITH TIMELIKE PRINCIPAL NORMAL ACCORDING TO DUAL BISHOP FRAMES IN THE DUAL LORENTZIAN SPACE

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Abstract. In this paper, we study dual spacelike biharmonic curves with timelike principal normal for dual variable in dual Lorentzian space $\mathbb{D}_1^3$. We consider differential equations of dual Bishop curvatures of dual spacelike biharmonic curves with timelike principal normal for dual variable in dual Lorentzian space $\mathbb{D}_1^3$. This equations are seperated into dual and real parts such that the dual part of the equation is the higher order differential of each term in the real part.

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1.Introduction

Differential equations arise in many areas of science and technology, specifically, whenever a deterministic relation involving some continuously varying quantities (modeled by functions) and their rates of change in space and/or time (expressed as derivatives) is known or postulated. This is illustrated in classical mechanics, where the motion of a body is described by its position and velocity as the time varies. Newton's laws allow one to relate the position, velocity, acceleration and various forces acting on the body and state this relation as a differential equation for the unknown position of the body as a function of time. In some cases, this differential equation (called an equation of motion) may be solved explicitly.

An example of modelling a real world problem using differential equations is determination of the velocity of a ball falling through the air, considering only gravity and air resistance. The ball's acceleration towards the ground is the acceleration due to gravity minus the deceleration due to air resistance. Gravity is constant but air resistance may be modelled as proportional to the ball's velocity. This means the ball's acceleration, which is the derivative of its velocity, depends on the velocity. Finding the velocity as a function of time involves solving a differential equation.

In this paper, we study dual spacelike biharmonic curves with timelike principal normal for dual variable in dual Lorentzian space $\mathbb{D}_1^3$. We consider differential equations of dual Bishop curvatures of dual spacelike biharmonic curves with timelike principal normal for dual variable in dual Lorentzian space $\mathbb{D}_1^3$. This equations are seperated into dual and real parts such that the dual part of the equation is the higher order differential of each term in the real part.

2.Preliminaries

In the Euclidean 3-Space $\mathbb{E}^3$, lines combined with one of their two directions can be represented by unit dual vectors over the the ring of dual numbers. The important properties of real vector analysis are valid for the dual vectors. The oriented lines $\mathbb{E}^3$ are in one to one correspondence with the points of the dual unit sphere $\mathbb{D}^3$.

A dual point on $\mathbb{D}^3$ corresponds to a line in $\mathbb{E}^3$, two different points of $\mathbb{D}^3$ represents two skew lines in $\mathbb{E}^3$. A differentiable curve on $\mathbb{D}^3$ represents a ruled surface $\mathbb{E}^3$. If φ and φ^* are real numbers and $\varepsilon^2 = 0$ the combination $\hat{\varphi} = \varphi + \varphi^*$ is called a dual number. The symbol ε designates the dual unit with the property $\varepsilon^2 = 0$. In analogy with the complex numbers W.K. Clifford defined the dual numbers and showed that they form an algebra, not a field. Later, E.Study introduced the dual angle subtended by two nonparallel lines $\mathbb{E}^3$, and defined it as $\hat{\varphi} = \varphi + \varphi^*$ in which φ and φ^* are, respectively, the projected angle and the shortest distance between the two lines.

By a dual number $\hat{x}$, we mean an ordered pair of the form (x, x^*) for all $x, x^* \in \mathbb{R}$. Let the set $\mathbb{R} \times \mathbb{R}$ be denoted as $\mathbb{D}$. Two inner operations and an equality on $\mathbb{D} = \{(x, x^*) | x, x^* \in \mathbb{R}\}$ are defined as follows:

(i) $\oplus : \mathbb{D} \times \mathbb{D} \longrightarrow \mathbb{D}$ for $\hat{x} = (x, x^*)$, $\hat{y} = (y, y^*)$ defined as

$$\hat{x} \oplus \hat{y} = (x, x^*) \oplus (y, y^*) = (x + y, x^* + y^*)$$

is called the addition in $\mathbb{D}$.

(ii) $\otimes : \mathbb{D} \times \mathbb{D} \longrightarrow \mathbb{D}$ for $\hat{x} = (x, x^*)$, $\hat{y} = (y, y^*)$ defined as

$$\hat{x} \otimes \hat{y} = (x, x^*) \otimes (y, y^*) = (xy, xy^* + x^*y)$$

is called the multiplication in $\mathbb{D}$.

The set $\mathbb{D}$ of dual numbers is a commutative ring.

(iii) If $x = y$, $x^* = y^*$ for $\hat{x} = (x, x^*)$, $\hat{y} = (y, y^*) \in \mathbb{D}$, $\hat{x}$ and $\hat{y}$ are equal, and it is indicated as $\hat{x} = \hat{y}$.

If the operations of addition, multiplication and equality on $\mathbb{D} = \mathbb{R} \times \mathbb{R}$ with set of real numbers $\mathbb{R}$ are defined as above, the set $\mathbb{D}$ is called the dual numbers system and the element (x, x^*) of $\mathbb{D}$ is called a dual number. In a dual number $\hat{x} = (x, x^*) \in \mathbb{D}$, the real number x is called the real part of $\hat{x}$ and the real number x^* is called the dual part of $\hat{x}$. The dual number $(1, 0) = 1$ is called unit element of multiplication operation in $\mathbb{D}$ or real unit in $\mathbb{D}$. The dual number $(0, 1)$ is to be denoted with ε in short, and the $(0, 1) = \varepsilon$ is to be called dual unit. In accordance with the definition of the operation of multiplication, it can easily be seen that $\varepsilon^2 = 0$. Also, the dual number $\hat{x} = (x, x^*) \in \mathbb{D}$ can be written as $\hat{x} = x + \varepsilon x^*$.

The set

$$\mathbb{D}^3 = \left\{ \hat{x} : \hat{x} = x + \varepsilon x^*, \ x, x^* \in \mathbb{E}^3 \right\}$$

is a module over the ring $\mathbb{D}$.

The Lorentzian inner product of $\hat{x}$ and $\hat{y}$ is defined by

$$\langle \hat{x}, \hat{y} \rangle = \langle x, y \rangle + \varepsilon \left(\langle x, y^* \rangle + \langle x^*, y \rangle \right).$$

We call the dual space $\mathbb{D}^3$ together with the Lorentzian inner product the dual Lorentzian space and denote it by $\mathbb{D}_1^3$. The norm $\|\hat{x}\|$ of $\hat{x}$ is defined by

$$\|\hat{x}\| = \sqrt{|\langle \hat{x}, \hat{x} \rangle|} = \|x\| + \varepsilon \frac{\langle x, x^* \rangle}{\|x\|}.$$

A dual vector $\overrightarrow{\hat{x}}$ with norm 1 is called a dual unit vector.

Let $\overrightarrow{\hat{x}} = \overrightarrow{x} + \varepsilon \overrightarrow{x^*} \in \mathbb{D}_1^3$. The set

$$\mathbb{S}_1^2 = \left\{ \hat{x} = x + \varepsilon x^* \mid \|\hat{x}\| = (1, 0) \, ; x, x^* \epsilon \mathbb{R}^3 \right\}$$

is called the dual unit sphere with the center $\hat{O}$ in $\mathbb{D}_1^3$.

If every $x_i(s)$ and $x_i^*(s)$, $1 \leq i \leq 3$, real valued functions,are differentiable, the dual space curve

$$\begin{aligned} \hat{x} \quad &: \quad I \subset \mathbb{R} \to \mathbb{D}_1^3. \\ s \quad &\to \quad \hat{x}(s) = \left(x_1(s) + \varepsilon x_1^*(s), x_2(s) + \varepsilon x_2^*(s), x_3(s) + \varepsilon x_3^*(s) \right), \end{aligned}$$

in $\mathbb{D}_1^3$ is differentiable.

3. Spacelike Dual Biharmonic Curves with Timelike Principal Normal in the Dual Lorentzian Space $\mathbb{D}_1^3$

Let $\hat{\gamma}$ dual spacelike curve with spacelike principal normal by the dual arc length parameter $\hat{s}$. Then the unit tangent vector $\hat{\gamma}' = \hat{\mathbf{t}}$ is defined, and the principal normal is $\hat{\mathbf{n}} = \frac{1}{\hat{\kappa}}\nabla_{\hat{\mathbf{t}}}\hat{\mathbf{t}}$, where $\hat{\kappa}$ is never a pure-dual. The function $\hat{\kappa} = \left\|\nabla_{\hat{\mathbf{t}}}\hat{\mathbf{t}}\right\| = \kappa + \varepsilon\kappa^*$ is called the dual curvature of the dual curve $\hat{\gamma}$. Then the binormal of $\hat{\gamma}$ is given by the dual vector $\hat{\mathbf{b}} = \hat{\mathbf{t}} \times \hat{\mathbf{n}}$. Hence, the triple $\left\{\hat{\mathbf{t}}, \hat{\mathbf{n}}, \hat{\mathbf{b}}\right\}$ is called the Frenet frame fields and the Frenet formulas may be expressed

$$\begin{aligned}
\nabla_{\hat{\mathbf{t}}(\hat{s})}\hat{\mathbf{t}}(\hat{s}) &= \hat{\kappa}(\hat{s})\,\hat{\mathbf{n}}(\hat{s}), \\
\nabla_{\hat{\mathbf{t}}(\hat{s})}\hat{\mathbf{n}}(\hat{s}) &= \hat{\kappa}(\hat{s})\,\hat{\mathbf{t}}(\hat{s}) + \hat{\tau}(\hat{s})\,\hat{\mathbf{b}}(\hat{s}), \\
\nabla_{\hat{\mathbf{t}}(\hat{s})}\hat{\mathbf{b}}(\hat{s}) &= \hat{\tau}(\hat{s})\,\hat{\mathbf{n}}(\hat{s}),
\end{aligned} \tag{3.1}$$

where $\hat{\tau}(\hat{s})$ is the dual torsion of the dual curve $\hat{\gamma}(\hat{s})$. Here, we suppose that the dual torsion $\hat{\tau}(\hat{s})$ is never pure-dual. In addition,

$$\begin{aligned}
g\left(\hat{\mathbf{t}}(\hat{s}), \hat{\mathbf{t}}(\hat{s})\right) &= 1,\ g\left(\hat{\mathbf{n}}(\hat{s}), \hat{\mathbf{n}}(\hat{s})\right) = -1,\ g\left(\hat{\mathbf{b}}(\hat{s}), \hat{\mathbf{b}}(\hat{s})\right) = 1, \\
g\left(\hat{\mathbf{t}}(\hat{s}), \hat{\mathbf{n}}(\hat{s})\right) &= g\left(\hat{\mathbf{t}}(\hat{s}), \hat{\mathbf{b}}(\hat{s})\right) = g\left(\hat{\mathbf{n}}(\hat{s}), \hat{\mathbf{b}}(\hat{s})\right) = 0.
\end{aligned} \tag{3.2}$$

In the rest of the paper, we suppose everywhere $\hat{\kappa}(\hat{s}) \neq 0$ and $\hat{\tau}(\hat{s}) \neq 0$.

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. The Bishop frame is expressed as

$$\begin{aligned}
\nabla_{\hat{\mathbf{t}}(\hat{s})}\hat{\mathbf{t}}(\hat{s}) &= \hat{k}_1(\hat{s})\,\hat{\mathbf{m}}_1(\hat{s}) - \hat{k}_2(\hat{s})\,\hat{\mathbf{m}}_2(\hat{s}), \\
\nabla_{\hat{\mathbf{t}}(\hat{s})}\hat{\mathbf{m}}_1(\hat{s}) &= \hat{k}_1(\hat{s})\,\hat{\mathbf{t}}(\hat{s}), \\
\nabla_{\hat{\mathbf{t}}(\hat{s})}\hat{\mathbf{m}}_2(\hat{s}) &= \hat{k}_2(\hat{s})\,\hat{\mathbf{t}}(\hat{s}),
\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}
g\left(\hat{\mathbf{t}}(\hat{s}), \hat{\mathbf{t}}(\hat{s})\right) &= 1,\ g\left(\hat{\mathbf{m}}_1(\hat{s}), \hat{\mathbf{m}}_1(\hat{s})\right) = -1,\ g\left(\hat{\mathbf{m}}_2(\hat{s}), \hat{\mathbf{m}}_2(\hat{s})\right) = 1, \\
g\left(\hat{\mathbf{t}}(\hat{s}), \hat{\mathbf{m}}_1(\hat{s})\right) &= g\left(\hat{\mathbf{t}}(\hat{s}), \hat{\mathbf{m}}_2(\hat{s})\right) = g\left(\hat{\mathbf{m}}_1(\hat{s}), \hat{\mathbf{m}}_2(\hat{s})\right) = 0.
\end{aligned} \tag{3.4}$$

Here, we shall call the set $\{\hat{\mathbf{t}}, \hat{\mathbf{m}}_1, \hat{\mathbf{m}}_1\}$ as Bishop trihedra, $\hat{k}_1$ and $\hat{k}_2$ as Bishop curvatures and $\tau(s) = \hat{\theta}'(s)$, $\hat{\kappa}(s) = \sqrt{\left|\hat{k}_2^2 - \hat{k}_1^2\right|}$. Thus, Bishop curvatures are defined by

$$\begin{aligned} \hat{k}_1(\hat{s}) &= \hat{\kappa}(\hat{s}) \sinh \hat{\theta}(\hat{s}), \\ \hat{k}_2(\hat{s}) &= \hat{\kappa}(\hat{s}) \cosh \hat{\theta}(\hat{s}). \end{aligned} \tag{3.5}$$

Theorem 3.1. *Let $\hat{\gamma}$ be spacelike dual curve with timelike principal normal parametrized by dual arc length. $\hat{\gamma}$ is a spacelike dual biharmonic curve if and only if*

$$\begin{gathered} \hat{k}_1^2(\hat{s}) - \hat{k}_2^2(\hat{s}) = \hat{\Omega}, \\ \hat{k}_1''(\hat{s}) + \hat{k}_1^3(\hat{s}) - \hat{k}_2^2(\hat{s}) \hat{k}_1 = 0, \\ -\hat{k}_2''(\hat{s}) + \hat{k}_2^3(\hat{s}) - \hat{k}_1^2(\hat{s}) \hat{k}_2 = 0, \end{gathered} \tag{3.6}$$

where $\hat{\Omega}$ is dual constant of integration.

Proof. Using dual Bishop frame, we obtain above system.

Lemma 3.2. *Let $\hat{\gamma}$ be a spacelike dual curve with timelike principal normal parametrized by dual arc length. $\hat{\gamma}$ is a spacelike dual biharmonic curve if and only if*

$$\begin{gathered} -\hat{k}_1^2(\hat{s}) + \hat{k}_2^2(\hat{s}) = \hat{\Omega}, \\ \hat{k}_1''(\hat{s}) + \hat{k}_1(\hat{s}) \left(-\hat{k}_2^2(\hat{s}) + \hat{k}_1^2(\hat{s})\right) = 0, \\ \hat{k}_2''(\hat{s}) + \hat{k}_2(\hat{s}) \left(-\hat{k}_2^2(\hat{s}) + \hat{k}_1^2(\hat{s})\right) = 0, \end{gathered} \tag{3.7}$$

where $\hat{\Omega}$ is constant of integration.

Definition 3.3. *If x and y are real variable and $\varepsilon^2 = 0$, the combination*

$$\begin{gathered} F\left(X, Y, Y', ..., Y^{(n)}\right) = G(X) \\ = F\left(x, f(x), f'(x), ..., f^{(n)}(x)\right) \\ +\varepsilon x^* [F\left(x, f(x), f'(x), ..., f^{(n)}(x)\right)] \\ = g(x) + \varepsilon x^* g'(x) \end{gathered} \tag{3.8}$$

is called a differential equation with dual variable. Here,

$$
\begin{aligned}
X &= x+\varepsilon x^{*}, \\
Y &= f(x)+\varepsilon x^{*} f'(x), \\
Y' &= f'(x)+\varepsilon x^{*} f''(x), \\
&\ldots \\
Y^{(n)} &= f^{(n)}(x)+\varepsilon x^{*} f^{(n+1)}(s).
\end{aligned}
$$

If $y=f(x)$ is solution for real part of differential equation (3.8), $y=f(x)+c$ ($c=$ constant) is solution of dual part of differential equation (3.8).

Lemma 3.4. *If Y_{g_i} is a general solution of a differential equation of order i, then*

$$
\begin{aligned}
&Y_{g_1} \\
&Y_{g_2}=Y_{g_1}+c_1, \\
&Y_{g_3}=Y_{g_2}+c_2 x, \\
&Y_{g_4}=Y_{g_3}+c_3 x^2, \\
&\ldots \\
&Y_{g_n}=Y_{g_{n-1}}+c_{n-1} x^{n-2},
\end{aligned}
$$

where dependent variable Y_{g_i} and independent variable s.

Theorem 3.5. *Let $\hat{\gamma}$ be a non-geodesic spacelike dual curve with timelike principal normal parametrized by dual arc length. Then, general solution of a differential equation*

$$
\begin{aligned}
\hat{k}_1(\hat{s})_g &= C_1\cos[\sqrt{k_1^2-k_2^2}s]+C_2\sin[\sqrt{k_1^2-k_2^2}s] \\
&\quad+\varepsilon s^{*}[C_1\cos[\sqrt{k_1^2-k_2^2}s]+C_2\sin[\sqrt{k_1^2-k_2^2}s]+C_3 s],
\end{aligned}
\tag{3.9}
$$

where C_1, C_2, C_3 are constants of integration.

Proof. Using second equation of (3.7), we have

$$
\frac{d^2\hat{k}_1(\hat{s})}{d\hat{s}^2}+\hat{k}_1(\hat{s})\left(\hat{k}_1^2(\hat{s})-\hat{k}_2^2(\hat{s})\right)=0. \tag{3.10}
$$

Using above method in (3.10) we obtain

$$
\frac{d^2k_1}{ds^2}+\Omega k_1+\varepsilon s^{*}[\frac{d^3k_1}{ds^3}+\left(k_1^2(\hat{s})-k_2^2(\hat{s})\right)\frac{dk_1}{ds}]=0.
$$

If we calculate the real and dual parts of this equation, we get the following relations

$$\begin{aligned}
\frac{d^2k_1}{ds^2} + \left(k_1^2 - k_2^2\right) k_1 &= 0, \\
\frac{d^3k_1}{ds^3} + \left(k_1^2 - k_2^2\right) \frac{dk_1}{ds} &= 0.
\end{aligned}$$

Using Mathematica in above equations we obtain

$$k_1 = C_1 \cos[\sqrt{k_1^2 - k_2^2}s] + C_2 \sin[\sqrt{k_1^2 - k_2^2}s], \tag{3.11}$$

where C_1, C_2 are constants of integration.

On the other hand, solution of dual part of differential equation we have

$$k_1^* = k_1 + C_3 s, \tag{3.12}$$

where C_3 is constant of integration.

From (3.12) we obtain

$$k_1^* = C_1 \cos[\sqrt{k_1^2 - k_2^2}s] + C_2 \sin[\sqrt{k_1^2 - k_2^2}s] + C_3 s.$$

By substituting (3.11) and (3.12) in the last equation we get

$$\begin{aligned}
\hat{k}_1 \left(\hat{s}\right)_g &= C_1 \cos[\sqrt{k_1^2 - k_2^2}s] + C_2 \sin[\sqrt{k_1^2 - k_2^2}s] \\
&\quad + \varepsilon s^* [C_1 \cos[\sqrt{k_1^2 - k_2^2}s] + C_2 \sin[\sqrt{k_1^2 - k_2^2}s] + C_3 s].
\end{aligned}$$

Therefore, we have the equations (3.9), which it completes the proof.

Using Mathematica above Theorem, we obtain

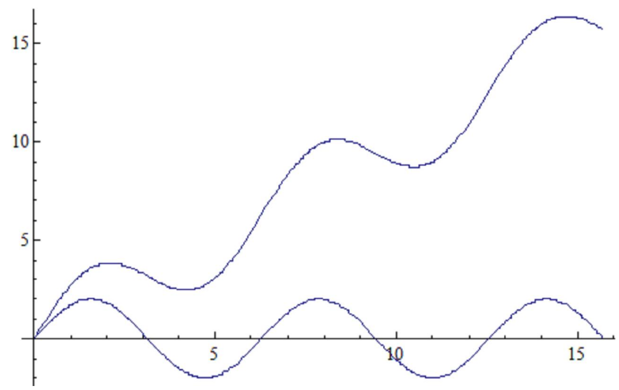

Fig 1.

Corollary 3.6. *Let $\hat{\gamma}$ be a non-geodesic spacelike dual curve with spacelike principal normal parametrized by dual arc length. Then, general solution of a differential equation*

$$\begin{aligned} \hat{k}_2\left(\hat{s}\right)_g &= C_4 \cos[\sqrt{k_1^2 - k_2^2} s] + C_5 \sin[\sqrt{k_1^2 - k_2^2} s] \\ &+ \varepsilon s^* [C_4 \cos[\sqrt{k_1^2 - k_2^2} s] + C_5 \sin[\sqrt{k_1^2 - k_2^2} s] + C_6 s], \end{aligned}$$

where C_4, C_5, C_6 are constants of integration.

According to Lemma 3.4, the proof of the Corollary 3.6 is similar to the proof of Theorem 3.5.

Smilarly, using Mathematica above Corollary, we obtain

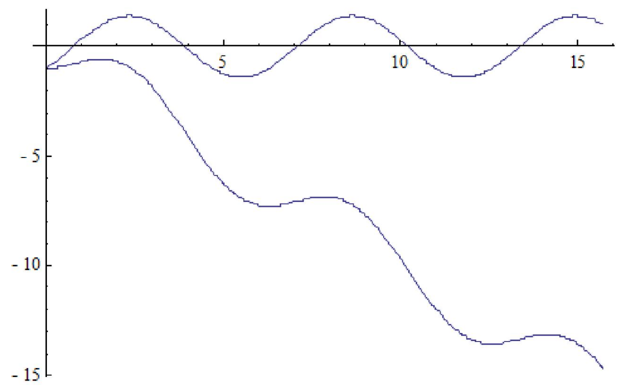

Fig 2.

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